Power Law Size Distributions

Principles of Complex Systems
Course 300, Fall, 2008

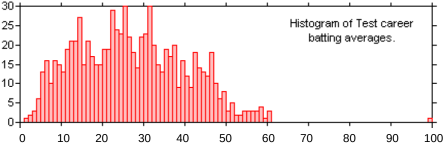
Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

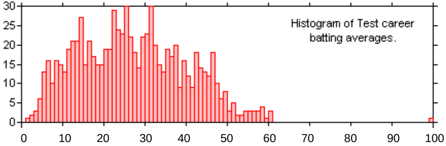
The Don

Extreme deviations in test cricket



The Don

Extreme deviations in test cricket



Don Bradman's batting average = 166% next best.

Outline

Size distributions

The sizes of many systems' elements appear to obey an inverse power-law size distribution:

$$P(\text{size} = x) \sim c\, x^{-\gamma}$$

$$\text{where} \quad x_{\min} < x < x_{\max}$$

$$\text{and} \quad \gamma > 1$$

- Typically, $2 < \gamma < 3$.
- $x_{\min}$ = lower cutoff
- $x_{\max}$ = upper cutoff

Size distributions

The sizes of many systems' elements appear to obey an inverse power-law size distribution:

$$P(\text{size} = x) \sim c\, x^{-\gamma}$$

$$\text{where} \quad x_{\min} < x < x_{\max}$$

$$\text{and} \quad \gamma > 1$$

- Typically, $2 < \gamma < 3$.
- $x_{\min}$ = lower cutoff
- $x_{\max}$ = upper cutoff

Size distributions

The sizes of many systems' elements appear to obey an inverse power-law size distribution:

$$P(\text{size} = x) \sim c\, x^{-\gamma}$$

$$\text{where} \quad x_{\min} < x < x_{\max}$$

$$\text{and} \quad \gamma > 1$$

- Typically, $2 < \gamma < 3$.
- $x_{\min}$ = lower cutoff
- $x_{\max}$ = upper cutoff

Size distributions

- Usually, only the tail of the distribution obeys a power law:

$$P(x) \sim c\,x^{-\gamma} \text{ as } x \rightarrow \infty.$$

- Still use term 'power law distribution'

Size distributions

- Usually, only the tail of the distribution obeys a power law:

$$P(x) \sim c\, x^{-\gamma} \text{ as } x \rightarrow \infty.$$

- Still use term 'power law distribution'

Size distributions

Many systems have discrete sizes k:

- Word frequency
- Node degree (as we have seen): # hyperlinks, etc.
- number of citations for articles, court decisions, etc.

$$P(k) \sim c\, k^{-\gamma}$$

$$\text{where } k_{\min} \le k \le k_{\max}$$

Size distributions

Power law size distributions are sometimes called Pareto distributions after Italian scholar Vilfredo Pareto.

- Pareto noted wealth in Italy was distributed unevenly (80–20 rule).
- Term used especially by economists

Size distributions

- Negative linear relationship in log-log space:

$$\log P(x) = \log c - \gamma \log x$$

Outline

Size distributions

Examples:

- Earthquake magnitude (Gutenberg Richter law): $P(M) \propto M^{-3}$
- Number of war deaths: $P(d) \propto d^{-1.8}$
- Sizes of forest fires
- Sizes of cities: $P(n) \propto n^{-2.1}$
- Number of links to and from websites

Size distributions

Examples:

- Number of citations to papers: $P(k) \propto k^{-3}$.
- Individual wealth (maybe): $P(W) \propto W^{-2}$.
- Distributions of tree trunk diameters: $P(d) \propto d^{-2}$.
- The gravitational force at a random point in the universe: $P(F) \propto F^{-5/2}$.
- Diameter of moon craters: $P(d) \propto d^{-3}$.
- Word frequency: e.g., $P(k) \propto k^{-2.2}$ (variable)

(Note: Exponents range in error; see M.E.J. Newman arxiv.org/cond-mat/0412004v3 (⊞))

Size distributions

Power-law distributions are..

- often called 'heavy-tailed'
- or said to have 'fat tails'

Important!:

- Inverse power laws aren't the only ones:
  - lognormals, stretched exponentials, ...

Size distributions

Power-law distributions are..

- often called 'heavy-tailed'
- or said to have 'fat tails'

Important!:

- Inverse power laws aren't the only ones:
  - lognormals, stretched exponentials, ...

Size distributions

Power-law distributions are..

- often called 'heavy-tailed'
- or said to have 'fat tails'

Important!:

- Inverse power laws aren't the only ones:
  - lognormals, stretched exponentials, ...

Size distributions

Power-law distributions are..

- often called 'heavy-tailed'
- or said to have 'fat tails'

Important!:

- Inverse power laws aren't the only ones:
  - lognormals, stretched exponentials, ...

Outline

Zipfian rank-frequency plots

George Kingsley Zipf:

- noted various rank distributions followed power laws, often with exponent -1 (word frequency, city sizes...) "Human Behaviour and the Principle of Least-Effort"[8] Addison-Wesley, Cambridge MA, 1949.
- We'll study Zipf's law in depth...

Zipfian rank-frequency plots

George Kingsley Zipf:

- noted various rank distributions followed power laws, often with exponent -1 (word frequency, city sizes...) "Human Behaviour and the Principle of Least-Effort"[8] Addison-Wesley, Cambridge MA, 1949.
- We'll study Zipf's law in depth...

Zipfian rank-frequency plots

Zipf's way:

- s_i = the size of the ith ranked object.
- $i = 1$ corresponds to the largest size.
- s_1 could be the frequency of occurrence of the most common word in a text.
- Zipf's observation:

$$s_i \propto i^{-\alpha}$$

Zipfian rank-frequency plots

Zipf's way:

- s_i = the size of the ith ranked object.
- $i = 1$ corresponds to the largest size.
- s_1 could be the frequency of occurrence of the most common word in a text.
- Zipf's observation:

$$s_i \propto i^{-\alpha}$$

Outline

Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See "The Black Swan" by Nassim Taleb [7])

Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See “The Black Swan” by Nassim Taleb [7])

Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See "The Black Swan" by Nassim Taleb [7])

Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See "The Black Swan" by Nassim Taleb [7])

Turkeys...

FIGURE 1: ONE THOUSAND AND ONE DAYS OF HISTORY

A turkey before and after Thanksgiving. The history of a process over a thousand days tells you nothing about what is to happen next. This naïve projection of the future from the past can be applied to anything.

From “The Black Swan” [7]

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Taleb's table [7]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Outline

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$

- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$

- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$

- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$

- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma + 1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of P follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of P follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of P follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of P follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.

Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

- Use integrals to approximate sums.

Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

$$= \sum_{k'=k}^{\infty} P(k)$$

- Use integrals to approximate sums.

Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

$$= \sum_{k'=k}^{\infty} P(k)$$

$$\propto k^{-\gamma+1}$$

- Use integrals to approximate sums.

Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

$$= \sum_{k'=k}^{\infty} P(k)$$

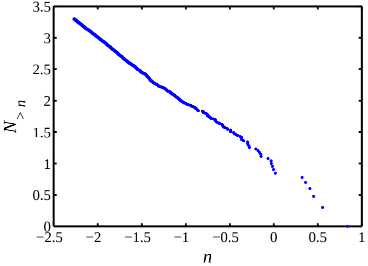
$$\propto k^{-\gamma+1}$$

- Use integrals to approximate sums.

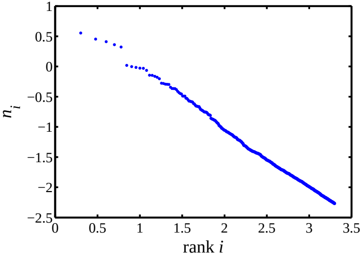
Size distributions

Brown Corpus (1,015,945 words):

CCDF:



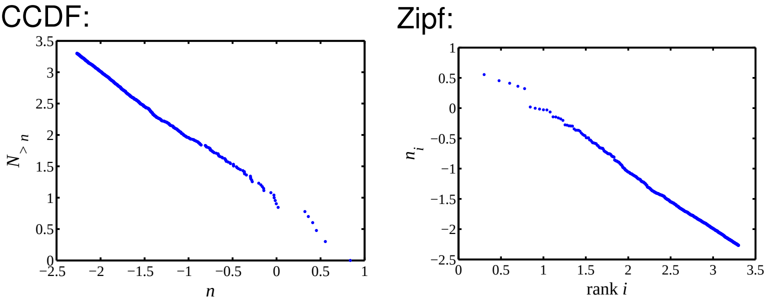
Zipf:



- The, of, and, to, a, ... = ‘objects’
- ‘Size’ = word frequency

Size distributions

Brown Corpus (1,015,945 words):



- The, of, and, to, a, ... = 'objects'
- 'Size' = word frequency
- Beep: CCDF and Zipf plots are related...

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Size distributions

Observe:

- $NP_{\geq}(x)$ = the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Details on the lack of scale:

Let's find the mean:

- $$\langle x \rangle = \int_{x=x_{\text{min}}}^{x_{\text{max}}} x P(x) \mathrm{d}x$$

$$= c \int_{x=x_{\text{min}}}^{x_{\text{max}}} x x^{-\gamma} \mathrm{d}x$$

$$= \frac{c}{2-\gamma} \left(x_{\text{max}}^{2-\gamma} - x_{\text{min}}^{2-\gamma} \right).$$

Details on the lack of scale:

Let's find the mean:

- $$\langle x \rangle = \int_{x=x_{\min}}^{x_{\max}} x P(x) \mathrm{d}x$$
  $$= c \int_{x=x_{\min}}^{x_{\max}} x x^{-\gamma} \mathrm{d}x$$
  $$= \frac{c}{2-\gamma} \left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma} \right).$$

Details on the lack of scale:

Let's find the mean:

- $$\langle x \rangle = \int_{x=x_{\min}}^{x_{\max}} x P(x) \mathrm{d}x$$
  $$= c \int_{x=x_{\min}}^{x_{\max}} x x^{-\gamma} \mathrm{d}x$$
  $$= \frac{c}{2-\gamma} \left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma} \right).$$

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma}\left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma}\right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma}\left(x_{\text{max}}^{2-\gamma} - x_{\text{min}}^{2-\gamma}\right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma} \left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma} \right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma}\left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma}\right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma}\left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma}\right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

And in general...

Moments:

- All moments depend only on cutoffs.
- No internal scale dominates (even matters).
- Compare to a Gaussian, exponential, etc.

And in general...

Moments:

- All moments depend only on cutoffs.
- No internal scale dominates (even matters).
- Compare to a Gaussian, exponential, etc.

And in general...

Moments:

- All moments depend only on cutoffs.
- No internal scale dominates (even matters).
- Compare to a Gaussian, exponential, etc.

Moments

For many real size distributions:

$$2 < \gamma < 3$$

- mean is finite (depends on lower cutoff)
- σ^2 = variance is ‘infinite’ (depends on upper cutoff)
- Width of distribution is ‘infinite’

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle| \rangle$$

- MAD is unpleasant analytically...

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle | \rangle$$

- MAD is unpleasant analytically...

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle| \rangle$$

- MAD is unpleasant analytically...

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle| \rangle$$

- MAD is unpleasant analytically...

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle | \rangle$$

- MAD is unpleasant analytically...

How sample sizes grow...

Given $P(x) \sim cx^{-\gamma}$:

- We can show that after n samples, we expect the largest sample to be

$$x_1 \gtrsim n^{1/(\gamma-1)}$$

- Sampling from a 'mild' distribution gives a much slower growth with n.
- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find

$$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$

How sample sizes grow...

Given $P(x) \sim cx^{-\gamma}$:

- We can show that after n samples, we expect the largest sample to be

$$x_1 \gtrsim n^{1/(\gamma-1)}$$

- Sampling from a 'mild' distribution gives a much slower growth with n.
- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find

$$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$

How sample sizes grow...

Given $P(x) \sim cx^{-\gamma}$:

- We can show that after n samples, we expect the largest sample to be

$$x_1 \gtrsim n^{1/(\gamma-1)}$$

- Sampling from a 'mild' distribution gives a much slower growth with n.
- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find

$$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$

References I

M. Abramowitz and I. A. Stegun, editors.
Handbook of Mathematical Functions.
Dover Publications, New York, 1974.

P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
Annu. Rev. Earth Planet. Sci., 28:571–610, 2000.
pdf (⊞)

W. Feller.
An Introduction to Probability Theory and Its Applications, volume I.
John Wiley & Sons, New York, third edition, 1968.

I. Gradshteyn and I. Ryzhik.
Table of Integrals, Series, and Products.
Academic Press, San Diego, fifth edition, 1994.

References II

J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957.

A. E. Scheidegger.
The algebra of stream-order numbers.
United States Geological Survey Professional Paper, 525-B:B187–B189, 1967.

N. N. Taleb.
The Black Swan.
Random House, New York, 2007.

G. K. Zipf.
Human Behaviour and the Principle of Least-Effort.
Addison-Wesley, Cambridge, MA, 1949.