Power Law Size Distributions

Principles of Complex Systems
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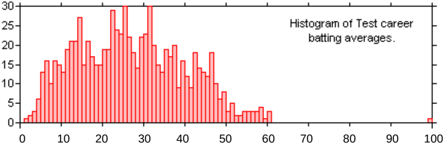
Outline

The Don

Extreme deviations in test cricket



Don Bradman's batting average = 166% next best.

Size distributions

The sizes of many systems' elements appear to obey an inverse power-law size distribution:

$$P(\text{size} = x) \sim c\,x^{-\gamma}$$

$$\text{where} \quad x_{\min} < x < x_{\max}$$

$$\text{and} \quad \gamma > 1$$

- Typically, $2 < \gamma < 3$.
- $x_{\min}$ = lower cutoff
- $x_{\max}$ = upper cutoff

Size distributions

- Usually, only the tail of the distribution obeys a power law:

$$P(x) \sim c\, x^{-\gamma} \text{ as } x \to \infty.$$

- Still use term 'power law distribution'

Size distributions

Many systems have discrete sizes k:

- Word frequency
- Node degree (as we have seen): # hyperlinks, etc.
- number of citations for articles, court decisions, etc.

$$P(k) \sim c\, k^{-\gamma}$$

$$\text{where } k_{\min} \le k \le k_{\max}$$

Size distributions

Power law size distributions are sometimes called Pareto distributions after Italian scholar Vilfredo Pareto.

- Pareto noted wealth in Italy was distributed unevenly (80–20 rule).
- Term used especially by economists

Size distributions

- Negative linear relationship in log-log space:

$$\log P(x) = \log c - \gamma \log x$$

Size distributions

Examples:

- Earthquake magnitude (Gutenberg Richter law): $P(M) \propto M^{-3}$
- Number of war deaths: $P(d) \propto d^{-1.8}$
- Sizes of forest fires
- Sizes of cities: $P(n) \propto n^{-2.1}$
- Number of links to and from websites

Size distributions

Examples:

- Number of citations to papers: $P(k) \propto k^{-3}$.
- Individual wealth (maybe): $P(W) \propto W^{-2}$.
- Distributions of tree trunk diameters: $P(d) \propto d^{-2}$.
- The gravitational force at a random point in the universe: $P(F) \propto F^{-5/2}$.
- Diameter of moon craters: $P(d) \propto d^{-3}$.
- Word frequency: e.g., $P(k) \propto k^{-2.2}$ (variable)

(Note: Exponents range in error; see M.E.J. Newman arxiv.org/cond-mat/0412004v3 (⊞))

Size distributions

Power-law distributions are..

- often called ‘heavy-tailed’
- or said to have ‘fat tails’

Important!:

- Inverse power laws aren’t the only ones:
  - lognormals, stretched exponentials, ...

Zipfian rank-frequency plots

George Kingsley Zipf:

- noted various rank distributions
  followed power laws, often with exponent -1
  (word frequency, city sizes...) “Human Behaviour and the Principle of Least-Effort”[2] Addison-Wesley, Cambridge MA, 1949.
- We’ll study Zipf’s law in depth...

Zipfian rank-frequency plots

Zipf's way:

- s_i = the size of the ith ranked object.
- $i = 1$ corresponds to the largest size.
- s_1 could be the frequency of occurrence of the most common word in a text.
- Zipf's observation:

$$s_i \propto i^{-\alpha}$$

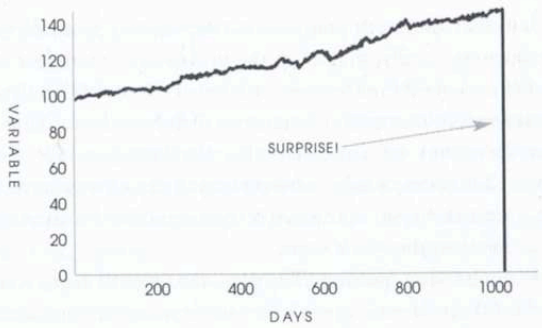
Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See “The Black Swan” by Nassim Taleb [1])

Turkeys...

FIGURE 1: ONE THOUSAND AND ONE DAYS OF HISTORY



A turkey before and after Thanksgiving. The history of a process over a thousand days tells you nothing about what is to happen next. This naïve projection of the future from the past can be applied to anything.

From “The Black Swan”[1]

Taleb's table [1]

Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/ It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the accidental

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$

- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$

- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$

- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$

- $$\propto x^{-\gamma+1}$$

Complementary Cumulative Distribution Function:

CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of P follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.

Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

$$= \sum_{k'=k}^{\infty} P(k)$$

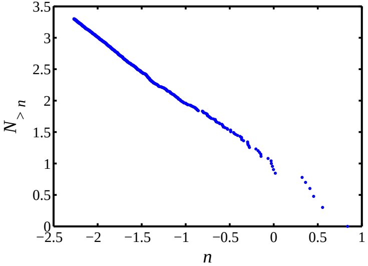
$$\propto k^{-\gamma+1}$$

- Use integrals to approximate sums.

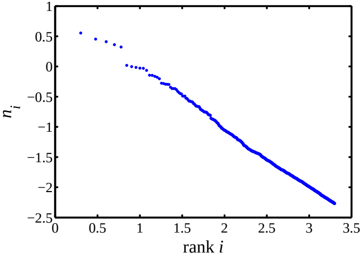
Size distributions

Brown Corpus (1,015,945 words):

CCDF:



Zipf:



- The, of, and, to, a, ... = 'objects'
- 'Size' = word frequency
- Beep: CCDF and Zipf plots are related...

Size distributions

Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least x where N = total number of objects.
- If an object has size x_i, then $NP_{\geq}(x_i)$ is its rank i.
- So

$$x_i \propto i^{-\alpha} = (NP_{\geq}(x_i))^{-\alpha}$$

$$\propto x_i^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\alpha = \frac{1}{\gamma - 1}$$

A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.

Details on the lack of scale:

Let's find the mean:

- $$\langle x \rangle = \int_{x=x_{\text{min}}}^{x_{\text{max}}} x P(x) \mathrm{d}x$$
  $$= c \int_{x=x_{\text{min}}}^{x_{\text{max}}} x x^{-\gamma} \mathrm{d}x$$
  $$= \frac{c}{2-\gamma} \left(x_{\text{max}}^{2-\gamma} - x_{\text{min}}^{2-\gamma} \right).$$

The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma}\left(x_{\max}^{2-\gamma} - x_{\min}^{2-\gamma}\right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.

And in general...

Moments:

- All moments depend only on cutoffs.
- No internal scale dominates (even matters).
- Compare to a Gaussian, exponential, etc.

Moments

For many real size distributions:

$$2 < \gamma < 3$$

- mean is finite (depends on lower cutoff)
- σ^2 = variance is 'infinite' (depends on upper cutoff)
- Width of distribution is 'infinite'

Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

  Mean average deviation (MAD) =

$$\langle |x - \langle x \rangle| \rangle$$

- MAD is unpleasant analytically...

How sample sizes grow...

Given $P(x) \sim cx^{-\gamma}$:

- We can show that after n samples, we expect the largest sample to be

$$x_1 \gtrsim n^{1/(\gamma-1)}$$

- Sampling from a 'mild' distribution gives a much slower growth with n.
- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find

$$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$

References I

N. N. Taleb.
The Black Swan.
Random House, New York, 2007.

G. K. Zipf.
Human Behaviour and the Principle of Least-Effort.
Addison-Wesley, Cambridge, MA, 1949.