Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems Course 300. Fall. 2008

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Department of Mathematics & Statistics University of Vermont



Random Walks

Variable transformation

Growth

Mechanisms

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The First Return Problem Examples

Variable transformation

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A powerful theme in complex systems:

structure arises out of randomness.

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A powerful theme in complex systems:

structure arises out of randomness.

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A powerful theme in complex systems:

- structure arises out of randomness.
- ► Exhibit A: Random walks... (⊞)

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The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin x = 0.
- ▶ Step at time *t* is ϵ_t :

$$\epsilon_t = \left\{ \begin{array}{ll} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{array} \right.$$



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$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2\\ -1 & \text{with probability } 1/2 \end{cases}$$



$$x_t = \sum_{i=1}^t \epsilon_i$$

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \left\langle \epsilon_i \right\rangle = 0$$

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$$Var(x_t) = Var\left(\sum_{i=1}^{t} \epsilon_i\right)$$
$$= \sum_{i=1}^{t} Var(\epsilon_i) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread

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Variances sum: (⊞)*

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So typical displacement from the origin scales as

 $\sigma = t^{1/2}$

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So typical displacement from the origin scales as

 $\sigma = t^{1/2}$

⇒ A non-trivial power-law arises out of additive aggregation or accumulation.

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For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- ▶ Think of a coin flip game with ten thousand tossess
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...

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Random walks are weirder than you might think...

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- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.

See Feller, [3] Intro to Probability Theory, Volume I



$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

Even crazier:

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$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

Even crazier:

The expected time between tied scores = ∞ !

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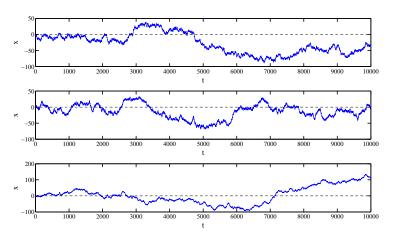




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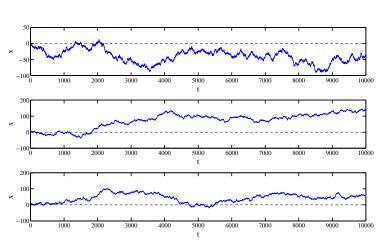






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- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- ▶ What about higher dimensions?

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- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?





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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?





- We will find a power-law size distribution with an interesting exponent
- Some physical structures may result from random walks
- We'll start to see how different scalings relate to each other

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- We will find a power-law size distribution with an interesting exponent
- Some physical structures may result from random walks
- 3. We'll start to see how different scalings relate to each other

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Reasons for caring:

- 1. We will find a power-law size distribution with an interesting exponent
- 2. Some physical structures may result from random walks

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Reasons for caring:

- 1. We will find a power-law size distribution with an interesting exponent
- 2. Some physical structures may result from random walks
- 3. We'll start to see how different scalings relate to each other

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Outline

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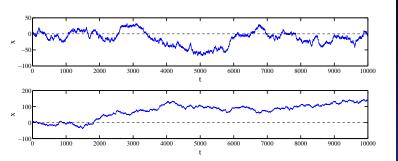
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Random Walks



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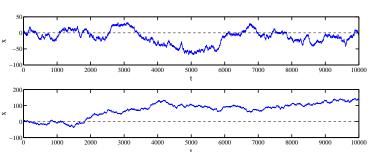
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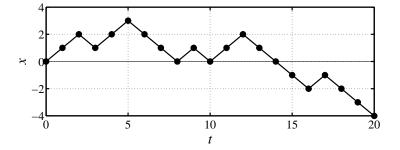


Again: expected time between ties = ∞ ... Let's find out why... [3]

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▶ Return can only happen when t = 2n.

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- ▶ Return can only happen when t = 2n.
- ► Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at t = 2n.

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- ▶ Return can only happen when t = 2n.
- ► Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at t=2n.
- Assume drunkard first lurches to x = 1.

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For random walks in 1-d:

- ▶ Return can only happen when t = 2n.
- ► Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at t = 2n.
- ightharpoonup Assume drunkard first lurches to x = 1.
- ► The problem

$$P_{\text{fr}}(2n) = Pr(x_t \ge 1, t = 1, \dots, 2n - 1, \text{ and } x_{2n} = 0)$$

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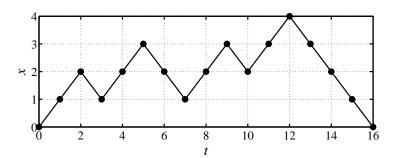
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- ▶ A useful restatement: $P_{fr}(2n) = \frac{1}{2} Pr(x_t \ge 1, t = 1, ..., 2n 1, \text{ and } x_1 = x_{2n-1} = 1)$
- ▶ Want walks that can return many times to x = 1.
- ► (The $\frac{1}{2}$ accounts for stepping to 2 or -2 instead of 0 at t = 2n.)

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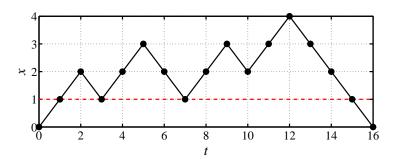
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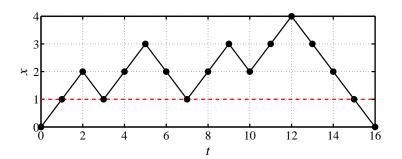
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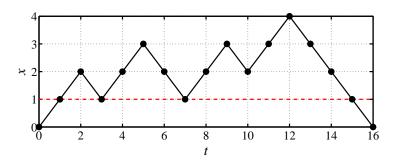
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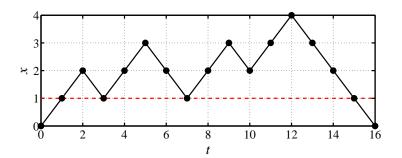
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- Use a method of images
- \triangleright Define N(i, j, t) as the # of possible walks between
- \triangleright Consider all paths starting at x = 1 and ending at x = 1 after t = 2n - 2 steps.
- Subtract how many hit x = 0.

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- Use a method of images
- ▶ Define N(i, j, t) as the # of possible walks between x = i and x = j taking t steps.
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- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- ▶ Define N(i, j, t) as the # of possible walks between x = i and x = i taking t steps.
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mechanics)

Use a method of images

x = i and x = i taking t steps.

Counting problem (combinatorics/statistical

▶ Define N(i, j, t) as the # of possible walks between

▶ Consider all paths starting at x = 1 and ending at



Key observation:

of t-step paths starting and ending at x = 1 and hitting x = 0 at least once

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```
# of t-step paths starting and ending at x = 1
and hitting x = 0 at least once
```

= # of t-step paths starting at x = -1 and ending at x = 1

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= # of t-step paths starting at
$$x = -1$$
 and ending at $x = 1$

$$= N(-1, 1, t)$$

of t-step paths starting and ending at x = 1and hitting x = 0 at least once

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of t-step paths starting and ending at x = 1

and hitting x = 0 at least once

= # of t-step paths starting at x = -1 and ending at x = 1

$$= N(-1, 1, t)$$

So
$$N_{\text{first return}}(2n) = \frac{1}{2} [N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)]$$

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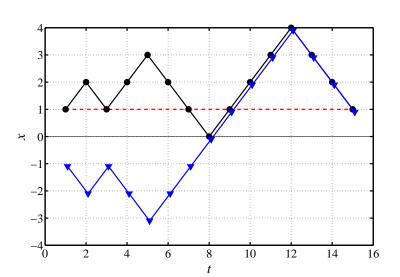
= # of t-step paths starting at x = -1 and ending at x = 1

= N(-1, 1, t)

So $N_{\text{first return}}(2n) = \frac{1}{2} [N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)]$

See this 1-1 correspondence visually...





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- For any path starting at x = 1 that hits 0, there is a unique matching path starting at x = -1.
- Matching path first mirrors and then tracks.

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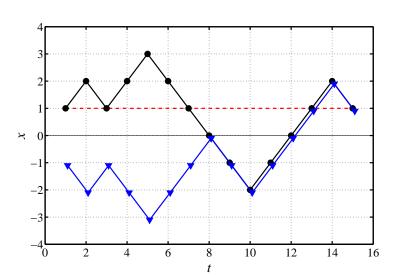
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Matching path first mirrors and then tracks.

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- \blacktriangleright # positive steps + # negative steps = t.
- \triangleright Random walk must displace by j-i after t steps
- ▶ # positive steps # negative steps = j i.
- # positive steps = (t + j i)/2.

$$N(i, j, t) = \begin{pmatrix} t \\ \# \text{ positive steps} \end{pmatrix} = \begin{pmatrix} t \\ (t + j - i)/2 \end{pmatrix}$$

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Next problem: what is N(i, j, t)?

 \blacktriangleright # positive steps = (t + i - i)/2.

positive steps + # negative steps = t.

▶ # positive steps - # negative steps = i - i.

$$N(i, j, t) = \begin{pmatrix} t \\ \# \text{ positive steps} \end{pmatrix} = \begin{pmatrix} t \\ (t + j - i)/2 \end{pmatrix}$$

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$$N_{\text{first return}}(2n) = \frac{1}{2} [N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)]$$

where

$$N(i,j,t) = \binom{t}{(t+j-i)/2}$$

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[Assignment question occurs]

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- ► Total number of possible paths = 2^{2n}

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$$

$$=\frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

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[Assignment question occurs]

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$$=\frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

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Find
$$N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$
.

- Normalized Number of Paths gives Probability
- ► Total number of possible paths = 2^{2n}

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$$

$$=\frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

Variable transformation

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Same scaling holds for continuous space/time walks.

$$P(t) \propto t^{-3/2}, \ \gamma = 3/2$$

- \triangleright P(t) is normalizable
- Recurrence: Random walker always returns to origin
- Moral: Repeated gambling against an infinitely

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Growth

- \blacktriangleright Walker in d=2 dimensions must also return
- ▶ For d = 1, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- ► Even though walker must return, expect a long wait...



Growth

- \blacktriangleright Walker in d=2 dimensions must also return
- ▶ Walker may not return in d > 3 dimensions
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Growth

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Higher dimensions:

- \blacktriangleright Walker in d=2 dimensions must also return
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- Even though walker must return, expect a long wait...



Growth

- In any finite volume, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a
- Non-trivial Invariant Densities arise in chaotic



Growth

On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
- ▶ Random walking ≡ Diffusion
- Call this probability the Invariant Density of a
- Non-trivial Invariant Densities arise in chaotic



Growth

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Growth

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On networks:

- ightharpoonup On networks, a random walker visits each node with frequency \propto node degree
- Equal probability still present: walkers traverse edges with equal frequency

Frame 30/88



Growth

- On networks, a random walker visits each node with frequency \propto node degree
- Equal probability still present: walkers traverse edges with equal frequency.

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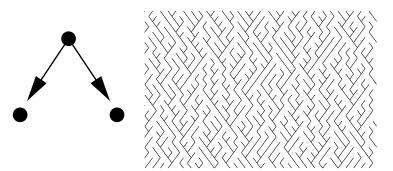
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- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Frame 32/88





- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

▶ Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$

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gives random walk with increments

Creates basins with random walk boundaries

Observe Subtracting one random walk from another

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Frame 33/88

- ▶ For a basin of length ℓ , width $\propto \ell^{1/2}$
- ▶ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- ► Invert: $\ell \propto a^{2/3}$
- $ightharpoonup \mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3a^{-1/3}\mathrm{d}a$
- ► Pr(basin area = a)da= $Pr(\text{basin length} = \ell)\text{d}\ell$ $\propto \ell^{-3/2}\text{d}\ell$ $\propto (a^{2/3})^{-3/2}a^{-1/3}\text{d}a$ = $a^{-4/3}\text{d}a$

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 - $= a^{-\tau} da$

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Growth

Mechanisms

- Both basin area and length obey power law distributions
- Observed for real river networks
- ▶ Typically: $1.3 < \beta < 1.5$ and $1.5 < \gamma < 2$
- \triangleright Smaller basins more allometric (h > 1/2)
- ▶ Larger basins more isometric (h = 1/2)

Frame 35/88



Growth

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Generalize relationship between area and length

► Hack's law ^[4]:

▶ Redo calc with γ , τ , and h.

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Generalize relationship between area and length

Hack's law [4]:

$$\ell \propto a^h$$

where
$$0.5 \lesssim h \lesssim 0.7$$

▶ Redo calc with γ , τ , and h.

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- Generalize relationship between area and length
- ► Hack's law [4]:

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$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \; {\rm and} \; P(\ell) \propto \ell^{-\gamma}$$

- $d\ell \propto d(a^h) = ha^{h-1}da$
- ► Pr(basin area = a)da= $Pr(\text{basin length} = \ell)d\ell$ $\propto \ell^{-1}d\ell$ $\propto (a^h)^{-1}a^{h-1}da$

$$\tau = 1 + h(\gamma - 1)$$

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$$| au = \mathbf{1} + h(\gamma - \mathbf{1})|$$

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$$\tau = 2 - h$$

$$\gamma = 1/h$$

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With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

Only one exponent is independent

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- Simplify system description

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With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

- Only one exponent is independent
- Simplify system description
- Expect scaling relations where power laws are found

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With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

- Only one exponent is independent
- Simplify system description
- Expect scaling relations where power laws are found
- Characterize universality class with independent exponents

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 \triangleright x_0 could be > 0.

Variable

Failure

Dispersion of suspended sediments in streams.

A very simple model of failure/death:

 $\rightarrow x_t$ = entity's 'health' at time t

Entity fails when x hits 0.

Long times for clearing.

Frame 39/88



 \triangleright x_0 could be > 0.

Examples

Variable

Failure

Streams

Dispersion of suspended sediments in streams.

A very simple model of failure/death:

 $\rightarrow x_t$ = entity's 'health' at time t

Entity fails when x hits 0.

Long times for clearing.

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▶ In 1-d.

Variable

Growth

Extensive memory of path now matters...

 Can generalize to Fractional Random Walks Levy flights, Fractional Brownian Motion



- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- ▶ In 1-d.

Extensive memory of path now matters...





Variable

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- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- In 1-d,

$$\sigma \sim t^{\alpha}$$

Extensive memory of path now matters...





Variable

- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- ▶ In 1-d,

$$\sigma \sim t^{\alpha}$$

$$\alpha > 1/2$$
 — superdiffusive $\alpha < 1/2$ — subdiffusive

Extensive memory of path now matters...



Growth

- Can generalize to Fractional Random Walks Levy flights, Fractional Brownian Motion
- ▶ In 1-d,

$$\sigma \sim t^{\alpha}$$

$$\alpha > 1/2$$
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Extensive memory of path now matters...



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Understand power laws as arising from

1. elementary distributions (e.g., exponentials)

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Understand power laws as arising from

- 1. elementary distributions (e.g., exponentials)
- 2. variables connected by power relationships

Frame 42/88



▶ Random variable X with known distribution P_x

▶ Second random variable *Y* with y = f(x).

$$\begin{aligned} & P_{y}(y) dy &= P_{x}(x) dx \\ &= \sum_{y \mid f(x) = y} P_{x}(f^{-1}(y)) \frac{dy}{\left| f'(f^{-1}(y)) \right|} \end{aligned}$$

Easier to do by hand...

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Variable

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Easier to do by hand...





- Random variable X with known distribution P_x
- ▶ Second random variable *Y* with y = f(x).

$$P_{y}(y)dy = P_{x}(x)dx$$

$$= \sum_{y|f(x)=y} P_{x}(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|}$$

Easier to do by hand...



Mechanisms

- Random variable X with known distribution P_x
- ▶ Second random variable *Y* with y = f(x).

$$P_{y}(y)dy = P_{x}(x)dx$$

$$= \sum_{y|f(x)=y} P_{x}(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|}$$

Easier to do by hand...



▶ Power-law relationship between variables:

$$y = cx^{-\alpha}, \, \alpha > 0$$

► Look at *y* large and *x* small

$$\mathrm{d}y = \mathrm{d}\left(\mathbf{c}\mathbf{x}^{-\alpha}\right)$$

Power-Law Mechanisms

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- ► Power-law relationship between variables:
 - $y = cx^{-\alpha}, \alpha > 0$
- Look at y large and x small

$$\mathrm{d}y = \mathrm{d}\left(\mathbf{c}x^{-\alpha}\right)$$

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Power-law relationship between variables:

$$y=cx^{-\alpha}, \, \alpha>0$$

Look at y large and x small

$$\mathrm{d}y=\mathrm{d}\left(\boldsymbol{c}\boldsymbol{x}^{-\alpha}\right)$$

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Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = ay^{-\alpha}$
 - $y = cx^{-\alpha}, \, \alpha > 0$
- Look at y large and x small

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

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- Assume relationship between x and y is 1-1.
 - Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
 - ► Look at y large and x small

 $\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

invert:
$$dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

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- Power-law relationship between variables: $y = cx^{-\alpha}, \alpha > 0$
- Look at y large and x small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$
invert:
$$dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}dy$$

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- Power-law relationship between variables: $y = cx^{-\alpha}, \alpha > 0$
- Look at y large and x small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$
invert:
$$dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy$$

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Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

► So $P_y(y) \propto y^{-1-1/\alpha}$ as $y \to \infty$ providing $P_x(x) \to \text{constant as } x \to 0$.

Frame 45/88



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_{y}(y)dy = P_{x}\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy$$

► So $P_v(y) \propto y^{-1-1/\alpha}$ as $y \to \infty$ $P_x(x) \to \text{constant as } x \to 0.$

Random Walks

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Frame 45/88



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_{y}(y)dy = P_{x}\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)\frac{dx}{\alpha}y^{-1-1/\alpha}dy$$

So $P_y(y) \propto y^{-1-1/\alpha}$ as $y \to \infty$ providing $P_x(x) \to \text{constant as } x \to 0$.

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$$P_{y}(y)dy = P_{x}\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy$$

▶ If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/lpha}$$
 as $y o \infty$

Frame 46/88



▶ If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then

$$P_{\nu}(y) \propto y^{-1-1/\alpha-\beta/\alpha}$$
 as $y \to \infty$

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Given
$$P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$$
 and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

Variable transformation Basics

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Frame 47/88



Exponential distribution

Given
$$P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$$
 and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

Exponentials arise from randomness...

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Exponential distribution

Given
$$P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$$
 and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness...
- More later when we cover robustness.

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Variable transformation

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▶ Select a random point in space \vec{x}

- ▶ Measure the force of gravity $F(\vec{x})$
- ▶ Observe that $P_F(F) \sim F^{-5/2}$.

Frame 49/88



Variable transformation

Holtsmark's Distribution

Growth

References

▶ Select a random point in space \vec{x}

- ▶ Measure the force of gravity $F(\vec{x})$
- ▶ Observe that $P_F(F) \sim F^{-5/2}$.

Frame 49/88



Holtsmark's Distribution

Growth

▶ Measure the force of gravity $F(\vec{x})$ ▶ Observe that $P_F(F) \sim F^{-5/2}$.

▶ Select a random point in space \vec{x}

Frame 49/88



- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = 0$

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space
- Assume only one star has significant effect at \vec{x} .
- Law of gravity:

$$F \propto r^{-2}$$

▶ invert:

$$r \propto F^{-1/2}$$

Random Walks

Variable transformation

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References





- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space
- ▶ Assume only one star has significant effect at \vec{x} .
- ► Law of gravity:

$$F \propto r^{-2}$$

▶ invert:

$$r \propto F^{-1/2}$$

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.....





- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

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▶ invert:

$$r \propto F^{-1/2}$$

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- F is distributed unevenly
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$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space
- Assume only one star has significant effect at \vec{x} .
- Law of gravity:

$$F \propto r^{-2}$$

invert:

$$r \propto F^{-1/2}$$

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$$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$

$$\propto r^{-3} dr$$

▶ invert:

$$dr \propto r^3 dF$$

$$\propto F^{-3/2} \mathrm{d}F$$

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Frame 51/88



$$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$

$$\propto r^{-3} dr$$

$$dr \propto r^3 dF$$

$$\propto F^{-3/2} \mathrm{d}F$$

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$$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$

$$\propto r^{-3} dr$$

invert:

$$\mathrm{d}r \propto r^3 \mathrm{d}F$$

ì

$$\propto F^{-3/2} \mathrm{d}F$$

Power-Law Mechanisms

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References

Frame 51/88



$$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$

$$\propto r^{-3} dr$$

invert:

$$\mathrm{d}r \propto r^3 \mathrm{d}F$$

$$\propto F^{-3/2} \mathrm{d}F$$

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 $P_F(F) dF = P_r(r) dr$

Frame 52/88



Variable transformation

Holtsmark's Distribution

References

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(F^{-1/2})F^{-3/2} \mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2} \mathrm{d}F$$

$$= F^{-1-3/2} dF$$

$$= F^{-5/2} dF$$

Frame 52/88



Using
$$r \propto F^{-1/2}$$
, $dr \propto F^{-3/2}dF$ and $P_r(r) \propto r^2$

$$|P_r(r) \propto r^2$$

$$P_F(F)dF = P_r(r)dr$$

$$\propto P_r(F^{-1/2})F^{-3/2} dF$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2} \mathrm{d}F$$

$$= F^{-1-3/2} dF$$

$$= F^{-5/2} dF$$

Variable transformation

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$$P_F(F)dF = P_r(r)dr$$

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$$= F^{-1-3/2} dF$$

$$= F^{-5/2} \mathrm{d}F$$

Variable transformation

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Frame 52/88



$$P_F(F)dF = P_r(r)dr$$

$$\propto P_r(F^{-1/2})F^{-3/2}dF$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2} \mathrm{d}F$$

$$= F^{-1-3/2} dF$$

$$= F^{-5/2} dF$$

Variable transformation

Holtsmark's Distribution

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References

Frame 52/88



$$P_F(F) = F^{-5/2} \mathrm{d}F$$

$$\gamma = 5/2$$

- Mean is finite
- ightharpoonup Variance = ∞
- A wild distribution
- Random sampling of space usually safe

Variable transformation

Holtsmark's Distribution

References



$$P_F(F) = F^{-5/2} \mathrm{d}F$$

$$\gamma = 5/2$$

- Mean is finite
- ightharpoonup Variance = ∞
- A wild distribution
- Random sampling of space usually safe

Variable transformation

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$$P_F(F) = F^{-5/2} \mathrm{d}F$$

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- Mean is finite
- ▶ Variance = ∞
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Variable transformation

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$$\gamma = 5/2$$

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- ▶ Variance = ∞
- A wild distribution
- Random sampling of space usually safe

Variable transformation

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$$\gamma = 5/2$$

- Mean is finite
- ▶ Variance = ∞
- A wild distribution
- Random sampling of space usually safe but can end badly...

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PLIPLO = Power law in, power law out

Random Walks

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PLIPLO = Power law in, power law out

Explain a power law as resulting from another unexplained power law.

Random Walks

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PLIPLO = Power law in, power law out

Explain a power law as resulting from another unexplained power law.

Don't do this!!! (slap, slap)

Random Walks

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PLIPLO = Power law in, power law out

Explain a power law as resulting from another unexplained power law.

Don't do this!!! (slap, slap)

We need mechanisms!

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- Mechanism: Random addition and subtraction
- Mechanism. Handom addition and Subtraction
- Compare across realizations, no competition
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
- Competing mechanisms (trickiness)

Variable transformation Basics

PLIPLO

Growth Mechanisms Random Copying Words, Cities, and the Web

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- Next: Random Additive/Copying Processes involving
- Widespread: Words, Cities, the Web, Wealth,
- Competing mechanisms (trickiness)



- Mechanism: Random addition and subtraction.
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving
- Widespread: Words, Cities, the Web, Wealth,
- Competing mechanisms (trickiness)





- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction.
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth,
- Competing mechanisms (trickiness)

Compare across realizations, no competition.

Random walks represent additive aggregation

- Next: Random Additive/Copying Processes involving Competition.
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- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction.
- Compare across realizations, no competition.
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- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
- Competing mechanisms (trickiness)



Work of Yore

- ► 1924: G. Udny Yule [13]: # Species per Genus
- ► 1926: Lotka [6]:
 # Scientific papers per author (Lotka's law)
- 1953: Mandelbrot^[7]: Optimality argument for Zipf's law; focus on language.
- 1955: Herbert Simon [11, 14]: Zipf's law for word frequency, city size, income, publications, and species per genus.
- ▶ 1965/1976: Derek de Solla Price [8, 9]: Network of Scientific Citations.
- ▶ 1999: Barabasi and Albert [1]: The World Wide Web, networks-at-large

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- ► 1924: G. Udny Yule [13]: # Species per Genus
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Variable transformation Basics

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Variable transformation Basics

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References



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▶ 1999: Barabasi and Albert [1]: The World Wide Web, networks-at-large. Random Walks
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Variable transformation Basics

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References



- 1. Start with 1 element of a particular flavor at t = 1
- At time t = 2,3,4,..., add a new element in one o two ways:
 - With probability ρ, create a new element with a new flavor

With probability $1 - \rho$, randomly choose from all existing elements, and make a copy.

Elements of the same flavor form a group

Random Walks
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Examples

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References





- 1. Start with 1 element of a particular flavor at t=1
- 2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
 - With probability ρ , create a new element with a new flavor

- With probability 1ρ , randomly choose from all
- Elements of the same flavor form a group



- 1. Start with 1 element of a particular flavor at t=1
- 2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
 - With probability ρ, create a new element with a new flavor

- With probability 1ρ , randomly choose from all existing elements, and make a copy.
- Elements of the same flavor form a group



Random Competitive Replication (RCR):

- 1. Start with 1 element of a particular flavor at t=1
- 2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
 - With probability ρ, create a new element with a new flavor

- With probability 1ρ , randomly choose from all existing elements, and make a copy.
- Elements of the same flavor form a group



- 1. Start with 1 element of a particular flavor at t=1
- 2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
 - With probability ρ , create a new element with a new flavor
 - ➤ Mutation/Innovation
 - With probability 1ρ , randomly choose from all existing elements, and make a copy.
 - Elements of the same flavor form a group



- 1. Start with 1 element of a particular flavor at t = 1
- 2. At time *t* = 2,3,4,..., add a new element in one of two ways:
 - With probability ρ , create a new element with a new flavor
 - ➤ Mutation/Innovation
 - With probability 1ρ , randomly choose from all existing elements, and make a copy.
 - ➤ Replication/Imitation
 - Elements of the same flavor form a group

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References



Example: Words in a text

- Consider words as they appear sequentially.

 \triangleright With probability 1 – ρ , randomly choose one word



Example: Words in a text

- Consider words as they appear sequentially.

 \triangleright With probability 1 – ρ , randomly choose one word



- Consider words as they appear sequentially.
- \triangleright With probability ρ , the next word has not previously appeared

 \triangleright With probability 1 – ρ , randomly choose one word





Example: Words in a text

- Consider words as they appear sequentially.
- \triangleright With probability ρ , the next word has not previously appeared

▶ With probability $1 - \rho$, randomly choose one word from all words that have come before, and reuse this word



- Consider words as they appear sequentially.
- \triangleright With probability ρ , the next word has not previously appeared
 - Mutation/Innovation
- ▶ With probability 1ρ , randomly choose one word from all words that have come before, and reuse this word



- Consider words as they appear sequentially.
- \triangleright With probability ρ , the next word has not previously appeared
 - Mutation/Innovation
- ▶ With probability 1ρ , randomly choose one word from all words that have come before, and reuse this word
 - Replication/Imitation



Growth Mechanisms Random Copying

References

Competition for replication between elements is random

- Competition for growth between groups is not
- Selection on groups is biased by size
- Rich-gets-richer story
- Random selection is easy
- No great knowledge of system needed





Growth Mechanisms Random Copying

References

- Competition for replication between elements is random
- Competition for growth between groups is not random
- Selection on groups is biased by size
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Growth Mechanisms Random Copying

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Growth Mechanisms Random Copying

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Growth Mechanisms Random Copying

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random

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Growth Mechanisms Random Copying

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Growth

Mechanisms Random Copying

References

- Steady growth of system: +1 element per unit time.
- We can incorporate



Growth

Mechanisms Random Copying

References

Steady growth of distinct flavors at rate ρ

Steady growth of system: +1 element per unit time.





Growth Mechanisms

- Steady growth of distinct flavors at rate ρ
- We can incorporate



Growth

Variable

Mechanisms Random Copying

References

- Steady growth of system: +1 element per unit time.
- Steady growth of distinct flavors at rate ρ
- We can incorporate
 - Element elimination



Variable

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Steady growth of distinct flavors at rate ρ

Elements moving between groups

We can incorporate

Element elimination

Steady growth of system: +1 element per unit time.

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Steady growth of system: +1 element per unit time.

- We can incorporate
 - Element elimination
 - Elements moving between groups

Steady growth of distinct flavors at rate ρ

- 3. Variable innovation rate ρ

Frame 62/88



Steady growth of system: +1 element per unit time.

Steady growth of distinct flavors at rate ρ

Elements moving between groups

4. Different selection based on group size

We can incorporate

Element elimination

3. Variable innovation rate ρ

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 - ~ 2 €

Variable

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References

- Steady growth of system: +1 element per unit time.
- Steady growth of distinct flavors at rate ρ
- We can incorporate
 - 1 Element elimination
 - Elements moving between groups
 - 3. Variable innovation rate ρ
 - Different selection based on group size (But mechanism for selection is not as simple...)

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- ▶ k_i = size of a group i
- \triangleright $N_k(t)$ = # groups containing k elements at time t.

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References





- $ightharpoonup k_i = \text{size of a group } i$
- $ightharpoonup N_k(t) = \#$ groups containing k elements at time t.

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References



- ▶ k_i = size of a group i
- $ightharpoonup N_k(t) = \#$ groups containing k elements at time t.

Basic question: How does $N_k(t)$ evolve with time?





- \triangleright k_i = size of a group i
- \triangleright $N_k(t)$ = # groups containing k elements at time t.

Basic question: How does $N_k(t)$ evolve with time?

First: $\sum kN_k(t) = t$ = number of elements at time t

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 $P_k(t)$ = Probability of choosing an element that belongs to

Variable

▶ t elements overall

 \triangleright $N_k(t)$ size k groups

a group of size k:



 $P_k(t)$ = Probability of choosing an element that belongs to

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t elements overall

 \triangleright $N_k(t)$ size k groups

a group of size k:



a group of size k:

 \triangleright $N_k(t)$ size k groups

t elements overall

 $P_k(t)$ = Probability of choosing an element that belongs to

Variable



Growth

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References

 $P_k(t)$ = Probability of choosing an element that belongs to a group of size k:

- \triangleright $N_k(t)$ size k groups
- ightharpoonup $\Rightarrow kN_k(t)$ elements in size k groups
- t elements overall



a group of size k:

 \triangleright $N_k(t)$ size k groups

t elements overall

Variable

Growth

References

 $P_k(t) = \frac{kN_k(t)}{t}$

 $P_k(t)$ = Probability of choosing an element that belongs to



 An element belonging to a group with k elements is replicated

References

An element belonging to a group with k − 1 elements is replicated

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 An element belonging to a group with k elements is replicated

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References

 An element belonging to a group with k - 1 elements is replicated



 An element belonging to a group with k elements is replicated

2. An element belonging to a group with k-1 elements is replicated

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References



 An element belonging to a group with k elements is replicated

$$N_k(t+1) = N_k(t) - 1$$

2. An element belonging to a group with k-1 elements is replicated

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References



 An element belonging to a group with k elements is replicated

$$N_k(t+1) = N_k(t) - 1$$

Happens with probability $(1 - \rho)kN_k(t)/t$

2. An element belonging to a group with k-1 elements is replicated

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 An element belonging to a group with k elements is replicated

$$N_k(t+1) = N_k(t) - 1$$

Happens with probability $(1 - \rho)kN_k(t)/t$

2. An element belonging to a group with k-1 elements is replicated

$$N_k(t+1) = N_k(t) + 1$$

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 An element belonging to a group with k elements is replicated

$$N_k(t+1) = N_k(t) - 1$$

Happens with probability $(1 - \rho)kN_k(t)/t$

2. An element belonging to a group with k-1 elements is replicated

$$N_k(t+1) = N_k(t) + 1$$

Happens with probability $(1 - \rho)(k-1)N_{k-1}(t)/t$

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References

Special case for $N_1(t)$:

The new element is a new flavor.

2. A unique element is replicated.





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References

Special case for $N_1(t)$:

1. The new element is a new flavor:





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References

Special case for $N_1(t)$:

1. The new element is a new flavor:

2. A unique element is replicated.





1. The new element is a new flavor:

$$N_1(t+1) = N_1(t) + 1$$

2. A unique element is replicated.

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References



Special case for $N_1(t)$:

- 1 The new element is a new flavor:
 - $N_1(t+1) = N_1(t) + 1$ Happens with probability ρ
- 2. A unique element is replicated.

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- 1. The new element is a new flavor:
 - $N_1(t+1) = N_1(t) + 1$ Happens with probability ρ
- 2. A unique element is replicated.

$$N_1(t+1) = N_1(t) - 1$$

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1 The new element is a new flavor:

- $N_1(t+1) = N_1(t) + 1$ Happens with probability ρ
- 2. A unique element is replicated.

$$N_1(t+1) = N_1(t) - 1$$

Happens with probability $(1 - \rho)N_1/t$

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For k > 1:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

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Put everything together:

For k > 1:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

For
$$k = 1$$
:

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

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Variable

Drop expectations

(Reasonable for t large)

Numbers of elements now fractional.

Assume distribution stabilizes: $N_k(t) = n_k t$

- Okay over large time scales
- $ightharpoonup n_k/\rho$ = the fraction of groups that have size k.



transformation

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References

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional.

Assume distribution stabilizes: $N_k(t) = n_k t$

- Okay over large time scales
- $ightharpoonup n_k/\rho$ = the fraction of groups that have size k.



Variable

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References

Assume distribution stabilizes: $N_k(t) = n_k t$ (Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional
- Okay over large time scales
- $ightharpoonup n_k/\rho$ = the fraction of groups that have size k.



Assume distribution stabilizes: $N_k(t) = n_k t$

transformation

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References

Drop expectations

(Reasonable for t large)

- Numbers of elements now fractional
- Okay over large time scales
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Variable

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References

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional

Assume distribution stabilizes: $N_k(t) = n_k t$

- Okay over large time scales
- $ightharpoonup n_k/\rho$ = the fraction of groups that have size k.



Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

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Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

$$n_k(\underline{t}+1-\underline{t})=(1-\rho)\left((k-1)\frac{n_{k-1}\underline{t}}{\underline{t}}-k\frac{n_k\underline{t}}{\underline{t}}\right)$$

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Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

$$n_k(t+1-t) = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

$$\Rightarrow n_k = (1-\rho)\left((k-1)n_{k-1} - kn_k\right)$$

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Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_{k}(t+1) - n_{k}t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_{k}t}{t}\right)$$

$$n_{k}(t+1-t) = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_{k}t}{t}\right)$$

$$\Rightarrow n_{k} = (1-\rho)\left((k-1)n_{k-1} - kn_{k}\right)$$

$$\Rightarrow n_{k}\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$

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$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- ▶ Interested in *k* large (the tail of the distribution)
- ightharpoonup Expand as a series of powers of 1/k
- ► [Assignment question occurs]

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$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- ▶ Interested in *k* large (the tail of the distribution)
- ► Expand as a series of powers of 1/*k*
- ► [Assignment question occurs]

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Growth

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in k large (the tail of the distribution)
- Expand as a series of powers of 1/k
- ► [Assignment question occurs]

Frame 70/88



$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- ▶ Interested in *k* large (the tail of the distribution)
- Expand as a series of powers of 1/k
- [Assignment question occurs]

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$$\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$$

 $\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\mu)}{(1-\mu)}}$

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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$$\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$$

•

$$\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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$$\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$$

•

$$\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

Þ

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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$$\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$$

$$\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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References

$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- ▶ Observe 2 < γ < ∞ as ρ varies.
- ▶ For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- ► Recalls Zipf's law: $s_r \sim r^{-\alpha}$ (s_r = size of the rth largest element)
- ▶ We found $\alpha = 1/(\gamma 1)$
- $ightharpoonup \gamma = 2$ corresponds to $\alpha = 1$

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- ▶ Observe $2 < \gamma < \infty$ as ρ varies.
- ▶ For $\rho \simeq 0$ (low innovation rate):

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References



$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- ▶ Observe $2 < \gamma < \infty$ as ρ varies.
- ▶ For $\rho \simeq 0$ (low innovation rate):

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- ► Recalls Zipf's law: $s_r \sim r^{-\alpha}$ (s_r = size of the rth largest element)
- ▶ We found $\alpha = 1/(\gamma 1)$
- $ightharpoonup \gamma = 2$ corresponds to $\alpha = 1$

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- ▶ Observe $2 < \gamma < \infty$ as ρ varies.
- ▶ For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- ► Recalls Zipf's law: $s_r \sim r^{-\alpha}$ (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma 1)$
- $ightharpoonup \gamma = 2$ corresponds to $\alpha = 1$

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- ▶ Observe 2 < γ < ∞ as ρ varies.
- ▶ For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- ► Recalls Zipf's law: $s_r \sim r^{-\alpha}$ (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma 1)$
- $ightharpoonup \gamma = 2$ corresponds to $\alpha = 1$

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References

- ▶ We (roughly) see Zipfian exponent [14] of $\alpha = 1$ for many real systems: city sizes, word distributions, ...
- ▶ Corresponds to $\rho \rightarrow 0$ (Krugman doesn't like it) [5]
- ▶ But still other mechanisms are possible.
- Must look at the details to see if mechanism makes sense...

Frame 73/88



Variable

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References

- ▶ We (roughly) see Zipfian exponent [14] of $\alpha = 1$ for many real systems: city sizes, word distributions, ...
- ▶ Corresponds to $\rho \rightarrow 0$ (Krugman doesn't like it) [5]
- But still other mechanisms are possible...

Frame 73/88



Variable

- ightharpoonup Corresponds to ho
 ightharpoonup 0 (Krugman doesn't like it) [5]
- But still other mechanisms are possible...
- Must look at the details to see if mechanism makes sense

Frame 73/88



•

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

▶ As before, set $N_1(t) = n_1 t$ and drop expectations

$$n_1(t+1) - n_1 t = \rho - (1-\rho)1 \cdot \frac{n_1}{t}$$

$$n_1 = \rho - (1 - \rho)n_1$$

▶ Rearrange:

$$n_1 + (1 - \rho)n_1 = \rho$$

$$n_1 = \frac{\rho}{2 - \rho}$$

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•

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$

▶ As before, set $N_1(t) = n_1 t$ and drop expectations

$$n_1(t+1) - n_1 t = \rho - (1-\rho)1 \cdot \frac{n_1}{t}$$

$$n_1 = \rho - (1 - \rho)n_1$$

► Rearrange:

$$n_1 + (1 - \rho)n_1 = \rho$$

$$n_1 = \frac{\rho}{2 - \rho}$$

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•

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$

▶ As before, set $N_1(t) = n_1 t$ and drop expectations

$$n_1(t+1) - n_1 t = \rho - (1-\rho)1 \cdot \frac{n_1 t}{t}$$

Þ

$$n_1 = \rho - (1 - \rho)n_1$$

Rearrange:

$$n_1 + (1 - \rho)n_1 = \rho$$

$$n_1 = \frac{\rho}{2 - \rho}$$

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Random Competitive Replication

We had one other equation:

•

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

▶ As before, set $N_1(t) = n_1 t$ and drop expectations

$$n_1(t+1) - n_1t = \rho - (1-\rho)1 \cdot \frac{n_1t}{t}$$

Þ

$$n_1 = \rho - (1 - \rho)n_1$$

Rearrange:

$$n_1 + (1 - \rho)n_1 = \rho$$

$$n_1 = \frac{\rho}{2 - \rho}$$

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•

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

▶ As before, set $N_1(t) = n_1 t$ and drop expectations

$$n_1(t+1) - n_1t = \rho - (1-\rho)1 \cdot \frac{n_1t}{t}$$

$$n_1 = \rho - (1 - \rho)n_1$$

Rearrange:

$$n_1 + (1 - \rho)n_1 = \rho$$

•

$$n_1 = \frac{
ho}{2-
ho}$$

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So...
$$N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$$

- ▶ Recall number of distinct elements = ρt .
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2 - \rho}$$

- ▶ For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ightharpoonup
 ho increases, fraction increases
- ightharpoonup Can show fraction of groups with two elements $\sim 1/6$
- ▶ Model does well at both ends of the distribution

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References



So...
$$N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$$

- ▶ Recall number of distinct elements = ρt .
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2 - \rho}$$

- \blacktriangleright For ρ small, fraction of unique elements \sim 1/2
- Roughly observed for real distributions
- ightharpoonup
 ho increases, fraction increases
- ► Can show fraction of groups with two elements ~ 1/6
- ▶ Model does well at both ends of the distribution

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So...
$$N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$$

- ▶ Recall number of distinct elements = ρt .
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Estimate $\rho_{\rm est} = \#$ unique words/# all words

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Estimate $\rho_{\rm est} = \#$ unique words/# all words

For Joyce's Ulysses: $\rho_{\rm est} \simeq 0.115$

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Estimate $\rho_{\rm est} = \#$ unique words/# all words

For Joyce's Ulysses: $\rho_{\rm est} \simeq 0.115$

N ₁ (real)	N ₁ (est)	N ₂ (real)	N ₂ (est)
16,432	15,850	4,776	4,870

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- Yule's paper (1924) [13]: "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
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From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economoic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only be a similarity in the structure of the underlying probability mechanisms.

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More on Herbert Simon (1916–2001):



Political scientist

- Involved in Cognitive Psychology, Computer Science, Public Administration, Economics, Management, Sociology
- Coined 'bounded rationality' and 'satisficing'
- Nearly 1000 publications
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
- Nobel Laureate in Economics

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- Derek de Solla Price was the first to study network evolution with these kinds of models.
- Citation network of scientific papers
- Price's term: Cumulative Advantage
- Directed network
- ► Two (surmountable) problems:

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 - 1. New papers have no citations
 - 2. Selection mechanism is more complicated

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- proportional to their existing # of citations Directed network
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Idea: papers receive new citations with probability

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Price's term: Cumulative Advantage



 Studied careers of scientists and found credit flowed disproportionately to the already famous Random Walks
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Matilda effect: women's scientific achievements are



Studied careers of scientists and found credit flowed. disproportionately to the already famous

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 Studied careers of scientists and found credit flowed disproportionately to the already famous

From the Gospel of Matthew:

"For to every one that hath shall be given...

 Matilda effect: women's scientific achievements are often overlooked



 Studied careers of scientists and found credit flowed disproportionately to the already famous

From the Gospel of Matthew: "For to every one that hath shall be given... (Wait! There's more....)

Matilda effect: women's scientific achievements are

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From the Gospel of Matthew:

"For to every one that hath shall be given... (Wait! There's more....) but from him that hath not, that also which he seemeth to have shall be taken away.

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From the Gospel of Matthew:

"For to every one that hath shall be given... (Wait! There's more....)

but from him that hath not, that also which he seemeth to have shall be taken away.

And cast the worthless servant into the outer darkness; there men will weep and gnash their teeth."

 Matilda effect: women's scientific achievements are often overlooked Random Walks
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Merton was a catchphrase machine:





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Merton was a catchphrase machine:

- self-fulfilling prophecy





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Merton was a catchphrase machine:

- self-fulfilling prophecy
- role model





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Merton was a catchphrase machine:

- 1. self-fulfilling prophecy
- 2. role model
- 3. unintended (or unanticipated) consequences
- 4. focused interview → focus group





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Merton was a catchphrase machine: self-fulfilling prophecy

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Merton was a catchphrase machine:

- self-fulfilling prophecy
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And just to rub it in...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.





Barabasi and Albert [1]—thinking about the Web

- Independent reinvention of a version of Simon and Price's theory for networks
- Another term: "Preferential Attachment"
- Considered undirected networks (not realistic but avoids 0 citation problem)
- Still have selection problem based on size (non-random)
- ► Solution: Randomly connect to a node (easy)
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- Scale-free networks = food on the table for physicists

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