Lognormals and friends

Principles of Complex Systems
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Outline

Alternative distributions

There are other heavy-tailed distributions:

1. Lognormal
2. Stretched exponential (Weibull)
3. ... (Gamma)

lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu+\frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

All moments of lognormals are finite.

Derivation from a normal distribution

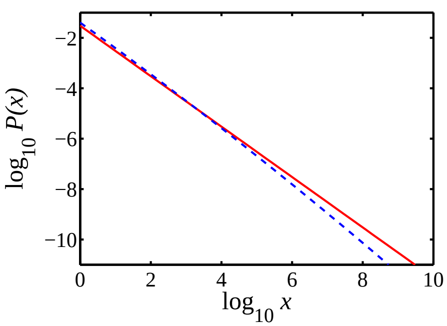
Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{d}y \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\alpha = 1$ and $c = 0.03$.

Confusion

What's happening:

- $$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$
- $$= -\ln x - \ln\sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$
- $$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$
- $\Rightarrow$ If $\sigma^2 \gg 1$ and μ,
- $$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

Confusion

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- This happens when (roughly)
- $$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$
- $$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2-\mu)\log_{10} e$$
- $$\simeq 0.05(\sigma^2-\mu)$$
- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

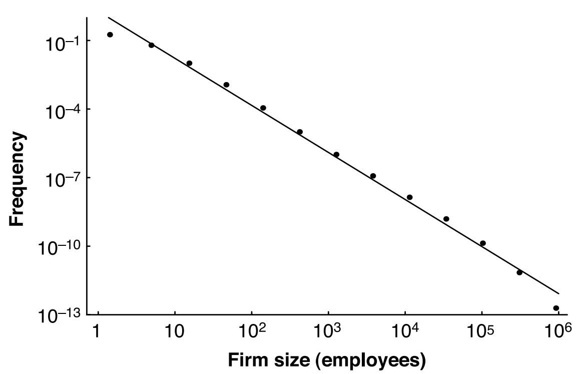
Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$
  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:
  $$\ln x_{n+1} = \ln r + \ln x_n$$
- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?

- Gibrat [2] (1931) uses this argument to explain lognormal distribution of firm sizes
- Robert Axtell (2001) shows power law fits the data very well [1] $\gamma \simeq 2$



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An explanation

- Axtel (mis)cites Malcai et al.'s (1999) argument [6] for why power laws appear with exponent $\gamma \simeq 1$
- The set up: N entities with size $x_i(t)$
- Generally:
  $$x_i(t+1) = r x_i(t)$$
  where r is drawn from some happy distribution
- Same as for lognormal but one extra piece:
- Each x_i cannot drop too low with respect to the other sizes:
  $$x_i(t+1) = \max(r x_i(t), c\,\langle x_i \rangle)$$

An explanation

Some math later...

- Find
$$P(x) \sim x^{-\gamma}$$
where
- $$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$
- Now, if $c/N \ll 1$,
$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$$
- Which gives
$$\gamma \sim 1 + \frac{1}{1-c}$$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [7]
$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$
(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)
- Now averaging different lognormal distributions.

Averaging lognormals

- $$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) \mathrm{d}t$$
- Substitute $t = u^2$:
$$P(x) = \frac{2\lambda}{\sqrt{2\pi}x}\int_{u=0}^{\infty} \exp\left(-\lambda u^2 - (\ln x/m)^2/2u^2\right) \mathrm{d}u$$
- We can (lazily) look this up: [3]
$$\int_0^{\infty} \exp\left(-au^2 - b/u^2\right) \mathrm{d}u = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp(-2\sqrt{ab})$$
- We have $a = \lambda$ and $b = (\ln x/m)^2/2$:
$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$
- Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [8, 9]
- Later: Huberman and Adamic [4, 5]: Number of pages per website

Quick summary of these exciting developments

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, double Pareto distribution appear
- Take home message: Be careful out there...

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