

# Lognormals and friends

## Principles of Complex Systems

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#### Lognormals

- Empirical Confusability
- Random Multiplicative Growth Model
- Random Growth with Variable Lifespan

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There are other heavy-tailed distributions:

1. Lognormal
2. Stretched exponential (Weibull)
3. ... (Gamma)

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- ▶  $\ln x$  is distributed according to a normal distribution with mean  $\mu$  and variance  $\sigma$ .
- ▶ Appears in economics and biology where growth increments are distributed normally.

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Standard form reveals the mean  $\mu$  and variance  $\sigma^2$  of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \quad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \quad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

All moments of lognormals are finite.

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# Derivation from a normal distribution

Take  $Y$  as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)$$

Set  $Y = \ln X$ :

▶ Transform according to  $P(x)dx = P(y)dy$ :



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

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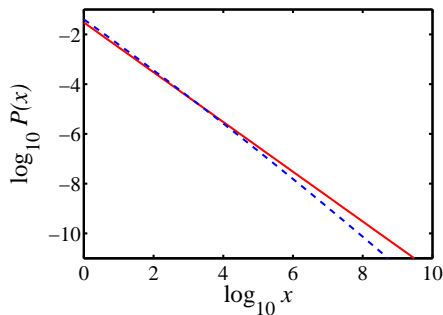
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# Confusion between lognormals and pure power laws

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Near agreement  
over four orders  
of magnitude!

- ▶ For lognormal (blue),  $\mu = 0$  and  $\sigma = 10$ .
- ▶ For power law (red),  $\alpha = 1$  and  $c = 0.03$ .

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left( \frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶  $\Rightarrow$  If  $\sigma^2 \gg 1$  and  $\mu$ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

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- ▶ Expect -1 scaling to hold until  $(\ln x)^2$  term becomes significant compared to  $(\ln x)$ .
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left( \frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶  $\Rightarrow$  If you find a -1 exponent, you may have a lognormal distribution...

# Generating lognormals:

## Random multiplicative growth:



$$X_{n+1} = rX_n$$

where  $r > 0$  is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶  $\Rightarrow \ln x_n$  is normally distributed
- ▶  $\Rightarrow x_n$  is lognormally distributed

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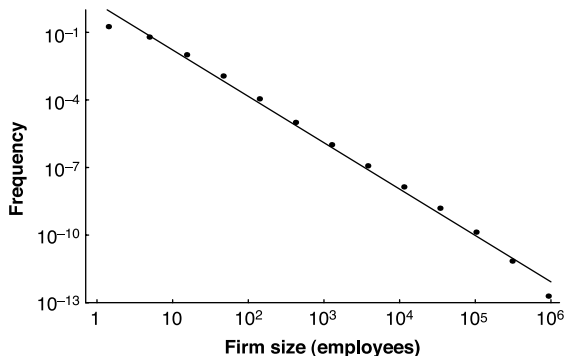
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# Lognormals or power laws?

- ▶ Gibrat<sup>[2]</sup> (1931) uses this argument to explain lognormal distribution of firm sizes
- ▶ Robert Axtell (2001) shows power law fits the data very well<sup>[1]</sup>

$$\gamma \simeq 2$$



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- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument<sup>[6]</sup> for why power laws appear with exponent  $\gamma \simeq 1$
- ▶ The set up:  $N$  entities with size  $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where  $r$  is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each  $x_i$  cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

# An explanation

## Some math later...

- ▶ Find

$$P(x) \sim x^{-\gamma}$$

where

- ▶

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

- ▶ Now, if  $c/N \ll 1$ ,

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{-1}{-(c/N)} \right]$$

- ▶ Which gives

$$\gamma \sim 1 + \frac{1}{1 - c}$$

- ▶ **Groovy...**  $c$  small  $\Rightarrow \gamma \simeq 2$

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## Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size  $x_i$  to vary
- ▶ Example:  $P(t)dt = ae^{-at}dt$
- ▶ Back to no bottom limit: each  $x_i$  follows a lognormal
- ▶ Sizes are distributed as <sup>[7]</sup>

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that  $\sigma \sim t$  and  $\mu = \ln m$ )

- ▶ Now averaging different lognormal distributions.

# Averaging lognormals



$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- ▶ Substitute  $t = u^2$ :

$$P(x) = \frac{2\lambda}{\sqrt{2\pi}x} \int_{u=0}^{\infty} \exp\left(-\lambda u^2 - (\ln x/m)^2/2u^2\right) du$$

- ▶ We can (lazily) look this up:<sup>[3]</sup>

$$\int_0^{\infty} \exp\left(-au^2 - b/u^2\right) du = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{ab})$$

- ▶ We have  $a = \lambda$  and  $b = (\ln x/m)^2/2$ :

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

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$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of  $\ln x/m$ , i.e., whether  $x/m > 1$  or  $x/m < 1$ .



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ **'Break' in scaling** (not uncommon)
- ▶ Double-Pareto distribution
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website



# Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, double Pareto distribution appear
- ▶ Take home message: Be careful out there...

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