Optimal Supply Networks

Complex Networks, Course 295A, Spring, 2008

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Outline

Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people,
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems

River network models

Optimality:

- Optimal channel networks [10]
- Thermodynamic analogy [11]

versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

Optimization approaches

Cardiovascular networks:

- Murray's law (1926) connects branch radii at forks: [8]

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

  where r_0 = radius of main branch
  and r_1 and r_2 are radii of sub-branches
- Calculation assumes Poiseuille flow
- Holds up well for outer branchings of blood networks
- Also found to hold for trees
- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

  where Δp = pressure difference, Φ = flux

Optimization approaches

Cardiovascular networks:

- Fluid mechanics: Poiseuille impedance for smooth flow in a tube of radius r and length ℓ:

$$Z = \frac{8\eta\ell}{\pi r^4}$$

  where η = dynamic viscosity

- Power required to overcome impedance:

$$P_{\mathrm{drag}} = \Phi\Delta p = \Phi^2 Z$$

- Also have rate of energy expenditure in maintaining blood:

$$P_{\mathrm{metabolic}} = c r^2 \ell$$

  where c is a metabolic constant.

Optimization approaches

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = rate work is done = $F \cdot v$
- ΔP = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
- So $\Phi \Delta P$ = Force · velocity

Optimization approaches

Murray's law:

- Total power (cost):

$$P = P_{\mathrm{drag}} + P_{\mathrm{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

Optimization

Murray's law:

- Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c 2 r \ell = 0$$

- Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.

Optimization

Murray's law:

- So we now have:

$$\Phi = kr^3$$

- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

Optimization

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$r_\omega^3 = 2 r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- Find Horton ratio for vessell radius $R_r = r_\omega / r_{\omega-1}$...

Optimization

Murray meets Tokunaga:

- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$\boxed{R_r^3 = R_n = R_v = R_n^3}$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Optimization

Murray meets Tokunaga:

- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_v = R_n}$$

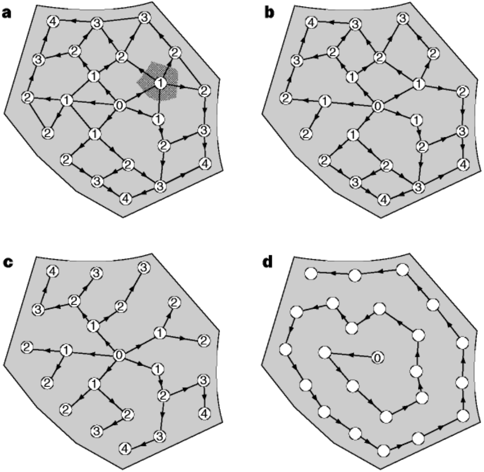
- We need one more constraint...
- West et al (1997) [16] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [15] using Tokunaga (sort of).

Optimization approaches

The bigger picture:

- Rashevsky (1960's) [9] showed using a network story that power output of heart should scale as $M^{2/3}$
- West et al. (1997 on) [16, 2] managed to find $M^{3/4}$ (a mess—super long story—see previous course...)
- Banavar et al. [1] attempt to derive a general result for all natural branching networks
- Again, something of a mess [2]
- We'll look at and build on Banavar et al.'s work...

Simple supply networks



- Banavar et al., Nature, (1999) [1]
- Very general attempt to find most efficient transportation networks.

Simple supply networks

- Banavar *et al.* find 'most efficient' networks with

$$P \propto M^{d/(d+1)}$$

- ... but also find

$$V_{\text{blood}} \propto M^{(d+1)/d}$$

- Consider a 3 g shrew with $V_{\text{blood}} = 0.1\,V_{\text{body}}$
- $\Rightarrow$ 3000 kg elephant with $V_{\text{blood}} = 10\,V_{\text{body}}$
- Such a pachyderm would be rather miserable.

Pachydermal sadness

Checking that last statement:

- For $d = 3$, we have $V_{\text{blood}} = cV^{(d+1)/d} = cV^{4/3}$
- If our shrew has $V_{\text{blood}}^{(\text{shrew})} = 0.1\,V^{(\text{shrew})}$ then $c = 0.1(V^{(\text{shrew})})^{-1/3}$.
- Assuming $V^{(\text{elephant})} = 10^6\,V^{\text{shrew}}$, we have

$$V_{\text{blood}}^{(\text{elephant})} = c(V^{(\text{elephant})})^{4/3}$$

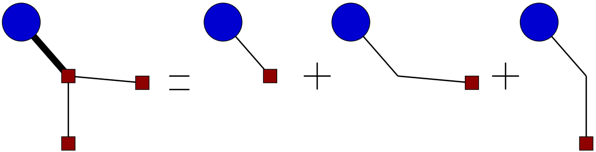
$$= \underbrace{0.1(V^{(\text{shrew})})^{-1/3}}_{c}\underbrace{(10^6\,V^{(\text{shrew})})}_{V^{(\text{elephant})}}{}^{4/3}$$

$$= 10^7\,V^{(\text{shrew})} = 10\,V^{(\text{elephant})}.$$

- Oops.

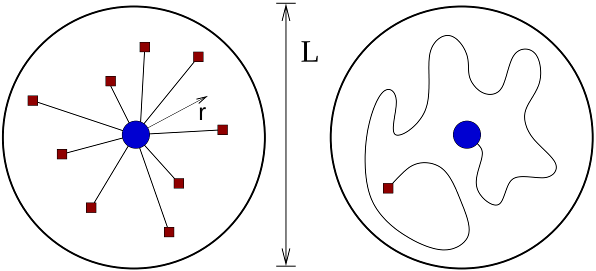
Geometric argument

- Consider one source supplying many sinks in a d dimensional volume
- Material draw by sinks is invariant.
- Assume some cap on flow speed of material, $v_{\max}$
- See network as a bundle of virtual vessels:



- The right question: how does number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- Or: what is highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

Geometric argument



- Best case: lengths of virtual vessels $\propto r$.
- Worst case: lengths of virtual vessels $\propto L^d$.

Geometric argument

- Banavar *et al.* assume sink density ρ is uniform
- If we allow ρ to vary, then we find

$$V_{\rm blood} \propto \rho L^{d+1}$$

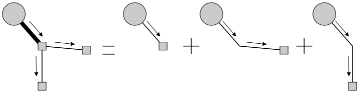
- Since $V_{\rm blood} \propto L^{d}$, we must have $\rho \propto L^{-1}$.
- $\Rightarrow$ capillary density must decrease as M increases (observed).

Geometric argument

- $$N_{\text{sinks}} \propto \rho L^d \propto L^{-1} L^d \propto M^{(d-1)/d}$$
- so for $d = 3$, we have $\alpha = 2/3$.
- for $d = 2$, we have $\alpha = 1/2$.
- Claim: If volume shapes change allometrically, the exponent decreases.
- Claim: Less Efficient networks have lower exponents too (b/c they must have lower densities of sinks).
- We'll work through these claims in detail...

Geometric argument

- Reminder: we break network up into virtual vessels:



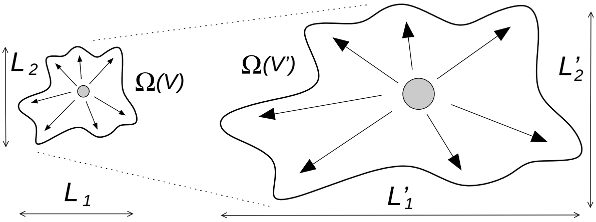
- Assume flow rate at each sink is independent of system size.
- Take the cross-sectional area *a* of virtual vessels to be constant.
- Minimizing the volume of the network is then equivalent to minimizing the sum of the path lengths from the source to all sinks.

Geometric argument

- Note: we are ignoring issues such as impedance.
- Changes in impedance (e.g., due to combining of flows) may change material speed but not overall flow rate
- Scaling of material volume must be $\propto$ system volume—it's a 0th order concern.

Geometric argument

- Consider families of systems that grow allometrically.
- Family = a basic shape Ω indexed by volume V.



- Orient shape to have dimensions $L_1 \times L_2 \times ... \times L_d$
- In 2-d, $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ where A = area.
- In general, have d lengths which scale as $L_i \propto V^{\gamma_i}$.
- For above example, width grows faster than height: $\gamma_1 > \gamma_2$.

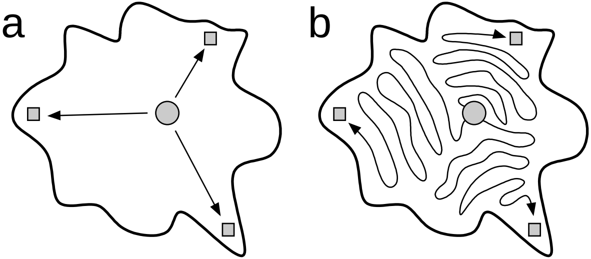
Geometric argument

Some generality:

- Consider d dimensional spatial regions living in D dimensional ambient spaces. Notation: $\Omega_{d,D}(V)$.
- River networks: $d = 2$ and $D = 3$
- Cardiovascular networks: $d = 3$ and $D = 3$
- Star-convexity of $\Omega_{d,D}(V)$: A spatial region is star-convex if from at least one point, all other points in the region can be reached by travelling along straight lines while remaining within the region.
- Assume source can be located at a point which has direct line of sight to all sources.
- We can generalize to a much broader class of shapes...

Geometric argument

- Reminder of best and worst configurations



- Basic idea: Minimum volume of material in system $V_{\text{net}} \propto$ sum of distance from the source to the sinks.
- See what this means for sink density ρ if sinks do not change their feeding habits with overall size.

Geometric argument

Assumptions in detail:

- Each region $\Omega_{d,D}(V)$ has overall dimensions $L_1 \times L_2 \times \cdots \times L_d$.
- Specifically, $V = cL_1 L_2 \cdots L_d$ where $c \leq 1$ is a shape factor dependent of Ω.
- We allow for arbitrary shape scaling:

$$L_i = c_i^{-1} V^{\gamma_i}$$

  where $\prod_{i=1}^{d} c_i = c$ and $\sum_{i=1}^{d} \gamma_i = 1$.
- For isometric growth, $\gamma_i = 1/d$.
- For allometric growth, we must have at least two of the $\{\gamma_i\}$ being different
- We choose the L_i so that $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d$

Computing the minimal network volume:

- $$\min V_{\mathrm{net}} \propto \int_{\Omega_{d,D}(V)} \rho \, ||\vec{x}|| \, \mathrm{d}\vec{x}$$
  $$= \rho \int_{\Omega_{d,D}(V)} (x_1^2 + x_2^2 + \ldots + x_d^2)^{1/2} \mathrm{d}\vec{x}$$
- Substituting $x_i = L_i u_i$, we have
  $$\min V_{\mathrm{net}} \propto \rho L_1 \cdots L_d \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + \ldots + L_d^2 u_d^2)^{1/2} \mathrm{d}\vec{u}$$
  $$\propto \rho V \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{1/2} \mathrm{d}\vec{u}$$
  where we have rescaled to a volume of size $c < 1$ where c is the shape factor.

Computing the minimal network volume:

- We are here:

$$\min V_{\mathrm{net}} \propto \rho V \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{1/2} \mathrm{d}\vec{u}$$

- Observe that the integrand will be dominated by the L_i that scale strongest with V.
- Assume first $k \leq d$ dimensions scale with equal strength, $L_i = c_i^{-1} V^{\gamma_*}$.
- Plug in scaling for L_i in terms of V and pull V^{γ_*} out to the front.

$$\min V_{\mathrm{net}} \propto \rho V V^{\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^{-2} u_1^2 + \ldots + c_k^2 u_k^2 + \ldots$$

$$c_{k+1}^{-2} V^{2(\gamma_{k+1} - \gamma_*)} u_{k+1}^2 + \ldots + c_d^{-2} V^{2(\gamma_d - \gamma_*)} u_d^2)^{1/2} \mathrm{d}\vec{u}$$

Computing the minimal network volume:

- Where we are now:

$$\min V_{\mathrm{net}} \propto \rho V^{1+\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^{-2} u_1^2 + \ldots + c_k^{-2} u_k^2 + \ldots$$

$$c_{k+1}^2 V^{2(\gamma_{k+1}-\gamma_*)} u_{k+1}^2 + \ldots + c_d^2 V^{2(\gamma_d-\gamma_*)} u_d^2)^{1/2} \mathrm{d}\vec{u}$$

- Now allow $V \to \infty$ and see that part of integrand vanishes:

$$\min V_{\mathrm{net}} \to \rho V^{1+\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^2 u_1^2 + \ldots + c_k^2 u_k^2)^{1/2} \mathrm{d}\vec{u}$$

$$\propto \rho V^{1+\gamma_*}$$

since integral is now nice and friendly and small.

Geometric argument

- Our general result:

$$\boxed{\min V_{\text{net}} \propto \rho V^{1+\gamma_*}}$$

- For scaling is isometric, we have $\gamma_* = \gamma_{\text{iso}} = 1/d$ and all the L_i scale as $V^{1/d}$:

$$\min V_{\text{net/iso}} \propto \rho V^{1+1/d} = \rho V^{(d+1)/d}$$

- If scaling is allometric, we have $\gamma_* = \gamma_{\text{allo}} = \max_i \gamma_i > 1/d$ and

$$\min V_{\text{net/allo}} \propto \rho V^{1+\gamma_{\text{allo}}}$$

- We see that isometrically scaling volumes require less network volume than allometrically scaling volumes:

$$\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \to 0 \text{ as } V \to \infty$$

Geometric argument

Blood networks

- Material costly $\Rightarrow$ expect lower optimal bound of $V_{\mathrm{net}} \propto \rho V^{(d+1)/d} \propto \rho L^{d+1}$ to be closely followed.
- For cardiovascular networks, $d = D = 3$.
- Know that volume of blood scales linearly with blood volume [12], $V_{\mathrm{net}} \propto V_{\Omega} \propto L^{d}$.
- Since we have shown $V_{\mathrm{net}} \propto \rho L^{d+1}$, sink density must also decrease as volume increases:

$$\rho \propto L^{-1} \propto V^{-1/d}.$$

Geometric argument

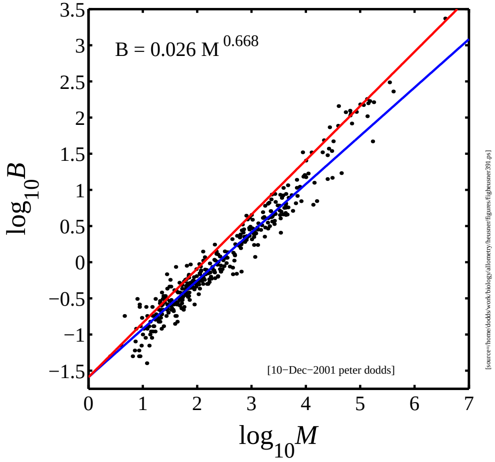
Blood networks

- We assume, reasonably, that $V \propto M$ where M is mass.
- It next follows that P, the rate of overall energy use in Ω, can at most scale with volume as

$$P \propto \rho V \propto \rho M \propto M^{(d-1)/d}$$

- For three dimensional organisms, we have $P \propto M^{2/3}$.
- Much controversy about all this [2] but for small mammals and birds, 2/3 scaling looks good for resting metabolic rate.

Some data on metabolic rates



- Heusner's data (1991) [5]
- 391 Mammals
- blue line: 2/3
- red line: 3/4.
- $B = P$ = power

Geometric argument

Interesting result from quantum mechanics:

- Homeothermic organisms need to keep their temperature static
- A good amount of heat loss is through infra-red radiation (when resting)
- For mammals with $M \leq 10$ kg:
  $P = 2.57 \times 10^5 M^{2/3}$erg/sec.
- Stefan-Boltzmann's law (⊞): $\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma \varepsilon S T^4$
  where T is absolute temperature, S is surface area, ε = emissivity < 1 and σ depends on Planck's constant, speed of light, π^5, these sorts of things.
- Rough estimates of these constants give

$$P \simeq 10^5 M^{2/3} \text{erg/sec}.$$

Not bad...

Geometric argument

Organisms at work:

- What about organisms working as hard as possible?
- For short bursts, power scales closer to mass.
- Energy is stored locally muscles and we have accounted for this.
- Also: apparently some capillaries are dormant during rest.

Geometric argument

- River networks can be seen as collection networks.
- Many sources and one sink.
- For river networks, we know ρ is constant so

$$V_{\mathrm{net}} \propto \rho V^{(d+1)/d} = \text{constant} \times V^{3/2}$$

- Hmmm: now network volume is growing faster than basin 'volume' (really area).
- It's all okay:
  Landscapes are 2-d surfaces living in 3-d.
- $D = 3$ and $d = 2$.
- Streams can grow not just in width but in depth...

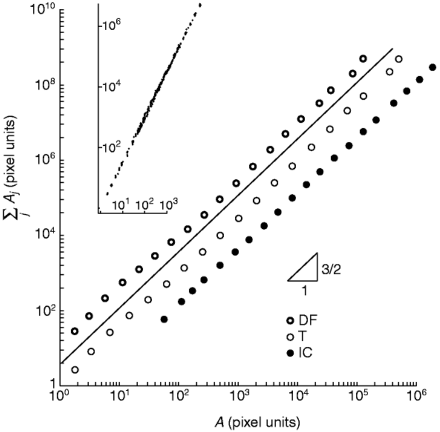
Geometric argument

- Volume of water in river network can be calculated by adding up basin areas
- (Discreteness of data means summing instead of integrating)
- Each site on discrete lattice is a source.
- Imagine a steady flow from each source to outlet.
- Flows sum in such a way that

$$V_{\text{net}} = \sum_{\text{all pixels}} a_{\text{pixel } i}$$

Geometric argument

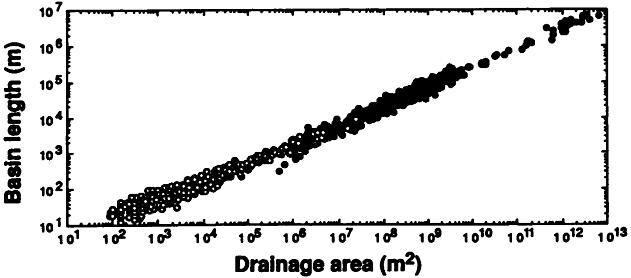
- Banavar et al.'s approach [1] is okay because ρ really is constant.
- The irony: shows optimal basins are isometric
- Optimal Hack's law: $a \sim \ell^h$ with $h = 1/2$
- (Zzzzz)



From Banavar et al. (1999) [1]

Geometric argument: evidence

Montgomery and Dietrich [7]

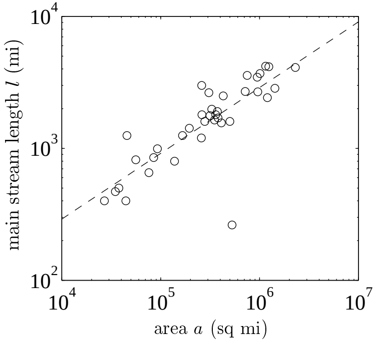
- 

- Composite data set: includes everything from unchanneled valleys up to world's largest rivers.
- Esimated fit:

$$L \simeq 1.78 a^{0.49}$$

- N.b., data is a mixture of basin and main stream lengths.

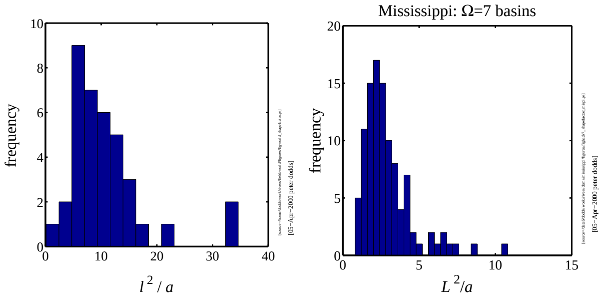
World's largest rivers only:



- Data from Leopold (1994) [6]
- Estimate of Hack exponent: $h = 0.50 \pm 0.06$

Optimal river networks

Large scale deviations in Hack's law

- 

- Rivers seem generally relatively long (but isometric).
- Measured width/length ratio unexplained.

Many sources, many sinks

How do we distribute sources?

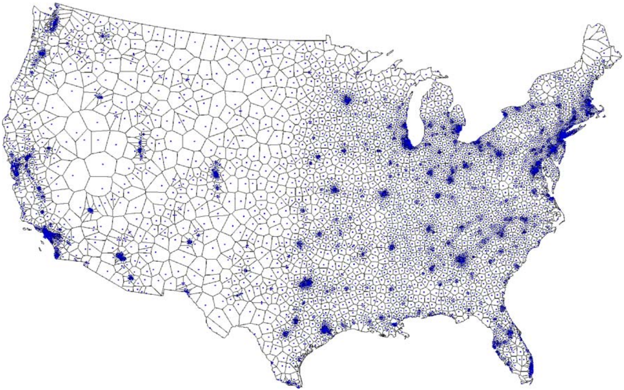
- Focus on 2-d (results generalize to higher dimensions)
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly
- Which lattice is optimal? The hexagonal lattice
  Q1: How big should the hexagons be?
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan [13, 14] and by Gastner and Newman (2006) [4] and work cited by them.

Optimal source allocation

Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

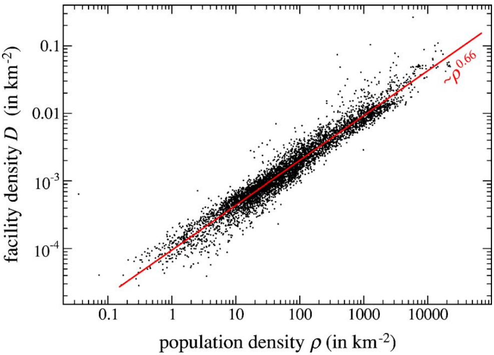
Optimal source allocation



From Gastner and Newman (2006) [4]

- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



From Gastner and Newman (2006) [4]

- Optimal facility density D vs. population density ρ.
- Fit is $D \propto \rho^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power...

Optimal source allocation

Size-density law:

- $$\boxed{D \propto \rho^{2/3}}$$
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources sinks are distributed throughout region...

Optimal source allocation

- We first examine Stephan's treatment (1977) [13, 14]
- "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer principle.

Optimal source allocation

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center is $\bar{d}$ and assume average speed of travel is $\bar{v}$.
- Note that average travel distance will be on the length scale of the region which is $A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

Optimal source allocation

- Next assume facility requires regular maintenance (person-hours per day)
- Call this quantity τ
- If burden of mainenance is shared then average cost per person is τ/P.
- Replace P by ρA where ρ is density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho A) = gA^{1/2}/\bar{v} + \tau/(\rho A).$$

- Now Minimize with respect to A...

Optimal source allocation

- Differentiating...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho A)\right)$$

$$= c/(2\bar{v}A^{1/2} - \tau/(\rho A^2) = 0$$

- Rearrange:

$$A = (2\bar{v}\tau/c\rho)^{2/3} \propto \rho^{-2/3}$$

- # facilities per unit area $\propto$

$$A^{-1} \propto \rho^{2/3}$$

- Groovy...

Optimal source allocation

An issue:

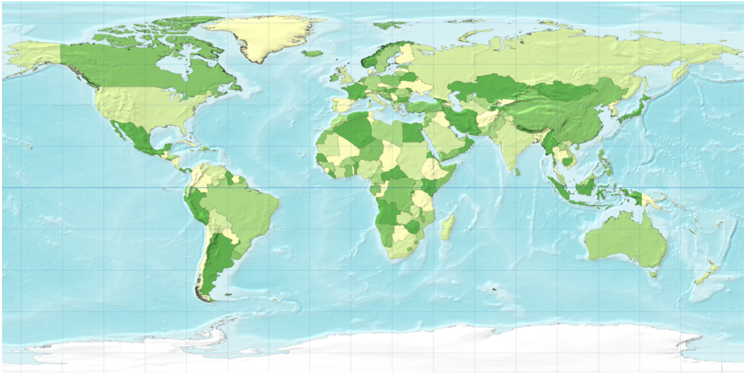
- Maintenance (τ) is assumed to be independent of population and area (P and A)

Optimal source allocation

Stephan's online book
"The Division of Territory in Society" is here (⊞).

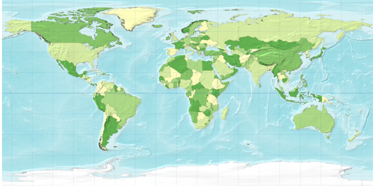
Cartograms

Standard world map:



Cartograms

Cartogram of countries 'rescaled' by population:



Cartograms

Diffusion-based cartograms:

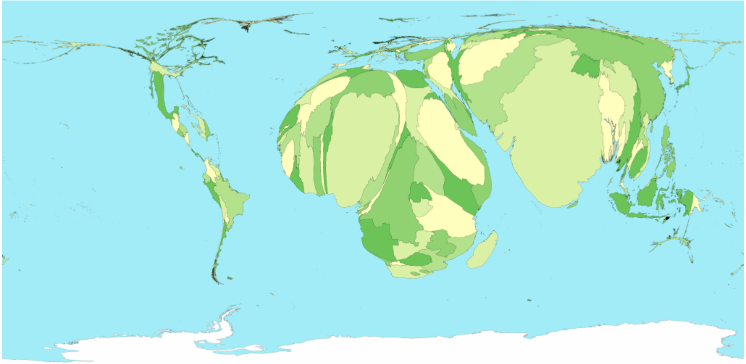
- Idea of cartograms is to distort areas to more accurately represent some local density ρ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [3] is based on standard diffusion:

$$\nabla^2 \rho - \frac{\partial \rho}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}$.

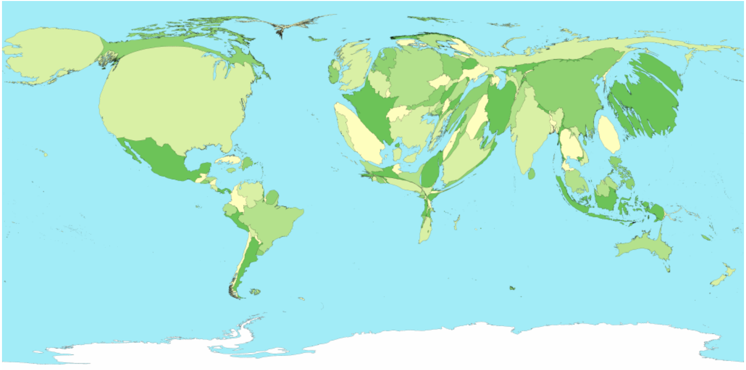
Cartograms

Child mortality:



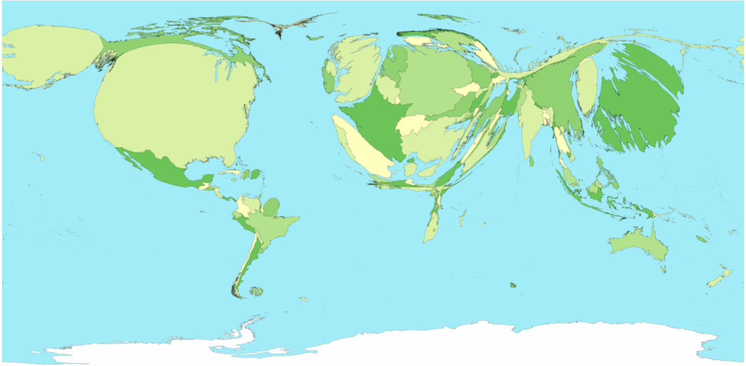
Cartograms

Energy consumption:



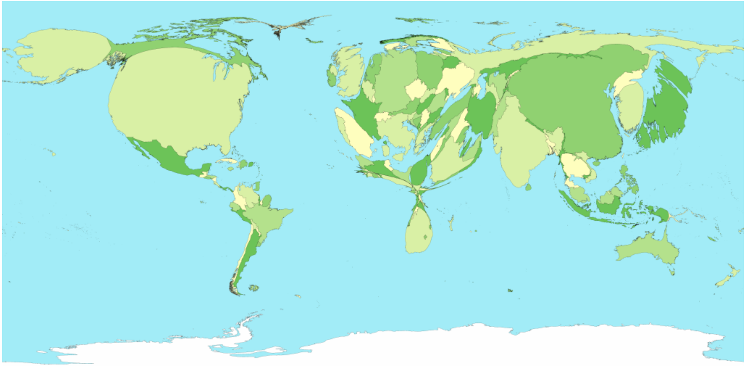
Cartograms

Gross domestic product:



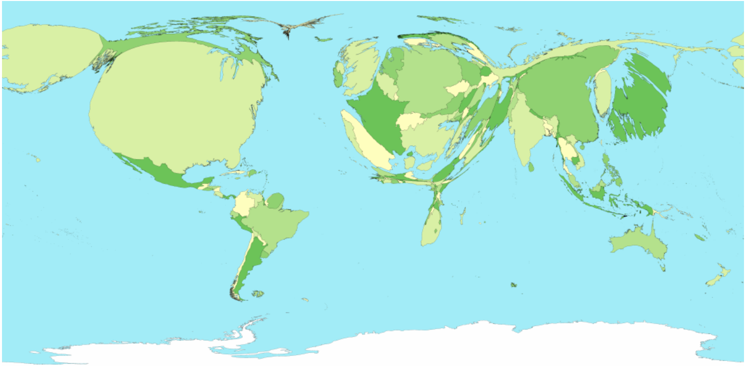
Cartograms

Greenhouse gas emissions:



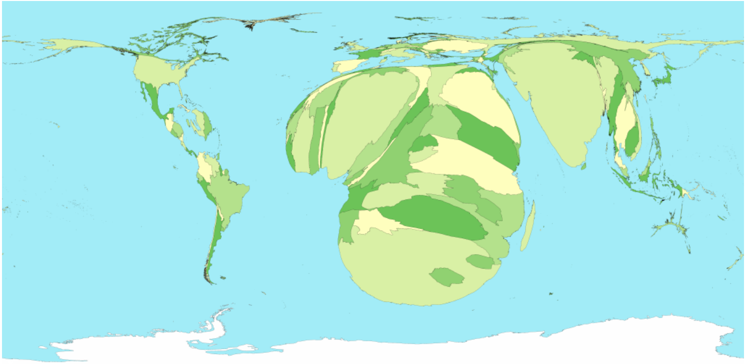
Cartograms

Spending on healthcare:



Cartograms

People living with HIV:

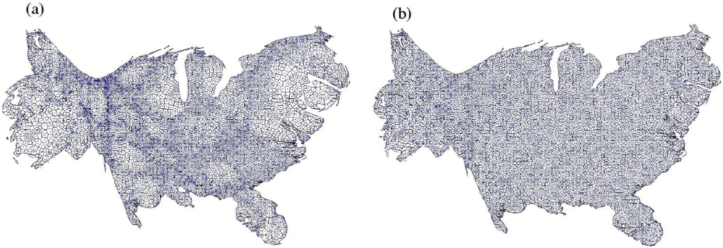


Cartograms

- The preceding sampling of Gastner & Newman's cartograms lives here (⊞).
- A larger collection can be found at worldmapper.org (⊞).

WORLDMAPPER The world as you've never seen it before

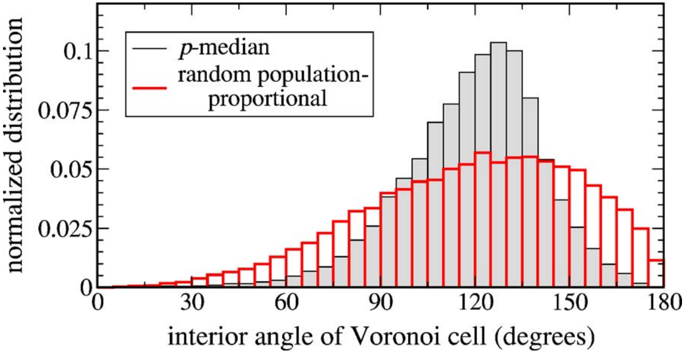
Size-density law



- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [4]

Size-density law



From Gastner and Newman (2006) [4]

- Cartogram's Voronoi cells are somewhat hexagonal.

Size-density law

Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility. [3]
- Assume given a fixed population density ρ defined on a spatial region Ω.
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \ldots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \ldots, \vec{x}_n\}) = \int_{\Omega} \rho(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \, .$$

- Also known as the p-median problem.
- Not easy... in fact this one is an NP-hard problem. [3]

Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (⊞), one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

  where c_i is a shape factor for the ith Voronoi cell.
- Approximate c_i as a constant c.

Size-density law

Carrying on:

- The cost function is now

$$F = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x}\,.$$

- We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$

- Within each cell, $A(\vec{x})$ is constant.
- So... integral over each of the n cells equals 1.

Size-density law

Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, ..., \vec{x}_n\}$, minimize

$$G(A) = c \int_\Omega \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_\Omega \left[A(\vec{x}) \right]^{-1} \mathrm{d}\vec{x} \right)$$

- Next compute $\delta G / \delta A$, the functional derivative (⊞) of the functional $G(A)$.
- This gives

$$\int_\Omega \left[\frac{-c}{2} \rho(\vec{x}) A(\vec{x})^{-1/2} + \lambda \left[A(\vec{x}) \right]^{-2} \right] \mathrm{d}\vec{x}$$

- Setting the integrand to be zilch, we have:

$$\rho(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

Size-density law

Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $D(\vec{x})$, an approximation of the local source density.
- Substituting $D = 1/A$, we have

$$D(\vec{x}) = \left(\frac{c}{2\lambda}\rho\right)^{2/3}.$$

- Normalizing (or solving for λ):

$$D(\vec{x}) = n \frac{[\rho(\vec{x})]^{2/3}}{\int_\Omega [\rho(\vec{x})]^{2/3} \mathrm{d}\vec{x}} \propto [\rho(\vec{x})]^{2/3}.$$

Global redistribution networks

One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gaster and Newman model: cost is a function of basic maintenance and travel time:

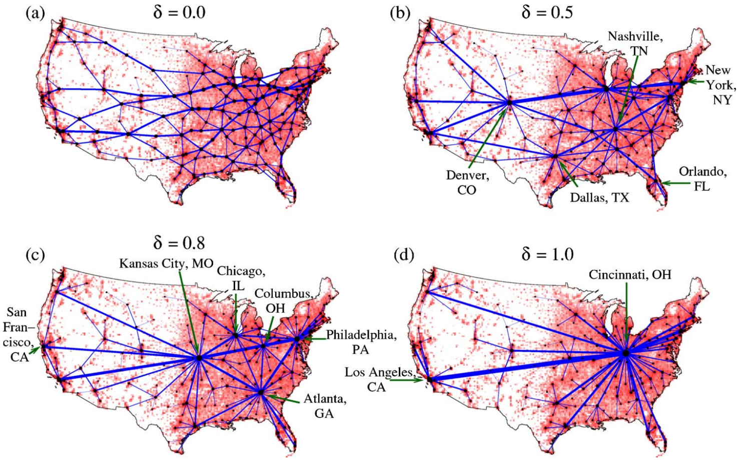
$$C_{\text{maint}} + \gamma C_{\text{travel}}.$$

- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1-\delta)\ell_{ij} + \delta(\#\text{hops}).$$

- When $\delta = 1$, only number of hops matters.

Global redistribution networks



From Gastner and Newman (2006) [4]

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