Scheidegger Networks—A Bonus Calculation

Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

First return random walk

References

Outline

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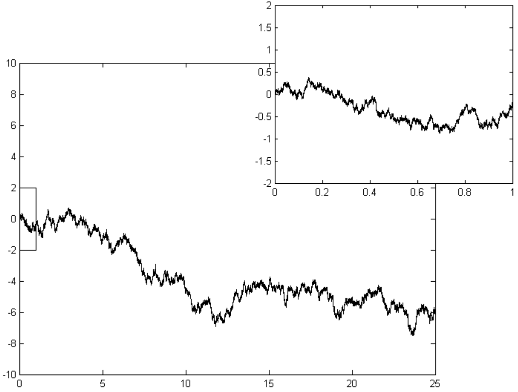
- A classic, delightful problem

Random walks



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Random walking on a sphere...

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- Continuous but nowhere differentiable

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- In words: Probability of returning at time t equals the integral of the probability of returning at time τ and then not returning until exactly $t-\tau$ time units later.

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- Rearrange:

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- For river networks: $P(\ell) \sim \ell^{-\gamma}$ so $\gamma = 3/2$ for Scheidegger networks.

References I

A. E. Scheidegger.
A stochastic model for drainage patterns into an intramontane trench.
Bull. Int. Assoc. Sci. Hydrol., 12(1):15–20, 1967.
.

A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.
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