Scheidegger Networks—A Bonus Calculation

Complex Networks, Course 295A, Spring, 2008

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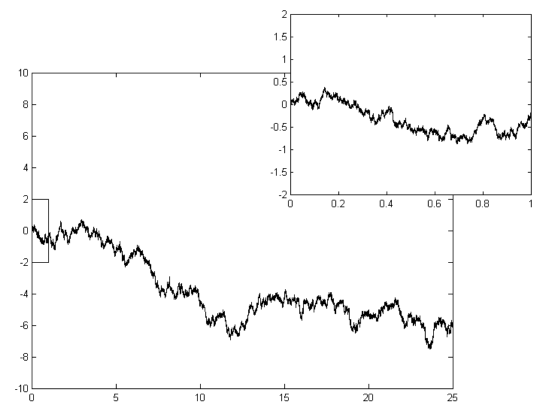
Outline

Random walks

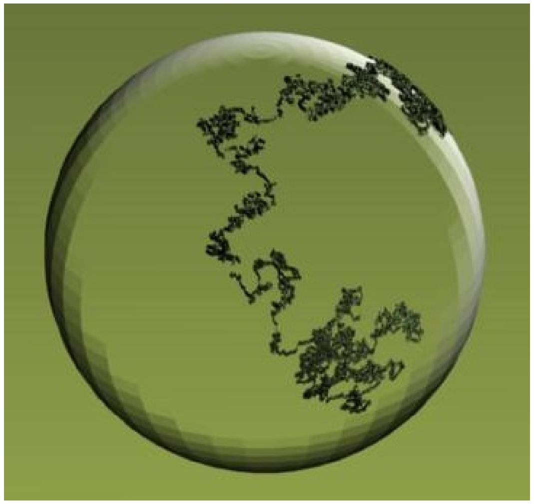
- We've seen that Scheidegger networks have random walk boundaries [1, 2]
- Determining expected shape of a 'basin' becomes a problem of finding the probability that a 1-d random walk returns to the origin after t time steps
- We solved this with a counting argument for the discrete random walk the preceding Complex Systems course
- For fun and the constitution, let's work on the continuous time Wiener process version
- A classic, delightful problem

Random walks



The Wiener process (⊞)

Random walking on a sphere...



The Wiener process (⊞)

Random walks

- Wiener process = Brownian motion
- $$x(t_2) - x(t_1) \sim \mathcal{N}(0, t_2 - t_1)$$
  where
  $$\mathcal{N}(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$
- Continuous but nowhere differentiable

First return

- Objective: find $g(t)$, the probability that Wiener process first returns to the origin at time t.
- Use what we know: the probability density for a return (not necessarily the first) at time t is
  $$f(t) = \frac{1}{\sqrt{2\pi t}} e^{-0/2t} = \frac{1}{\sqrt{2\pi t}}$$
- Observe that f and g are connected like this:
  $$f(t) = \int_{\tau=0}^{t} f(\tau) g(t-\tau) \mathrm{d}\tau + \underbrace{\delta(t)}_{\text{Dirac delta function}}$$
- In words: Probability of returning at time t equals the integral of the probability of returning at time τ and then not returning until exactly $t - \tau$ time units later.

First return

- Next see that right hand side of $f(t) = \int_{\tau=0}^{t} f(\tau) g(t-\tau) \mathrm{d}\tau + \delta(t)$ is a juicy convolution.
- So we take the Laplace transform:
  $$\mathcal{L}[f(t)] = F(s) = \int_{t=0-}^{\infty} f(t) e^{-st} \mathrm{d}t$$
- and obtain
  $$F(s) = F(s)G(s) + 1$$
- Rearrange:
  $$G(s) = 1 - 1/F(s)$$

First return

- We are here: $G(s) = 1 - 1/F(s)$
- Now we want to invert $G(s)$ to find $g(t)$
- Use calculation that $F(s) = (2s)^{-1/2}$
- $$G(s) = 1 - (2s)^{1/2} \simeq e^{-(2s)^{1/2}}$$

First return

Groovy aspects of $g(t) \sim t^{-3/2}$:

- Variance is infinite (weird but okay...)
- Mean is also infinite (just plain crazy...)
- Distribution is normalizable so process always returns to 0.
- For river networks: $P(\ell) \sim \ell^{-\gamma}$ so $\gamma = 3/2$ for Scheidegger networks.

References I

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