# Scale-Free Networks Complex Networks, Course 295A, Spring, 2008

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Networks with power-law degree distributions have become known as scale-free networks.

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 Scale-free refers specifically to the degree distribution having a power-law decay in its tail: Original model Introduction

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 Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

 $P_k \sim k^{-\gamma}$  for 'large' k

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- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large' k

One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2] Introduction
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- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large'  $k$ 

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

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Scale-free networks are not fractal in any sense.

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Scale-free networks are not fractal in any sense.

abstract, relational, informational, ... (non-physical)

Usually talking about networks whose links are

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- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ... (non-physical)
- Primary example: hyperlink network of the Web

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'scale-free' or not...

Scale-free networks are not fractal in any sense.

abstract, relational, informational, ... (non-physical)

Usually talking about networks whose links are

Primary example: hyperlink network of the Web Much arguing about whether or networks are

References









$$\gamma$$
 = 2.5  $\langle k \rangle$  = 1.8



$$\gamma = 2.5$$
  $\langle k \rangle = 1.66667$ 











$$\gamma = 2.5$$
 $\langle k \rangle = 1.6$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.50667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.62667$ 

$$\gamma$$
 = 2.5  $\langle k \rangle$  = 1.8

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### The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

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## The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

How does the exponent γ depend on the mechanism? Original model
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## The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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## Work that presaged scale-free networks

▶ 1924: G. Udny Yule [9]: # Species per Genus

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## Work that presaged scale-free networks

- 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author

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# Heritage

# Work that presaged scale-free networks

- 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot [5]): Zipf's law for word frequency through optimization

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- ▶ 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot <sup>[5]</sup>): Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus

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# Heritage

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- ▶ 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot <sup>[5]</sup>): Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]: Network of Scientific Citations

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Barabási-Albert model = BA model.

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Key ingredients:

Barabási-Albert model = BA model.

Growth and Preferential Attachment (PA).

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- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.

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### BA model

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- Step 2:

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### BA model

- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- Step 1: start with m₀ disconnected nodes.
- Step 2:
  - 1. Growth—a new node appears at each time step  $t = 0, 1, 2, \dots$

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- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
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  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - Each new node makes m links to nodes already present.

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### **BA** model

- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*<sub>0</sub> disconnected nodes.
- Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - Each new node makes m links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\propto k_i$ .

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### BA model

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- ▶ Step 1: start with  $m_0$  disconnected nodes.
- Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - Each new node makes m links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\propto k_i$ .
- In essence, we have a rich-gets-richer scheme.

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 $\triangleright$  Definition:  $A_k$  is the attachment kernel for a node with degree k.

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For the original model:

$$A_k = k$$

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### BA model

- $\triangleright$  Definition:  $A_k$  is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

**Definition:**  $P_{\text{attach}}(k, t)$  is the attachment probability.

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- For the original model:

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- **Definition:**  $P_{\text{attach}}(k, t)$  is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t

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$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

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- For the original model:

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where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

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When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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Assumes probability of being connected to is small.

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- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.

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- ▶ Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

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## Approximate analysis

When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

where  $t = N(t) - m_0$ .

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# Approximate analysis

▶ Deal with denominator: each added node brings *m* new edges.

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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{d}{dt}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)} = m \frac{k_i(t)}{2mt}$$

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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{i=1}^{N(t)}k_i(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

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## Approximate analysis

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{i=1}^{N(t)}k_i(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

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Rearrange and solve:

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## Approximate analysis

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Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}.$$

▶ Next find *c<sub>i</sub>* . . .

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$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

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Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

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$$k_i(t) = m \left(\frac{t}{t_{i \text{ start}}}\right)^{1/2} \text{ for } t \geq t_{i, \text{start}}.$$

All node degrees grow as t<sup>1/2</sup>

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## Approximate analysis

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▶ So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

All node degrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i,\text{start}}$  which flattens out growth curve.

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▶ So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i \text{ start}}}\right)^{1/2} \text{ for } t \geq t_{i, \text{start}}.$$

- All node degrees grow as t<sup>1/2</sup> but later nodes have larger  $t_{i.start}$  which flattens out growth curve.
- Early nodes do best (First-mover advantage).

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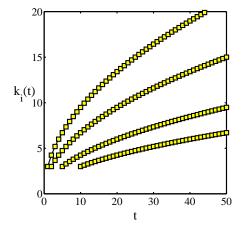
m = 3 $ightharpoonup t_{i,start} =$ 

1, 2, 5, and 10.

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- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

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$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

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## Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}).$$

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$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

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Using the uniformity of start times:

$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

Differentiate to find Pr(k):

$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k} \mathbf{Pr}(k_i < k)$$

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Using the uniformity of start times:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

Differentiate to find Pr(k):

$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k} \mathbf{Pr}(k_i < k) = \frac{2m^2t}{(t + m_0)k^3}$$

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Using the uniformity of start times:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

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$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k} \mathbf{Pr}(k_i < k) = \frac{2m^2t}{(t + m_0)k^3}$$

$$\sim 2m^2k^{-3}$$
 as  $m\to\infty$ .

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We thus have a very specific prediction of

▶ Typical for real networks:  $2 < \gamma < 3$ .

 $Pr(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .

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- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.

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- We thus have a very specific prediction of  $Pr(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.
- ▶  $2 < \gamma < 3$ : finite mean and 'infinite' variance

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- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.
- ightharpoonup 2 <  $\gamma$  < 3: finite mean and 'infinite' variance
- In practice,  $\gamma$  < 3 means variance is governed by upper cutoff.

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- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.
- **2**  $< \gamma <$  3: finite mean and 'infinite' variance
- In practice,  $\gamma$  < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance

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- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.
- ▶  $2 < \gamma < 3$ : finite mean and 'infinite' variance (wild)
- In practice,  $\gamma$  < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance

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- We thus have a very specific prediction of  $\Pr(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance (wild)
- In practice,  $\gamma$  < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance (mild)

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WWW  $\gamma \simeq 2.1$  for in-degree WWW  $\gamma \simeq 2.45$  for out-degree

Movie actors  $\gamma \simeq 2.3$ 

Words (synonyms)  $\gamma \simeq 2.8$ 

The Internets is a different business...

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### Real data

### From Barabási and Albert's original paper [2]:

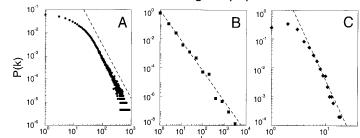


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle=28.78$ . (B) WWW, N=325,729,  $\langle k \rangle=5.46$  (6). (C) Power grid data, N=4941,  $\langle k \rangle=2.67$ . The dashed lines have slopes (A)  $\gamma_{\rm actor}=2.3$ , (B)  $\gamma_{\rm www}=2.1$  and (C)  $\gamma_{\rm power}=4$ .

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Vary attachment kernel.

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Scale-Free Networks

- Vary attachment kernel.
- Vary mechanisms:

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- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion

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- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - Add node deletion

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- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - Add node deletion
  - Add edge rewiring

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - Add node deletion
  - Add edge rewiring
- Deal with directed versus undirected networks.

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - Add node deletion
  - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - Add node deletion
  - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?

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- Vary attachment kernel.
- Vary mechanisms:
  - Add edge deletion
  - 2 Add node deletion
  - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- O.: Do model details matter?
- The answer is (surprisingly) yes.

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► Let's look at preferential attachment (PA) a little more closely.

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▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.

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### Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.

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### Preferential attachment

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- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
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- ▶ We need to know what everyone's degree is...

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- For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- ► PA is : an outrageous assumption of node capability.

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### Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ► For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- ► PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

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 Instead of attaching preferentially, allow new nodes to attach randomly.

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- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.

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- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.

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- Instead of attaching preferentially, allow new nodes to attach randomly.
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- Can also do this at random.
- We know that friends are weird...

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- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird...
- ► Assuming the existing network is random, we know probability of a random friend having degree *k* is

 $Q_k \propto kP_k$ 

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- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- We know that friends are weird...
- ► Assuming the existing network is random, we know probability of a random friend having degree *k* is

$$Q_k \propto kP_k$$

So rich-gets-richer scheme can now be seen to work in a natural way. Original model

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▶ We've looked at some aspects of contagion on scale-free networks:

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We've looked at some aspects of contagion on

1. Facilitate disease-like spreading.





We've looked at some aspects of contagion on

 Facilitate disease-like spreading. 2. Inhibit threshold-like spreading.





We've looked at some aspects of contagion on

Another simple story concerns system robustness.

 Facilitate disease-like spreading. 2. Inhibit threshold-like spreading.



Albert et al., Nature, 2000:

We've looked at some aspects of contagion on

Another simple story concerns system robustness.

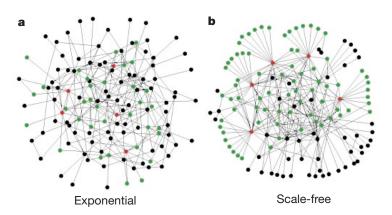
"Error and attack tolerance of complex networks" [1]

 Facilitate disease-like spreading. 2. Inhibit threshold-like spreading.

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 Standard random networks (Erdös-Rényi) versus
 Scale-free networks



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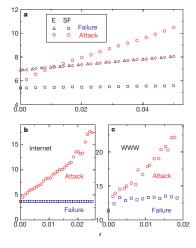
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 Plots of network diameter as a function of fraction of nodes removed

- Erdös-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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from Albert et al., 2000

 Scale-free networks are thus robust to random failures yet fragile to targeted ones.

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Scale-free networks are thus robust to random

failures yet fragile to targeted ones.

All very reasonable: Hubs are a big deal.



Scale-free networks are thus robust to random.

But: next issue is whether hubs are vulnerable or not.

failures yet fragile to targeted ones.

All very reasonable: Hubs are a big deal.



- Scale-free networks are thus robust to random. failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)

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- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
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- Most connected nodes are either:

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  - Physically larger nodes that may be harder to 'target'

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- Scale-free networks are thus robust to random. failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
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  - 2. or subnetworks of smaller, normal-sized nodes.

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- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  - 1. Physically larger nodes that may be harder to 'target'
  - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

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# Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

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### Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK)<sup>[3]</sup> explored the general attachment kernel:

**Pr**(attach to node i)  $\propto A_k = k_i^{\nu}$ 

where  $A_k$  is the attachment kernel and  $\nu > 0$ .

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# Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

**Pr**(attach to node 
$$i$$
)  $\propto A_k = k_i^{\nu}$ 

where  $A_k$  is the attachment kernel and  $\nu > 0$ .

RK also looked at changing the details of the attachment kernel.

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▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

**Pr**(attach to node 
$$i$$
)  $\propto A_k = k_i^{\nu}$ 

where  $A_k$  is the attachment kernel and  $\nu > 0$ .

- RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (⊞).

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$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

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Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

1. The first term corresponds to degree k-1 nodes becoming degree k nodes.

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- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.

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► Here's the set up:

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where  $N_k$  is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail:  $A_0 = 0$

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- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail:  $A_0 = 0$
- 4. One node is added per unit time.

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- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree *k* nodes becoming degree k-1 nodes.
- 3. Detail:  $A_0 = 0$
- One node is added per unit time.
- Seed with some initial network

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Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

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- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail:  $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

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$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

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▶ In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

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▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .

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- For  $A_k = k$ , we have

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- ▶ For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t)$$

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$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- ▶ For  $A_k = k$ , we have

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In general, probability of attaching to a specific node of degree k at time t is

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▶ Detail: we are ignoring initial seed network's edges.

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► So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[ (k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

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As for BA method, look for steady-state growing solution:

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- We replace  $dN_k/dt$  with  $dn_kt/dt = n_k$ .

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- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .
- We replace  $dN_k/dt$  with  $dn_kt/dt = n_k$ .
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2!} [(k-1)n_{k-1}! - kn_k!] + \delta_{k1}$$

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$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

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$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

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## Generalized model

Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

▶ Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

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## Generalized model

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► Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$

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$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}n_1$$

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Original model

Now find  $n_k$ :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)(k-2)(k-3)(k-3)(k-4)(k-5)\cdots 5}{k+2} \frac{4}{k+1} \frac{3}{k} \frac{2}{(k-1)(k-2)\cdots 8} \frac{1}{7} \frac{3}{7} \frac{2}{5} \frac{1}{4} n_1$$

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Now find  $n_k$ :

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$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{87}\frac{4}{87}\frac{3}{57}\frac{2}{4}n_1$$



# Generalized model

Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 54321}{k+2k+1k} \frac{(k-1)(k-2)\cdots 54321}{(k-1)(k-2)\cdots 87854} \frac{3}{8} \frac{2}{5} \frac{1}{4} n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1$$

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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 54321}{k+2k+1} n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)}$$

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$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

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As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large  $k$ .

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# Universality?

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- ▶ Keep  $A_k$  linear in k but tweak details.

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- ▶ Natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .
- ▶ But we'll first explore a more subtle modification of  $A_k$  made by Redner/Krapivsky [3]
- ► Keep  $A_k$  linear in k but tweak details.
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

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$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large  $t$ .

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$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large  $t$ .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ .

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▶ We assume that  $A = \mu t$ 

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- We'll find µ later and make sure that our assumption is consistent.

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where we only know the asymptotic behavior of  $A_k$ .

- ▶ We assume that  $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- ▶ As before, also assume  $N_k(t) = n_k t$ .

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$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

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▶ For  $A_k = k$  we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

▶ This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_k n_k] + \delta_{k1}$$

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For  $A_k = k$  we had

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This now becomes

$$n_{k} = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_{k} n_{k}] + \delta_{k1}$$

$$\Rightarrow (A_{k} + \mu) n_{k} = A_{k-1} n_{k-1} + \mu \delta_{k1}$$

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▶ This now becomes

$$n_{k} = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_{k} n_{k}] + \delta_{k1}$$
$$\Rightarrow (A_{k} + \mu) n_{k} = A_{k-1} n_{k-1} + \mu \delta_{k1}$$

Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

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ightharpoonup For  $A_k = k$  we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

▶ This now becomes

$$n_{k} = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_{k} n_{k}] + \delta_{k1}$$
$$\Rightarrow (A_{k} + \mu) n_{k} = A_{k-1} n_{k-1} + \mu \delta_{k1}$$

Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k-1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$
$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left( 1 + \frac{\mu}{A_1} \right) \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}}$$

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$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left( 1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}} \text{ since } n_1 = \mu/(\mu + A_1)$$

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Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- ► For large *k*:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{A_j}{A_j + \mu}$$

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{K} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{K} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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$$= \frac{\mu}{\cancel{A_k}} \frac{A_1}{(A_1 + \mu)} \frac{A_2}{(A_2 + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{\cancel{K}}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)}$$

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- ▶ For large k:

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$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi} k^{k+1/2} e^{-k}}{\sqrt{2\pi} (k+\mu+1)^{k+\mu+1+1/2} e^{-(k+\mu+1)}}$$

 $\propto k^{-\mu-1}$ 

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- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

 $\propto k^{-\mu-1}$ 

▶ Since  $\mu$  depends on  $A_k$ , details matter...

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▶ Now we need to find  $\mu$ .

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▶ Now we need to find  $\mu$ .

▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$ 

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- Now we need to find μ.
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$

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- Now we need to find  $\mu$ .
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- Now substitute in our expression for  $n_k$ :

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- Now we need to find  $\mu$ .
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$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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- Now we need to find  $\mu$ .
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$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{\mathcal{A}_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{\mathcal{A}_j}} \mathcal{A}_k$$

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- ▶ Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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- ▶ Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
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$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

▶ Closed form expression for  $\mu$ .

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- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
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$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for  $\mu$ .
- ▶ We can solve for  $\mu$  in some cases.

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- ▶ Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t)A_k$
- Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for  $\mu$ .
- We can solve for  $\mu$  in some cases.
- ▶ Our assumption that  $A = \mu t$  is okay.

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Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .

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- ightharpoonup Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .
- ho  $\gamma$  = 2 is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

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- ightharpoonup Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .
- ho  $\gamma$  = 2 is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

Let's now look at a specific example of  $A_k$  to see this range of  $\gamma$  is possible.

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- ▶ Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- **Expression** for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- ▶ Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- **Expression** for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$
$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for k > 2.
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- **Expression** for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- **Expression** for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_{1}}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_{1}}} = \frac{1}{1 + \frac{\mu}{A_{1}}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} \text{ since } A_{1} = \alpha$$

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► Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{\alpha}}$$

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► Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{\alpha}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and  $b = \mu + 1$ .

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

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with a = 1 and  $b = \mu + 1$ .

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

▶ Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and  $b = \mu + 1$ .

 $\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$  $\Rightarrow \mu(\mu - 1) = 2\alpha$ 

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# Universality?

 $\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$ 

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▶ Since  $\gamma = \mu + 1$ , we have

 $\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$ 

 $0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$ 



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$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

▶ Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness...

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Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

#### Original model

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Sublinear attachment

References





► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

► General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$
.

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## Sublinear attachment kernels

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.

Stretched exponentials (truncated power laws).

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- Stretched exponentials (truncated power laws).
- aka Weibull distributions.

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- Universality: now details of kernel do not matter.

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## Sublinear attachment kernels

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$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing  $\nu$  < 1.

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### Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

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### Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

▶ For  $1/3 < \nu < 1/2$ :

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

#### Original model

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## Sublinear attachment kernels

### Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

▶ For  $1/3 < \nu < 1/2$ :

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

And for  $1/(r+1) < \nu < 1/r$ , we have r pieces in exponential.

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 $A_{k} \sim k^{\nu}$  with  $\nu > 1$ .

Rich-get-much-richer:

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Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

Now a winner-take-all mechanism.

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Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.

#### Original model

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# Superlinear attachment kernels

Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes
- For  $\nu > 2$ , all but a finite # of nodes connect to one node.

#### Original model

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