## Random Networks Complex Networks, Course 295A, Spring, 2008

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Department of Mathematics & Statistics University of Vermont



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### Random Networks

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### Basics

### Definitions

- How to build Some visual examples
- Structure
  - Clustering
  - Degree distributions
  - Configuration model
  - Largest component
- **Generating Functions** 
  - Definitions
  - Basic Properties
  - **Giant Component Condition**
  - Component sizes
  - Useful results
  - Size of the Giant Component Average Component Size
- References

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Pure, abstract random networks:

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### Pure, abstract random networks:

 Consider set of all networks with N labelled nodes and m edges.

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### Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.

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### Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.

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### Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.

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### Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdös-Rényi random networks or ER graphs.

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### Some features:

Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

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### Some features:

Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$

Given *m* edges, there are 
 <sup>(N)</sup>
 <sub>m</sub>) different possible networks.

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Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$

- Given *m* edges, there are 
   <sup>N</sup>
   <sup>(N)</sup>
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- Crazy factorial explosion for  $1 \ll m \ll \binom{N}{2}$ .

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- Limit of m = 0: empty graph.

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### Some features:

Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$

- Given *m* edges, there are 
   <sup>N</sup>
   <sup>2</sup>
   <sup>N</sup>
   <sup>N</sup>
- Crazy factorial explosion for  $1 \ll m \ll \binom{N}{2}$ .
- Limit of m = 0: empty graph.
- Limit of  $m = \binom{N}{2}$ : complete or fully-connected graph.

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Number of possible edges:

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- Crazy factorial explosion for  $1 \ll m \ll \binom{N}{2}$ .
- Limit of m = 0: empty graph.
- Limit of  $m = \binom{N}{2}$ : complete or fully-connected graph.
- Real world: links are usually costly so real networks are almost always sparse.

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### How to build standard random networks:

▶ Given *N* and *m*.

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### How to build standard random networks:

- ▶ Given *N* and *m*.
- Two probablistic methods

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### How to build standard random networks:

- ▶ Given *N* and *m*.
- Two probablistic methods (we'll see a third later on)

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### How to build standard random networks:

- ▶ Given *N* and *m*.
- Two probablistic methods (we'll see a third later on)
- 1. Connect each of the  $\binom{N}{2}$  pairs with appropriate probability *p*.

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- ▶ Given *N* and *m*.
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- 1. Connect each of the  $\binom{N}{2}$  pairs with appropriate probability *p*.
- 2. Take *N* nodes and add exactly *m* links by selecting edges without replacement.

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- ▶ Given *N* and *m*.
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  - Useful for theoretical work.
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- 1. Connect each of the  $\binom{N}{2}$  pairs with appropriate probability *p*.
  - Useful for theoretical work.
- Take N nodes and add exactly m links by selecting edges without replacement.
  - ► Algorithm: Randomly choose a pair of nodes *i* and *j*, *i* ≠ *j*, and connect if unconnected; repeat until all *m* edges are allocated.

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  - Best for adding small numbers of links (most cases).

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- Take N nodes and add exactly m links by selecting edges without replacement.
  - ► Algorithm: Randomly choose a pair of nodes *i* and *j*, *i* ≠ *j*, and connect if unconnected; repeat until all *m* edges are allocated.
  - Best for adding small numbers of links (most cases).
  - 1 and 2 are effectively equivalent for large N.

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### A few more things:

► For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2}$$

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### A few more things:

► For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

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### A few more things:

► For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

So the expected or average degree is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

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$$=\frac{2}{N}p\frac{1}{2}N(N-1)$$

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$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

So the expected or average degree is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{2}{N} p \frac{1}{2} N(N-1) = p(N-1).$$

Which is what it should be...

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$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

So the expected or average degree is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$=\frac{2}{N}p\frac{1}{2}N(N-1)=\frac{2}{N}p\frac{1}{2}N(N-1)=p(N-1).$$

- Which is what it should be...
- If we keep  $\langle k \rangle$  constant then  $p \propto 1/N \rightarrow 0$  as  $N \rightarrow \infty$ .

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### Next slides:

Example realizations of random networks

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### Next slides:

### Example realizations of random networks

► *N* = 500

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### Next slides:

Example realizations of random networks

- ► *N* = 500
- ► Vary *m*, the number of edges from 100 to 1000.

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### Next slides:

Example realizations of random networks

- ► *N* = 500
- ▶ Vary *m*, the number of edges from 100 to 1000.
- Average degree  $\langle k \rangle$  runs from 0.4 to 4.

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### Next slides:

Example realizations of random networks

- ► *N* = 500
- ▶ Vary *m*, the number of edges from 100 to 1000.
- Average degree  $\langle k \rangle$  runs from 0.4 to 4.
- Look at full network plus the largest component.

#### Random Networks

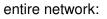
#### Basics Definitions How to build Some visual examples

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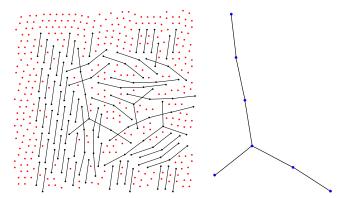
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N = 500, number of edges m = 100average degree  $\langle k \rangle = 0.4$ 

#### Random Networks

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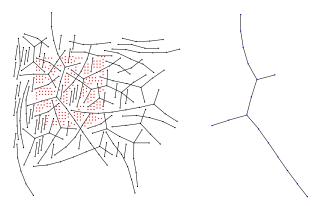
Structure Clustering Degree distributions Configuration model Largest component

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entire network:



N = 500, number of edges m = 200average degree  $\langle k \rangle = 0.8$ 

largest component:

#### Random Networks

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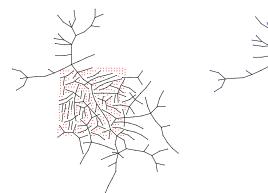
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entire network:



largest component:

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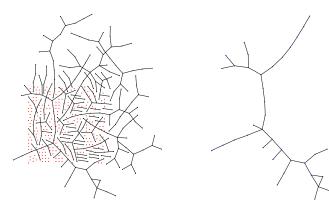
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N = 500, number of edges m = 230average degree  $\langle k \rangle = 0.92$ 

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entire network:



N = 500, number of edges m = 240average degree  $\langle k \rangle = 0.96$ 

largest component:

#### Random Networks

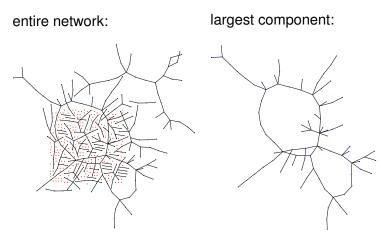
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N = 500, number of edges m = 250average degree  $\langle k \rangle = 1$ 

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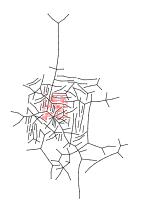
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entire network:



largest component:

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N = 500, number of edges m = 260average degree  $\langle k \rangle = 1.04$  Random Networks

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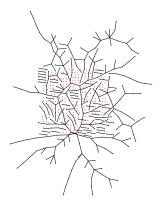
Structure Clustering Degree distributions Configuration model Largest component

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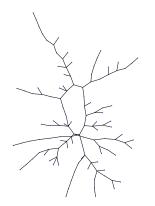
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entire network:



largest component:



N = 500, number of edges m = 280average degree  $\langle k \rangle = 1.12$ 

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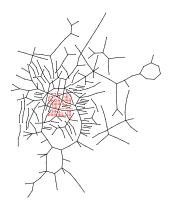
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entire network:



N = 500, number of edges m = 300average degree  $\langle k \rangle = 1.2$ 

largest component:

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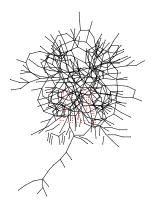
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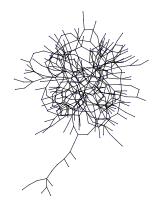
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entire network:



largest component:



N = 500, number of edges m = 500average degree  $\langle k \rangle = 2$ 

#### Random Networks

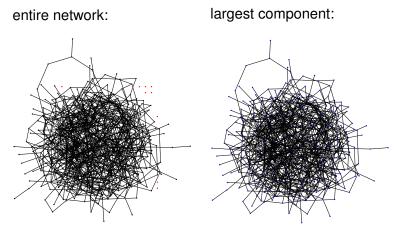
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N = 500, number of edges m = 1000average degree  $\langle k \rangle = 4$ 

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## Random networks: examples for N=500











m = 100 $\langle k \rangle = 0.4$ 



m = 230 $\langle k \rangle = 0.92$  m = 240 $\langle k \rangle = 0.96$  m = 250 $\langle k \rangle = 1$ 



m = 260

 $\langle k \rangle = 1.04$ 



m = 200

m = 280

 $\langle k \rangle = 1.12$ 

 $\langle k \rangle = 0.8$ 



m = 300

 $\langle k \rangle = 1.2$ 



m = 500

 $\langle k \rangle = 2$ 

m = 1000 $\langle k \rangle = 4$ 

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## Random networks: largest components







m = 260

 $\langle k \rangle = 1.04$ 



m = 230 $\langle k \rangle = 0.92$ 



m = 240 $\langle k \rangle = 0.96$  *m* = 250  $\langle k \rangle = 1$ 

m = 1000

 $\langle k \rangle = 4$ 



m = 280

 $\langle k \rangle = 1.12$ 



m = 300

 $\langle k \rangle = 1.2$ 



m = 500

 $\langle k \rangle = 2$ 





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## Random networks: examples for N=500







m = 250

 $\langle k \rangle = 1$ 



 $\langle k \rangle = 1$ 

m = 250 $\langle k \rangle = 1$ 



m = 250

 $\langle k \rangle = 1$ 

*m* = 250

 $\langle k \rangle = 1$ 



m = 250 $\langle k \rangle = 1$ 





*m* = 250  $\langle k \rangle = 1$ 



m = 250 $\langle k \rangle = 1$ 



m = 250 $\langle k \rangle = 1$ 

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## Random networks: largest components





m = 250 $\langle k \rangle = 1$ 

m = 250 $\langle k \rangle = 1$ 

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m = 250

 $\langle k \rangle = 1$ 

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 $\langle k \rangle = 1$ 

m = 250 $\langle k \rangle = 1$ 

*m* = 250

m = 250 $\langle k \rangle = 1$ 



*m* = 250  $\langle k \rangle = 1$ 

m = 250 $\langle k \rangle = 1$ 

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### Clustering:

For method 1, what is the clustering coefficient for a finite network?

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### **Clustering:**

- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient (Newman<sup>[1]</sup>):

$$C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$$

#### Random Networks

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- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient (Newman<sup>[1]</sup>):

 $C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$ 

Recall: C<sub>2</sub> = probability that two nodes are connected given they have a friend in common.

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### **Clustering:**

- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient (Newman<sup>[1]</sup>):

 $C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$ 

- Recall: C<sub>2</sub> = probability that two nodes are connected given they have a friend in common.
- For standard random networks, we have simply that

 $C_2 = p.$ 

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### Clustering:

So for large random networks (N → ∞), clustering drops to zero.

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### **Clustering:**

- So for large random networks (N → ∞), clustering drops to zero.
- Key structural feature of random networks is that they locally look like branching networks (no loops).

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### Degree distribution:

Recall p<sub>k</sub> = probability that a randomly selected node has degree k.

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Degree distribution:

- Recall p<sub>k</sub> = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.

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Degree distribution:

- Recall p<sub>k</sub> = probability that a randomly selected node has degree k.
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- Now consider one node: there are 'N choose k' ways the node can be connected to k of the other N − 1 nodes.

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- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are 'N choose k' ways the node can be connected to k of the other N − 1 nodes.
- ► Each connection occurs with probability *p*, each non-connection with probability (1 − *p*).

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Degree distribution:

- Recall p<sub>k</sub> = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are 'N choose k' ways the node can be connected to k of the other N − 1 nodes.
- ► Each connection occurs with probability *p*, each non-connection with probability (1 − *p*).
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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Limiting form of P(k; p, N):

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### Limiting form of P(k; p, N):

• Our degree distribution:  $P(k; p, N) = {\binom{N-1}{k}}p^k(1-p)^{N-1-k}.$ 



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### Limiting form of P(k; p, N):

- Our degree distribution:  $P(k; p, N) = {\binom{N-1}{k}}p^k(1-p)^{N-1-k}.$
- What happens as  $N \to \infty$ ?

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# Random networks

## Limiting form of P(k; p, N):

- Our degree distribution:  $P(k; p, N) = {\binom{N-1}{k}}p^k(1-p)^{N-1-k}.$
- What happens as  $N \to \infty$ ?

### We must end up with the normal distribution right?

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# Random networks

## Limiting form of P(k; p, N):

- Our degree distribution:  $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$
- What happens as  $N \to \infty$ ?
- We must end up with the normal distribution right?
- If p is fixed, then we would end up with a Gaussian with average degree ⟨k⟩ ≃ pN → ∞.

### Random Networks

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# Random networks

## Limiting form of P(k; p, N):

- Our degree distribution:  $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$
- What happens as  $N \to \infty$ ?
- We must end up with the normal distribution right?
- If p is fixed, then we would end up with a Gaussian with average degree ⟨k⟩ ≃ pN → ∞.
- But we want to keep  $\langle k \rangle$  fixed...
- ► So examine limit of P(k; p, N) when  $p \to 0$  and  $N \to \infty$  with  $\langle k \rangle = p(N-1) = \text{constant.}$

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Substitute  $p = \frac{\langle k \rangle}{N-1}$  into P(k; p, N) and hold k fixed:

$$P(k; p, N) = \binom{N-1}{k} \left(\frac{\langle k \rangle}{N-1}\right)^k \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k}$$

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$$=\frac{(N-1)!}{k!(N-1-k)!}\frac{\langle k\rangle^k}{(N-1)^k}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

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$$=\frac{(N-1)!}{k!(N-1-k)!}\frac{\langle k\rangle^k}{(N-1)^k}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

$$=\frac{(N-1)(N-2)\cdots(N-k)}{k!}\frac{\langle k\rangle^k}{(N-1)^k}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

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$$=\frac{(N-1)(N-2)\cdots(N-k)}{k!}\frac{\langle k\rangle^k}{(N-1)^k}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

$$=\frac{N^{k}(1-\frac{1}{N})\cdots(1-\frac{k}{N})}{k!N^{k}}\frac{\langle k\rangle^{k}}{(1-\frac{1}{N})^{k}}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1}$$

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$$=\frac{(N-1)(N-2)\cdots(N-k)}{k!}\frac{\langle k\rangle^k}{(N-1)^k}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

$$=\frac{\mathcal{M}^{k}(1-\frac{1}{N})\cdots(1-\frac{k}{N})}{k!\mathcal{M}^{k}}\frac{\langle k\rangle^{k}}{(1-\frac{1}{N})^{k}}\left(1-\frac{\langle k\rangle}{N-1}\right)^{N-1-k}$$

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$$=\frac{\cancel{k}}{\cancel{k!}}\left(1-\frac{\cancel{k}}{\cancel{N}}\right)\cdots\left(1-\frac{\cancel{k}}{\cancel{N}}\right)}{\cancel{k!}}\frac{\cancel{k}}{\cancel{k}}\left(1-\frac{\cancel{k}}{\cancel{N}}\right)^{\cancel{k}}\left(1-\frac{\cancel{k}}{\cancel{N}}\right)^{\cancel{k}}$$

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We are now here:

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k}$$

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We are now here:

$$P(k; p, N) \simeq rac{\langle k 
angle^k}{k!} \left(1 - rac{\langle k 
angle}{N-1}
ight)^{N-1-k}$$

Now use the excellent result:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$$

### Random Networks

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Now use the excellent result:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$$

(Use l'Hôpital's rule to prove.)

• Identifying n = N - 1 and  $x = -\langle k \rangle$ :

$$P(k;\langle k\rangle) \simeq \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle} \left(1 - \frac{\langle k\rangle}{N-1}\right)^{-k}$$

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$$P(k;\langle k\rangle) \simeq \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle} \left(1 - \frac{\langle k\rangle}{N-1}\right)^{-k} \to \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle}$$

▶ This is a Poisson distribution ( $\boxplus$ ) with mean  $\langle k \rangle$ .

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 So... standard random networks have a Poisson degree distribution Random Networks

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- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution  $P_k$ .

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- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution  $P_k$ .
- Also known as the configuration model<sup>[1]</sup>.

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- So... standard random networks have a Poisson degree distribution
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- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution  $P_k$ .
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- Assign each node a weight w from some distribution P<sub>w</sub> and form links with probability

 $P(\text{link between } i \text{ and } j) \propto w_i w_j.$ 

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 $P(\text{link between } i \text{ and } j) \propto w_i w_j.$ 

But we'll be more interested in

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- But we'll be more interested in
  - 1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.

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 $P(\text{link between } i \text{ and } j) \propto w_i w_j.$ 

- But we'll be more interested in
  - 1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
  - Examining mechanisms that lead to networks with certain degree distributions.

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## Coming up:

Example realizations of random networks with power law degree distributions:

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## Coming up:

Example realizations of random networks with power law degree distributions:

► *N* = 1000.

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## Coming up:

Example realizations of random networks with power law degree distributions:

- ► *N* = 1000.
- $P_k \propto k^{-\gamma}$  for  $k \ge 1$ .

### Random Networks

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## Coming up:

Example realizations of random networks with power law degree distributions:

- ► *N* = 1000.
- $P_k \propto k^{-\gamma}$  for  $k \ge 1$ .
- Set  $P_0 = 0$  (no isolated nodes).

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- Vary exponent  $\gamma$  between 2.10 and 2.91.

### Random Networks

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- Again, look at full network plus the largest component.

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Example realizations of random networks with power law degree distributions:

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- Vary exponent  $\gamma$  between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

### Random Networks

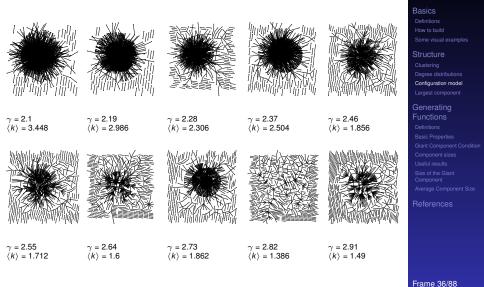
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# Random networks: examples for N=1000



### Random Networks

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# Random networks: largest components





 $\gamma = 2.19$ 







 $\gamma = 2.1$ (k) = 3.448

 $\langle k \rangle = 2.986$ 

 $\gamma = 2.28$  $\langle k \rangle = 2.306$ 

 $\gamma = 2.37$  $\langle k \rangle = 2.504$ 

 $\gamma = 2.46$  $\langle k \rangle = 1.856$ 













 $\gamma = 2.73$  $\langle k \rangle = 1.862$ 

 $\gamma = 2.82$  $\langle k \rangle = 1.386$ 

 $\gamma = 2.91$  $\langle k \rangle = 1.49$ 

### Random Networks

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Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

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Normalization: we must have

-

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

► Checking:

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

### Random Networks

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### Random Networks

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Mean degree: we must have

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### Random Networks

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$$= \langle k \rangle e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)^k}$$

$$= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^{i}}{i!}$$

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 $= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^{i}}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle}$ 

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### Random Networks

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We'll get to a better way of doing this...

### Random Networks

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The variance of degree distributions for random networks turns out to be very important.

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- The variance of degree distributions for random networks turns out to be very important.
- Use calculation similar to one for finding (k) to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

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- The variance of degree distributions for random networks turns out to be very important.
- Use calculation similar to one for finding (k) to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

Variance is then

 $\sigma^2 = \langle \mathbf{k}^2 \rangle - \langle \mathbf{k} \rangle^2$ 

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$$\sigma^{2} = \langle k^{2} \rangle - \langle k \rangle^{2} = \langle k \rangle^{2} + \langle k \rangle - \langle k \rangle^{2}$$

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So standard deviation  $\sigma$  is equal to  $\sqrt{\langle \mathbf{k} \rangle}$ .

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Variance is then

$$\sigma^{2} = \langle \mathbf{k}^{2} \rangle - \langle \mathbf{k} \rangle^{2} = \langle \mathbf{k} \rangle^{2} + \langle \mathbf{k} \rangle - \langle \mathbf{k} \rangle^{2} = \langle \mathbf{k} \rangle.$$

- So standard deviation  $\sigma$  is equal to  $\sqrt{\langle \mathbf{k} \rangle}$ .
- Note: This is a special property of Poisson distribution and can trip us up...

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The degree distribution P<sub>k</sub> is fundamental for our description of many complex networks

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- The degree distribution P<sub>k</sub> is fundamental for our description of many complex networks
- Again:  $P_k$  is the degree of randomly chosen node.

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- The degree distribution P<sub>k</sub> is fundamental for our description of many complex networks
- Again:  $P_k$  is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.

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- The degree distribution P<sub>k</sub> is fundamental for our description of many complex networks
- Again:  $P_k$  is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q<sub>k</sub> to be the probability the node at a random end of a randomly chosen edge has degree k.

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- Now choosing nodes based on their degree (i.e., size):



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- Define Q<sub>k</sub> to be the probability the node at a random end of a randomly chosen edge has degree k.

 $Q_k \propto k P_k$ 

Now choosing nodes based on their degree (i.e., size):

Normalized form:

$$\mathbf{Q}_{k} = \frac{k\mathbf{P}_{k}}{\sum_{k'=0}^{\infty} k'\mathbf{P}_{k'}}$$

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For random networks, Q<sub>k</sub> is also the probability that a friend (neighbor) of a random node has k friends.

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- For random networks, Q<sub>k</sub> is also the probability that a friend (neighbor) of a random node has k friends.
- Useful variant on Q<sub>k</sub>:

 $R_k$  = probability that a friend of a random node has *k* other friends.

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- Useful variant on Q<sub>k</sub>:

 $R_k$  = probability that a friend of a random node has *k* other friends.

$$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}}$$

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$$R_{k} = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

• Equivalent to friend having degree k + 1.

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- For random networks, Q<sub>k</sub> is also the probability that a friend (neighbor) of a random node has k friends.
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 $R_k$  = probability that a friend of a random node has *k* other friends.

$$R_{k} = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

- Equivalent to friend having degree k + 1.
- Natural question: what's the expected number of other friends that one friend has?

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 Given R<sub>k</sub> is the probability that a friend has k other friends, then the average number of friends' other friends is

$$\langle k \rangle_R = \sum_{k=0}^{\infty} k R_k$$

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Given R<sub>k</sub> is the probability that a friend has k other friends, then the average number of friends' other friends is

$$\langle k \rangle_R = \sum_{k=0}^{\infty} kR_k = \sum_{k=0}^{\infty} k \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

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angle} \ &= rac{1}{\langle k 
angle} \sum_{k=1}^\infty k (k+1) P_{k+1} \end{aligned}$$

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Given R<sub>k</sub> is the probability that a friend has k other friends, then the average number of friends' other friends is

$$\begin{split} \langle k \rangle_{R} &= \sum_{k=0}^{\infty} k R_{k} = \sum_{k=0}^{\infty} k \frac{(k+1)P_{k+1}}{\langle k \rangle} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1)P_{k+1} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} \left( (k+1)^{2} - (k+1) \right) P_{k+1} \end{split}$$

(where we have sneakily matched up indices)

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Given R<sub>k</sub> is the probability that a friend has k other friends, then the average number of friends' other friends is

$$\langle k \rangle_{R} = \sum_{k=0}^{\infty} kR_{k} = \sum_{k=0}^{\infty} k \frac{(k+1)P_{k+1}}{\langle k \rangle}$$
$$= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1)P_{k+1}$$

$$=\frac{1}{\langle k\rangle}\sum_{k=1}^{\infty}\left((k+1)^2-(k+1)\right)P_{k+1}$$

(where we have sneakily matched up indices)

$$= \frac{1}{\langle k \rangle} \sum_{j=0}^{\infty} (j^2 - j) P_j \quad \text{(using j = k+1)}$$

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► Note: our result, \langle k \rangle\_R = \frac{1}{\langle k \rangle} \left( \langle k^2 \rangle - \langle k \rangle \right), is true for all random networks, independent of degree distribution.

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- ► Note: our result, \langle k \rangle\_R = \frac{1}{\langle k \rangle} \left( \langle k^2 \rangle \langle k \rangle \right), is true for all random networks, independent of degree distribution.
- For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

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Therefore:

$$\langle k \rangle_{R} = \frac{1}{\langle k \rangle} \left( \langle k \rangle^{2} + \langle k \rangle - \langle k \rangle \right)$$

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 Again, neatness of results is a special property of the Poisson distribution.

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- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have  $\langle k \rangle$  other friends, and  $\langle k \rangle + 1$  total friends...

### Random Networks

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Reason #1:

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Reason #1:

Average # friends of friends per node is

$$\langle {\it k}_2 
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## Reason #1:

Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} \left( \langle k^2 \rangle - \langle k \rangle \right)$$

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Key: Average depends on the 1st and 2nd moments of P<sub>k</sub> and not just the 1st moment.

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- Key: Average depends on the 1st and 2nd moments of P<sub>k</sub> and not just the 1st moment.
- Three peculiarities:
  - 1. We might guess  $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle 1)$  but it's actually  $\langle k(k-1) \rangle$ .

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  - 2. If  $P_k$  has a large second moment, then  $\langle k_2 \rangle$  will be big.

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  - If *P<sub>k</sub>* has a large second moment, then ⟨*k*<sub>2</sub>⟩ will be big. (e.g., in the case of a power-law distribution)

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  - 2. If *P<sub>k</sub>* has a large second moment, then ⟨*k*<sub>2</sub>⟩ will be big.
    (e.g., in the case of a power-law distribution)
  - 3. Your friends are different to you...

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More on peculiarity #3:

A node's average # of friends: (k)

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## More on peculiarity #3:

- A node's average # of friends: (k)
- Friend's average # of friends:  $\frac{\langle k^2 \rangle}{\langle k \rangle}$

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- A node's average # of friends: (k)
- Friend's average # of friends:  $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- Comparison:

$$\frac{\langle \boldsymbol{k}^2 \rangle}{\langle \boldsymbol{k} \rangle} = \langle \boldsymbol{k} \rangle \frac{\langle \boldsymbol{k}^2 \rangle}{\langle \boldsymbol{k} \rangle^2}$$

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So only if everyone has the same degree (variance= σ<sup>2</sup> = 0) can a node be the same as its friends.

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- So only if everyone has the same degree (variance= σ<sup>2</sup> = 0) can a node be the same as its friends.
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

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## (Big) Reason #2:

► ⟨k⟩<sub>R</sub> is key to understanding how well random networks are connected together.

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## (Big) Reason #2:

- ⟨k⟩<sub>R</sub> is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.

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- As N → ∞, does our network have a giant component?

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- Defn: Component = connected subnetwork of nodes such that ∃ path between each pair of nodes in the subnetwork, and no node out side of the subnetwork is connected to it.

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- Defn: Component = connected subnetwork of nodes such that ∃ path between each pair of nodes in the subnetwork, and no node out side of the subnetwork is connected to it.
- ▶ Defn: Giant component = component that comprises a non-zero fraction of a network as  $N \rightarrow \infty$ .

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- Note: Component = Cluster

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## Giant component:

A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.

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## Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.

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- All of this is the same as requiring  $\langle k \rangle_R > 1$ .

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- Giant component condition (or percolation condition):

$$\langle k 
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 Again, see that the second moment is an essential part of the story.

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- Again, see that the second moment is an essential part of the story.
- Equivalent statement: (k<sup>2</sup>) > 2(k)

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# Giant component

## Standard random networks:

• Recall 
$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$
.

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### Standard random networks:

• Recall 
$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$
.

Condition for giant component:

$$\langle k \rangle_R = rac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}$$

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References

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#### Random Networks

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► Therefore when ⟨k⟩ > 1, standard random networks have a giant component.

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Standard random networks:

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- Therefore when (k) > 1, standard random networks have a giant component.
- When  $\langle k \rangle < 1$ , all components are finite.

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- Therefore when (k) > 1, standard random networks have a giant component.
- When  $\langle k \rangle < 1$ , all components are finite.
- ► Fine example of a continuous phase transition (⊞).

#### Random Networks

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- Therefore when (k) > 1, standard random networks have a giant component.
- When  $\langle k \rangle < 1$ , all components are finite.
- ► Fine example of a continuous phase transition (⊞).
- We say  $\langle k \rangle = 1$  marks the critical point of the system.

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### Random networks with skewed $P_k$ :

• e.g, if  $P_k = ck^{-\gamma}$  with 2 <  $\gamma$  < 3 then

$$\langle k^2 \rangle = c \sum_{k=0}^{\infty} k^2 k^{-\gamma}$$

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### Random networks with skewed $P_k$ :

• e.g, if  $P_k = ck^{-\gamma}$  with  $2 < \gamma < 3$  then

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$$\sim \int_{x=0}^{\infty} x^{2-\gamma} \mathrm{d}x$$

#### Random Networks

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$$\sim \int_{x=0}^{\infty} x^{2-\gamma} \mathrm{d}x$$

$$\propto x^{3-\gamma}\Big|_{x=0}^{\infty}$$

#### Random Networks

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$$\propto x^{3-\gamma}\Big|_{x=0}^{\infty} = \infty$$

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$$\propto x^{3-\gamma}\Big|_{x=0}^{\infty} = \infty \quad (>\langle k \rangle).$$

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### Random networks with skewed $P_k$ :

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• e.g, if  $P_k = ck^{-\gamma}$  with 2 <  $\gamma$  < 3 then

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$$\propto x^{3-\gamma} \Big|_{x=0}^{\infty} = \infty \quad (> \langle k \rangle).$$

 So giant component always exists for these kinds of networks.

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### Random networks with skewed $P_k$ :

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• e.g, if  $P_k = ck^{-\gamma}$  with 2 <  $\gamma$  < 3 then

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$$\sim \int_{x=0}^{\infty} x^{2-\gamma} \mathrm{d}x$$
$$\propto x^{3-\gamma} \Big|_{x=0}^{\infty} = \infty \quad (> \langle k \rangle).$$

.....

- So giant component always exists for these kinds of networks.
- ► Cutoff scaling is k<sup>-3</sup>: if γ > 3 then we have to look harder at ⟨k⟩<sub>R</sub>.

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And how big is the largest component?

• Define  $S_1$  as the size of the largest component.

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And how big is the largest component?

- Define S<sub>1</sub> as the size of the largest component.
- ► Consider an infinite ER random network with average degree (k).

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And how big is the largest component?

- Define S<sub>1</sub> as the size of the largest component.
- ► Consider an infinite ER random network with average degree (k).
- Let's find S<sub>1</sub> with a back-of-the-envelope argument.

#### Random Networks

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- Consider an infinite ER random network with average degree (k).
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- Define δ as the probability that a randomly chosen node does not belong to the largest component.

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- Simple connection:  $\delta = 1 S_1$ .

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- Simple connection:  $\delta = 1 S_1$ .
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.

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- Simple connection:  $\delta = 1 S_1$ .
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.

So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

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- Define δ as the probability that a randomly chosen node does not belong to the largest component.
- Simple connection:  $\delta = 1 S_1$ .
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

Substitute in Poisson distribution...

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### Carrying on:

$$\boldsymbol{\delta} = \sum_{k=0}^{\infty} \boldsymbol{P}_k \boldsymbol{\delta}^k$$

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Carrying on:

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k$$

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Giant Component Sizes Component sizes Useful results Size of the Giant Component Average Component Size

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Carrying on:

$$\boldsymbol{\delta} = \sum_{k=0}^{\infty} \boldsymbol{P}_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle \boldsymbol{k} \rangle^k}{k!} \boldsymbol{e}^{-\langle \boldsymbol{k} \rangle} \delta^k$$

$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}$$

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$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}$$

$$= e^{-\langle k 
angle} e^{\langle k 
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$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.$$

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$$=e^{-\langle k
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angle \delta)^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}$$

.

▶ Now substitute in  $\delta = 1 - S_1$  and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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# • We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$ .

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- We can figure out some limits and details for S<sub>1</sub> = 1 − e<sup>-⟨k⟩S<sub>1</sub></sup>.
- First, we can write  $\langle k \rangle$  in terms of  $S_1$ :

$$\langle k 
angle = rac{1}{S_1} \ln rac{1}{1-S_1}$$

.

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$$\langle k \rangle = rac{1}{S_1} \ln rac{1}{1-S_1}.$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 \rightarrow 0$ .

#### Random Networks

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- First, we can write  $\langle k \rangle$  in terms of  $S_1$ :

$$\langle k \rangle = rac{1}{S_1} \ln rac{1}{1-S_1}.$$

As 
$$\langle k \rangle$$
 → 0, S<sub>1</sub> → 0.
As  $\langle k \rangle$  → ∞, S<sub>1</sub> → 1.

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$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 \rightarrow 0$ .

• As  $\langle k \rangle \to \infty$ ,  $S_1 \to 1$ .

• Notice that at  $\langle k \rangle = 1$ , the critical point,  $S_1 = 0$ .

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$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$

- As  $\langle k \rangle \rightarrow 0$ ,  $S_1 \rightarrow 0$ .
- As  $\langle k \rangle \to \infty$ ,  $S_1 \to 1$ .
- Notice that at  $\langle k \rangle = 1$ , the critical point,  $S_1 = 0$ .
- Only solvable for S > 0 when  $\langle k \rangle > 1$ .

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- As  $\langle k \rangle \rightarrow 0$ ,  $S_1 \rightarrow 0$ .
- As  $\langle k \rangle \to \infty$ ,  $S_1 \to 1$ .
- Notice that at  $\langle k \rangle = 1$ , the critical point,  $S_1 = 0$ .
- Only solvable for S > 0 when  $\langle k \rangle > 1$ .
- Really a transcritical bifurcation<sup>[2]</sup>.

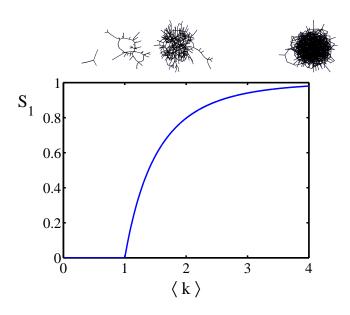
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Turns out we were lucky...

Our dirty trick only works for ER random networks.

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Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.

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Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...

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### Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .

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### Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .
- We need a separate probability δ' for the chance that a node at the end of a random edge is part of the largest component.

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### Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .
- We need a separate probability δ' for the chance that a node at the end of a random edge is part of the largest component.
- ► We can do this but we need to enhance our toolkit with Generatingfunctionology...<sup>[3]</sup>

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Idea: Given a sequence a<sub>0</sub>, a<sub>1</sub>, a<sub>2</sub>,..., associate each element with a distinct function or other mathematical object.

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- Idea: Given a sequence a<sub>0</sub>, a<sub>1</sub>, a<sub>2</sub>,..., associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

#### Random Networks

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- Idea: Given a sequence a<sub>0</sub>, a<sub>1</sub>, a<sub>2</sub>,..., associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

## Definition:

• The generating function (g.f.) for a sequence  $\{a_n\}$  is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

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- Idea: Given a sequence a<sub>0</sub>, a<sub>1</sub>, a<sub>2</sub>,..., associate each element with a distinct function or other mathematical object.
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- Idea: Given a sequence a<sub>0</sub>, a<sub>1</sub>, a<sub>2</sub>,..., associate each element with a distinct function or other mathematical object.
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- ► Roughly: transforms a vector in R<sup>∞</sup> into a function defined on R<sup>1</sup>.
- Related to Fourier, Laplace, Mellin, ...

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Take a degree distribution with exponential decay:

 $P_k = ce^{-\lambda k}$ 

where  $c = 1 - e^{-\lambda}$ .

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• Notice that 
$$F(1) = c/(1 - e^{-\lambda}) = 1$$
.

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• Notice that  $F(1) = c/(1 - e^{-\lambda}) = 1$ .

For probability distributions, we must always have F(1) = 1 since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$
$$= \frac{d}{dx} F(x) \bigg|_{x=1}$$

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Average degree:

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 In general, many calculations become simple, if a little abstract.

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$
$$= \frac{d}{dx} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x)=\frac{(1-e^{-\lambda})e^{-\lambda}}{(1-xe^{-\lambda})^2}.$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$
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- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x)=rac{(1-e^{-\lambda})e^{-\lambda}}{(1-xe^{-\lambda})^2}.$$

So:

$$\langle k 
angle = F'(1) = rac{e^{-\lambda}}{(1-e^{-\lambda})}.$$

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Useful pieces for probability distributions:

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Useful pieces for probability distributions:

F(1) = 1

Normalization:



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## Useful pieces for probability distributions:

Normalization:

First moment:

 $\langle k \rangle = F'(1)$ 

F(1) = 1

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## Useful pieces for probability distributions:

Normalization:

First moment:

 $\langle k \rangle = F'(1)$ 

F(1) = 1

Higher moments:

 $\langle k^n \rangle = \left( x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$ 

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## Useful pieces for probability distributions:

Normalization:

F(1) = 1

First moment:

 $\langle k \rangle = F'(1)$ 

Higher moments:

 $\langle k^n \rangle = \left( x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$ 

kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$

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# Edge-degree distribution

Recall our condition for a giant component:

$$\langle k \rangle_{R} = rac{\langle k^{2} \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

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# Edge-degree distribution

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Recall our condition for a giant component:

 $\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$ 

- We first need the g.f. for  $R_k$ .

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Recall our condition for a giant component:

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- We'll now use this notation:

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- We'll now use this notation:

 $F_P(x)$  is the g.f. for  $P_k$ .  $F_R(x)$  is the g.f. for  $R_k$ .

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- Condition in terms of g.f. is:

 $\langle k \rangle_R = F'_R(1) > 1.$ 

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- Condition in terms of g.f. is:

 $\langle k \rangle_R = F'_R(1) > 1.$ 

Now find how F<sub>R</sub> is related to F<sub>P</sub>...

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► We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k x^k}{k} = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to j = k + 1 and pull out  $\frac{1}{\langle k \rangle}$ :

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$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left( F_P(x) - P_0 \right)$$

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Shift index to j = k + 1 and pull out  $\frac{1}{\langle k \rangle}$ :

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$$=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{\infty}P_{j}x^{j}=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{P}(x)-P_{0}\right)=\frac{1}{\langle k\rangle}F_{P}'(x).$$

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# Basics

# Giant Component Condition

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k x^k}{k} = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to j = k + 1 and pull out  $\frac{1}{\langle k \rangle}$ :

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{\infty}P_{j}x^{j}=\frac{1}{\langle k\rangle}\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{P}(x)-P_{0}\right)=\frac{1}{\langle k\rangle}F_{P}'(x).$$

Finally, since  $\langle k \rangle = F'_P(1)$ ,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$

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# • Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1.$

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- Recall giant component condition is  $\langle k \rangle_R = F'_R(1) > 1.$
- Since we have  $F_R(x) = F'_P(x)/F'_P(1)$ ,

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- Recall giant component condition is  $\langle k \rangle_R = F'_R(1) > 1.$
- Since we have  $F_R(x) = F'_P(x)/F'_P(1)$ ,

$$F'_{R}(x) = rac{F''_{P}(x)}{F'_{P}(1)}$$

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- Recall giant component condition is  $\langle k \rangle_R = F'_R(1) > 1.$
- Since we have  $F_R(x) = F'_P(x)/F'_P(1)$ ,

$$F'_{R}(x) = rac{F''_{P}(x)}{F'_{P}(1)}$$

Setting x = 1, our condition becomes



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To figure out the size of the largest component  $(S_1)$ , we need more resolution on component sizes.

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To figure out the size of the largest component  $(S_1)$ , we need more resolution on component sizes.

### Definitions:

*π<sub>n</sub>* = probability that a random node belongs to a finite component of size *n* < ∞.</p>

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To figure out the size of the largest component  $(S_1)$ , we need more resolution on component sizes.

### Definitions:

- *π<sub>n</sub>* = probability that a random node belongs to a finite component of size *n* < ∞.</p>
- *ρ<sub>n</sub>* = probability a random link leads to a finite subcomponent of size *n* < ∞.</p>

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To figure out the size of the largest component  $(S_1)$ , we need more resolution on component sizes.

### Definitions:

- *π<sub>n</sub>* = probability that a random node belongs to a finite component of size *n* < ∞.</p>

Local-global connection:

 $P_k, R_k \Leftrightarrow \pi_n, \rho_n$ neighbors  $\Leftrightarrow$  components

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G.f.'s for component size distributions:

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G.f.'s for component size distributions:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n$$
 and  $F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$ 

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G.f.'s for component size distributions:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n$$
 and  $F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$ 

### The largest component:

Subtle key: F<sub>π</sub>(1) is the probability that a node belongs to a finite component.

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G.f.'s for component size distributions:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n$$
 and  $F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$ 

### The largest component:

- Subtle key: F<sub>π</sub>(1) is the probability that a node belongs to a finite component.
- Therefore:  $S_1 = 1 F_{\pi}(1)$ .

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G.f.'s for component size distributions:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n$$
 and  $F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$ 

### The largest component:

- Subtle key: F<sub>π</sub>(1) is the probability that a node belongs to a finite component.
- Therefore:  $S_1 = 1 F_{\pi}(1)$ .

### Our mission, which we accept:

Find the four generating functions

$$F_P, F_R, F_\pi$$
, and  $F_\rho$ 

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### Sneaky Result 1:

Consider two random variables U and V whose values may be 0, 1, 2, ...

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### Sneaky Result 1:

- Consider two random variables U and V whose values may be 0, 1, 2, ...
- Write probability distributions as U<sub>k</sub> and V<sub>k</sub> and g.f.'s as F<sub>U</sub> and F<sub>V</sub>.

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### Sneaky Result 1:

- Consider two random variables U and V whose values may be 0, 1, 2, ...
- ► Write probability distributions as U<sub>k</sub> and V<sub>k</sub> and g.f.'s as F<sub>U</sub> and F<sub>V</sub>.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{V} U^{(i)}$$
 with each  $U^{(i)} \stackrel{d}{=} U$ 

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### Sneaky Result 1:

- Consider two random variables U and V whose values may be 0, 1, 2, ...
- ► Write probability distributions as U<sub>k</sub> and V<sub>k</sub> and g.f.'s as F<sub>U</sub> and F<sub>V</sub>.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{V} U^{(i)}$$
 with each  $U^{(i)} \stackrel{d}{=} U$ 

then

$$F_W(x)=F_V(F_U(x))$$

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### Proof of SN1:

#### Write probability that variable *W* has value *k* as $W_k$ .

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## Write probability that variable *W* has value *k* as $W_k$ .

$$W_k = \sum_{j=0}^{\infty} V_j \times \Pr(\text{sum of } j \text{ draws of variable } U = k)$$

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Write probability that variable *W* has value *k* as  $W_k$ .

$$W_k = \sum_{j=0}^{\infty} V_j imes ext{Pr}( ext{sum of } j ext{ draws of variable } U = k)$$

$$=\sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1},i_{2},\ldots,i_{j}\} \mid \\ i_{1}+i_{2}+\ldots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}}$$

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Write probability that variable *W* has value *k* as  $W_k$ .

$$W_k = \sum_{j=0}^{\infty} V_j imes$$
 Pr(sum of  $j$  draws of variable  $U$  =  $k$ )

$$=\sum_{j=0}^{\infty} V_j \sum_{\substack{\{i_1,i_2,...,i_j\} \mid \\ i_1+i_2+...+i_j=k}} U_{i_1} U_{i_2} \cdots U_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k$$

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Write probability that variable *W* has value *k* as  $W_k$ .

 $\sim$ 

$$W_k = \sum_{j=0}^{\infty} V_j imes ext{Pr}( ext{sum of } j ext{ draws of variable } U = k)$$

$$=\sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1},i_{2},\ldots,i_{j}\} \mid \\ i_{1}+i_{2}+\ldots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}}$$

$$\therefore F_{W}(x) = \sum_{k=0}^{\infty} W_{k} x^{k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1}, i_{2}, \dots, i_{j}\} \mid \\ i_{1}+i_{2}+\dots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}} x^{k}$$

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Write probability that variable *W* has value *k* as  $W_k$ .

$$W_k = \sum_{j=0}^{\infty} V_j imes ext{Pr}( ext{sum of } j ext{ draws of variable } U = k)$$

$$=\sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1},i_{2},\ldots,i_{j}\} \mid \\ i_{1}+i_{2}+\ldots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}}$$

$$\therefore F_{W}(x) = \sum_{k=0}^{\infty} W_{k} x^{k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1}, i_{2}, \dots, i_{j}\} \mid \\ i_{1}+i_{2}+\dots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}} x^{k}$$

$$=\sum_{j=0}^{\infty}V_{j}\sum_{k=0}^{\infty}$$

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Write probability that variable *W* has value *k* as  $W_k$ .

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$$W_k = \sum_{j=0}^{\infty} V_j imes ext{Pr}( ext{sum of } j ext{ draws of variable } U = k)$$

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$$\therefore F_{W}(x) = \sum_{k=0}^{\infty} W_{k} x^{k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} V_{j} \sum_{\substack{\{i_{1}, i_{2}, \dots, i_{j}\} \mid \\ i_{1}+i_{2}+\dots+i_{j}=k}} U_{i_{1}} U_{i_{2}} \cdots U_{i_{j}} x^{k}$$

$$= \sum_{j=0}^{\infty} V_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1,i_2,\ldots,i_j\} \mid \\ i_1+i_2+\ldots+i_j=k}} U_{i_1} x^{i_1} U_{i_2} x^{i_2} \cdots U_{i_j} x^{i_j}$$

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With some concentration, observe:

$$F_{W}(x) = \sum_{j=0}^{\infty} V_{j} \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_{1},i_{2},\dots,i_{k}\} \mid \\ i_{1}+i_{2}+\dots+i_{k}=j \\ x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j}}_{x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j}}$$

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With some concentration, observe:

$$F_{W}(x) = \sum_{j=0}^{\infty} V_{j} \sum_{k=0}^{\infty} \sum_{\substack{\{i_{1}, i_{2}, \dots, i_{k}\} \mid \\ i_{1}+i_{2}+\dots+i_{k}=j}} U_{i_{1}} x^{i_{1}} U_{i_{2}} x^{i_{2}} \dots U_{i_{j}} x^{i_{j}}}{x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j}} \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j} = (F_{U}(x))^{j}}$$

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With some concentration, observe:

$$F_{W}(x) = \sum_{j=0}^{\infty} V_{j} \sum_{k=0}^{\infty} \sum_{\substack{\{i_{1},i_{2},...,i_{k}\} \mid \\ i_{1}+i_{2}+...+i_{k}=j}} U_{i_{1}} x^{i_{1}} U_{i_{2}} x^{i_{2}} \cdots U_{i_{j}} x^{i_{j}}} \underbrace{x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j}}_{\left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^{j} = (F_{U}(x))^{j}} = \sum_{j=0}^{\infty} V_{j} (F_{U}(x))^{j}$$

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With some concentration, observe:

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Sneaky Result 2:

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# Sneaky Result 2:

Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

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# Sneaky Result 2:

- Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)
- SNR2: If a second random variable is defined as

*V* = *U* + 1

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# Sneaky Result 2:

Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

## SNR2: If a second random variable is defined as

$$V = U + 1$$
 then  $|F_V(x) = xF_U(x)|$ 

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## Sneaky Result 2:

Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

## SNR2: If a second random variable is defined as

$$V = U + 1$$
 then  $|F_V(x) = xF_U(x)|$ 

• Reason: 
$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

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## Sneaky Result 2:

Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

## SNR2: If a second random variable is defined as

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• Reason: 
$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

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Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

## SNR2: If a second random variable is defined as

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• Reason: 
$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

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$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$
$$= x \sum_{j=0}^{\infty} U_j x^j$$

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Start with a random variable U with distribution  $U_k$  (k = 0, 1, 2, ...)

## SNR2: If a second random variable is defined as

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 then  $F_V(x) = xF_U(x)$ 

• Reason: 
$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$
$$= x \sum_{j=0}^{\infty} U_j x^j = x F_U(x).$$

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## Sneaky Result 2:

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## SNR2: If a second random variable is defined as

$$V = U + 1$$
 then  $F_V(x) = xF_U(x)$ 

• Reason: 
$$V_k = U_{k-1}$$
 for  $k \ge 1$  and  $V_0 = 0$ .

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$
$$= x \sum_{j=0}^{\infty} U_j x^j = x F_U(x) \cdot \checkmark$$

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Generalization of SN2:

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## Generalization of SN2:

(1) If 
$$V = U + i$$
 then

 $F_V(x) = x^i F_U(x).$ 

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## Generalization of SN2:

• (1) If 
$$V = U + i$$
 then

 $F_V(x) = x^i F_U(x).$ 

• (2) If V = U - i then

$$F_V(x) = x^{-i} \left( F_U(x) - U_0 - U_1 x - \ldots - U_{i-1} x^{i-1} \right)$$

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## Generalization of SN2:

• (1) If 
$$V = U + i$$
 then

 $F_V(x) = x^i F_U(x).$ 

• (2) If V = U - i then

$$F_V(x) = x^{-i} \left( F_U(x) - U_0 - U_1 x - \dots - U_{i-1} x^{i-1} \right)$$

$$=x^{-i}\sum_{k=i}^{\infty}U_kx^k$$

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 Goal: figure out forms of the component generating functions, *F<sub>π</sub>* and *F<sub>ρ</sub>*.

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- Goal: figure out forms of the component generating functions, *F<sub>π</sub>* and *F<sub>ρ</sub>*.
- $\pi_n$  = probability that a random node belongs to a finite component of size *n*

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- Goal: figure out forms of the component generating functions, *F<sub>π</sub>* and *F<sub>ρ</sub>*.
- $\pi_n$  = probability that a random node belongs to a finite component of size *n*

 $= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$ 

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Therefore:

$$F_{\pi}(x) = \underbrace{F_{P}(F_{
ho}(x))}_{SN1}$$

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 $= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$ 

Therefore:

$$F_{\pi}(x) = \underbrace{x}_{SN2} \underbrace{F_{P}(F_{\rho}(x))}_{SN1}$$

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Therefore: 
$$F_{\pi}(x) = \underbrace{x}_{SN2} \underbrace{F_{P}(F_{\rho}(x))}_{SN1}$$

Extra factor of x accounts for random node itself.

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*ρ<sub>n</sub>* = probability that a random link leads to a finite subcomponent of size *n*.

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- *ρ<sub>n</sub>* = probability that a random link leads to a finite subcomponent of size *n*.
- Invoke one step of recursion: ρ<sub>n</sub> = probability that a random node arrived along a random edge is part of a finite subcomponent of size n.

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- Invoke one step of recursion: ρ<sub>n</sub> = probability that a random node arrived along a random edge is part of a finite subcomponent of size n.

 $= \sum_{k=0}^{\infty} R_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$ 

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Therefore: 
$$F_{\rho}(x) = \underbrace{x}_{SN2} \underbrace{F_R(F_{\rho}(x))}_{SN1}$$

 Again, extra factor of x accounts for random node itself.

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References

We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$  and  $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ 

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$  and  $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ 

► Taking stock: We know  $F_P(x)$  and  $F_R(x) = F'_P(x)/F'_P(1)$ .

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$  and  $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ 

- ► Taking stock: We know  $F_P(x)$  and  $F_R(x) = F'_P(x)/F'_P(1)$ .
- We first untangle the second equation to find F<sub>ρ</sub>

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$  and  $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ 

- ► Taking stock: We know  $F_P(x)$  and  $F_R(x) = F'_P(x)/F'_P(1)$ .
- We first untangle the second equation to find F<sub>ρ</sub>
- We can do this because it only involves  $F_{\rho}$  and  $F_{R}$ .

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$  and  $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ 

- ► Taking stock: We know  $F_P(x)$  and  $F_R(x) = F'_P(x)/F'_P(1)$ .
- We first untangle the second equation to find F<sub>ρ</sub>
- We can do this because it only involves  $F_{\rho}$  and  $F_{R}$ .
- The first equation then immediately gives us F<sub>π</sub> in terms of F<sub>ρ</sub> and F<sub>R</sub>.

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Remembering vaguely what we are doing:

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Remembering vaguely what we are doing:
 Finding *F<sub>P</sub>* to obtain the size of the largest component *S*<sub>1</sub> = 1 - *F*<sub>π</sub>(1).

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Remembering vaguely what we are doing:
 Finding *F<sub>P</sub>* to obtain the size of the largest component *S*<sub>1</sub> = 1 - *F*<sub>π</sub>(1).

Set x = 1 in our two equations:

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Remembering vaguely what we are doing:
 Finding *F<sub>P</sub>* to obtain the size of the largest component *S*<sub>1</sub> = 1 - *F*<sub>π</sub>(1).

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$  and  $F_{\rho}(1) = F_{R}(F_{\rho}(1))$ 

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Remembering vaguely what we are doing:
 Finding *F<sub>P</sub>* to obtain the size of the largest component *S*<sub>1</sub> = 1 - *F*<sub>π</sub>(1).

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$  and  $F_{\rho}(1) = F_{R}(F_{\rho}(1))$ 

Solve second equation numerically for  $F_{\rho}(1)$ .

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Remembering vaguely what we are doing:
 Finding *F<sub>P</sub>* to obtain the size of the largest component *S*<sub>1</sub> = 1 - *F*<sub>π</sub>(1).

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$  and  $F_{\rho}(1) = F_{R}(F_{\rho}(1))$ 

- Solve second equation numerically for  $F_{\rho}(1)$ .
- Plug  $F_{\rho}(1)$  into first equation to obtain  $F_{\pi}(1)$ .

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### Example: Standard random graphs.

• We can show 
$$F_P(x) = e^{-\langle k \rangle (1-x)}$$

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### Example: Standard random graphs.

• We can show 
$$F_P(x) = e^{-\langle k \rangle (1-x)}$$

$$\therefore F_R(x) = F'_P(x)/F'_P(1)$$

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### Example: Standard random graphs.

• We can show 
$$F_P(x) = e^{-\langle k \rangle (1-x)}$$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

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$$= e^{-\langle k \rangle (1-x)}$$

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$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$
 ...aha!

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 ...aha!

RHS's of our two equations are the same.

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P

### Example: Standard random graphs.

• We can show 
$$F_P(x) = e^{-\langle k \rangle (1-x)}$$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$=e^{-\langle k \rangle(1-x)}=F_P(x)$$
 ...aha!

RHS's of our two equations are the same.

• So  $F_{\pi}(x) = F_{\rho}(x) = xF_{R}(F_{\rho}(x)) = xF_{R}(F_{\pi}(x))$ 

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Sac.

### Example: Standard random graphs.

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$$F_P(x) = e^{-\langle k \rangle (1-x)}$$

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$$=e^{-\langle k\rangle(1-x)}=F_P(x)$$
 ...aha!

- RHS's of our two equations are the same.
- ► So  $F_{\pi}(x) = F_{\rho}(x) = xF_R(F_{\rho}(x)) = xF_R(F_{\pi}(x))$
- Why our dirty (but wrong) trick worked earlier...

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# • We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = xe^{-\langle k \rangle(1-x)}$ .

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• We are down to 
$$F_{\pi}(x) = xF_R(F_{\pi}(x))$$
 and  $F_R(x) = xe^{-\langle k \rangle (1-x)}$ .

$$\therefore$$
  $F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ 

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• We are down to 
$$F_{\pi}(x) = xF_R(F_{\pi}(x))$$
 and  $F_R(x) = xe^{-\langle k \rangle (1-x)}$ .

$$\therefore$$
  $F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ 

We're first after S<sub>1</sub> = 1 − F<sub>π</sub>(1) so set x = 1 and replace F<sub>π</sub>(1) by 1 − S<sub>1</sub>:

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We're first after S<sub>1</sub> = 1 − F<sub>π</sub>(1) so set x = 1 and replace F<sub>π</sub>(1) by 1 − S<sub>1</sub>:

$$1-S_1=e^{-\langle k\rangle S_1}$$

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We're first after S<sub>1</sub> = 1 − F<sub>π</sub>(1) so set x = 1 and replace F<sub>π</sub>(1) by 1 − S<sub>1</sub>:

$$1-S_1=e^{-\langle k\rangle S_1}$$

Just as we found with our dirty trick...

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• We are down to 
$$F_{\pi}(x) = xF_R(F_{\pi}(x))$$
 and  $F_R(x) = xe^{-\langle k \rangle (1-x)}$ .

$$\therefore$$
  $F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ 

• We're first after  $S_1 = 1 - F_{\pi}(1)$  so set x = 1 and replace  $F_{\pi}(1)$  by  $1 - S_1$ :

$$1-S_1=e^{-\langle k\rangle S_1}$$

- Just as we found with our dirty trick...
- Again, have to resort to numerics at this point.

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• Next: find average size of finite components  $\langle n \rangle$ .

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- Next: find average size of finite components  $\langle n \rangle$ .
- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .

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- Next: find average size of finite components  $\langle n \rangle$ .
- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .
- Try to avoid finding  $F_{\pi}(x)$ ...

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- Next: find average size of finite components  $\langle n \rangle$ .
- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .
- Try to avoid finding  $F_{\pi}(x)$ ...
- Starting from  $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$ , we differentiate:

 $F_{\pi}'(x)=F_{\mathcal{P}}\left(F_{
ho}(x)
ight)+xF_{
ho}'(x)F_{\mathcal{P}}'\left(F_{
ho}(x)
ight)$ 

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- Next: find average size of finite components  $\langle n \rangle$ .
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 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\rho}(x)
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ho}(x)
ight)$ 

• While 
$$F_{\rho}(x) = xF_{R}(F_{\rho}(x))$$
 gives

 $F_{
ho}'(x)=F_R(F_{
ho}(x))+xF_{
ho}'(x)F_R'(F_{
ho}(x))$ 

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- Next: find average size of finite components  $\langle n \rangle$ .
- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .
- Try to avoid finding  $F_{\pi}(x)$ ...
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ho}(x)F'_{\mathcal{P}}\left(F_{
ho}(x)
ight)$ 

• While 
$$F_{\rho}(x) = xF_R(F_{\rho}(x))$$
 gives

$$F'_{\rho}(x) = F_{R}(F_{\rho}(x)) + xF'_{\rho}(x)F'_{R}(F_{\rho}(x))$$

Now set x = 1 in both equations.

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- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .
- Try to avoid finding  $F_{\pi}(x)$ ...
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 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\rho}(x)
ight) + xF'_{
ho}(x)F'_{\mathcal{P}}\left(F_{
ho}(x)
ight)$ 

• While 
$$F_{\rho}(x) = xF_{R}(F_{\rho}(x))$$
 gives

 $F'_{\rho}(x) = F_{R}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{R}\left(F_{\rho}(x)\right)$ 

- Now set x = 1 in both equations.
- We solve the second equation for F'<sub>ρ</sub>(1) (we must already have F<sub>ρ</sub>(1)).

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- Next: find average size of finite components  $\langle n \rangle$ .
- Using standard G.F. result:  $\langle n \rangle = F'_{\pi}(1)$ .
- Try to avoid finding  $F_{\pi}(x)$ ...
- Starting from  $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$ , we differentiate:

 $F_{\pi}'(x) = F_{\mathcal{P}}\left(F_{\rho}(x)
ight) + xF_{\rho}'(x)F_{\mathcal{P}}'\left(F_{\rho}(x)
ight)$ 

• While 
$$F_{\rho}(x) = xF_{R}(F_{\rho}(x))$$
 gives

 $F'_{\rho}(x) = F_{R}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{R}\left(F_{\rho}(x)\right)$ 

- Now set x = 1 in both equations.
- We solve the second equation for F'<sub>ρ</sub>(1) (we must already have F<sub>ρ</sub>(1)).
- Plug  $F'_{\rho}(1)$  and  $F_{\rho}(1)$  into first equation to find  $F'_{\pi}(1)$ .

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Example: Standard random graphs.

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Example: Standard random graphs.

• Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .

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Average Component Size

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Example: Standard random graphs.

- Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .
- Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{\mathcal{P}}\left(F_{\pi}(x)\right)$$

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Example: Standard random graphs.

• Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .

Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{P}(F_{\pi}(x)) + xF'_{\pi}(x)F'_{P}(F_{\pi}(x))$$
  
Rearrange:  $F'_{\pi}(x) = \frac{F_{P}(F_{\pi}(x))}{1 - xF'_{P}(F_{\pi}(x))}$ 

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Example: Standard random graphs.

- Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .
- Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{\mathcal{P}}\left(F_{\pi}(x)\right)$$

Rearrange: 
$$F'_{\pi}(x) = \frac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

Simplify denominator using  $F'_{\pi}(x) = \langle k \rangle F_{\pi}(x)$ 

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Example: Standard random graphs.

- Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .
- Two differentiated equations reduce to only one:

$$F_{\pi}'(x) = F_{\mathcal{P}}\left(F_{\pi}(x)
ight) + xF_{\pi}'(x)F_{\mathcal{P}}'\left(F_{\pi}(x)
ight)$$

Rearrange: 
$$F'_{\pi}(x) = \frac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

- Simplify denominator using  $F'_{\pi}(x) = \langle k \rangle F_{\pi}(x)$
- Replace  $F_P(F_\pi(x))$  using  $F_\pi(x) = xF_P(F_\pi(x))$ .

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- Use fact that  $F_P = F_R$  and  $F_{\pi} = F_{\rho}$ .
- Two differentiated equations reduce to only one:

$$F_{\pi}'(x) = F_{\mathcal{P}}\left(F_{\pi}(x)
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Rearrange: 
$$F'_{\pi}(x) = \frac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

- Simplify denominator using  $F'_{\pi}(x) = \langle k \rangle F_{\pi}(x)$
- Replace  $F_P(F_\pi(x))$  using  $F_\pi(x) = xF_P(F_\pi(x))$ .
- Set x = 1 and replace  $F_{\pi}(1)$  with  $1 S_1$ .

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ight)$$

Rearrange: 
$$F'_{\pi}(x) = \frac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

- Simplify denominator using  $F'_{\pi}(x) = \langle k \rangle F_{\pi}(x)$
- Replace  $F_P(F_\pi(x))$  using  $F_\pi(x) = xF_P(F_\pi(x))$ .
- Set x = 1 and replace  $F_{\pi}(1)$  with  $1 S_1$ .

End result: 
$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

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Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

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Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k 
angle(1-S_1)}$$

Recall that (k) = 1 is the critical value of average degree for standard random networks.

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Our result for standard random networks:

$$\langle n 
angle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k 
angle(1-S_1)}$$

- Recall that (k) = 1 is the critical value of average degree for standard random networks.
- Look at what happens when we increase (k) to 1 from below.

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Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

- Recall that (k) = 1 is the critical value of average degree for standard random networks.
- Look at what happens when we increase (k) to 1 from below.
- We have  $S_1 = 0$  for all  $\langle k \rangle < 1$

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Our result for standard random networks:

$$\langle n 
angle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k 
angle(1-S_1)}$$

- Recall that (k) = 1 is the critical value of average degree for standard random networks.
- Look at what happens when we increase (k) to 1 from below.
- We have  $S_1 = 0$  for all  $\langle k \rangle < 1$  so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

• This blows up as  $\langle k \rangle \rightarrow 1$ .

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Our result for standard random networks:

$$\langle n 
angle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k 
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- Recall that (k) = 1 is the critical value of average degree for standard random networks.
- Look at what happens when we increase (k) to 1 from below.
- We have  $S_1 = 0$  for all  $\langle k \rangle < 1$  so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as  $\langle k \rangle \rightarrow 1$ .
- ► Reason: we have a power law distribution of component sizes at ⟨k⟩ = 1.

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Our result for standard random networks:

$$\langle n 
angle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k 
angle(1-S_1)}$$

- Recall that (k) = 1 is the critical value of average degree for standard random networks.
- Look at what happens when we increase (k) to 1 from below.
- We have  $S_1 = 0$  for all  $\langle k \rangle < 1$  so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as  $\langle k \rangle \rightarrow 1$ .
- Reason: we have a power law distribution of component sizes at (k) = 1.
- Typical critical point behavior....

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### • Limits of $\langle k \rangle = 0$ and $\infty$ make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

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• Limits of  $\langle k \rangle = 0$  and  $\infty$  make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 = 0$ , and  $\langle n \rangle \rightarrow 1$ .

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• Limits of  $\langle k \rangle = 0$  and  $\infty$  make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 = 0$ , and  $\langle n \rangle \rightarrow 1$ .

All nodes are isolated.

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• Limits of  $\langle k \rangle = 0$  and  $\infty$  make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 = 0$ , and  $\langle n \rangle \rightarrow 1$ .

All nodes are isolated.

• As 
$$\langle k \rangle \to \infty$$
,  $S_1 \to 1$  and  $\langle n \rangle \to 0$ .

#### Random Networks

# Basics

Average Component Size

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• Limits of  $\langle k \rangle = 0$  and  $\infty$  make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

• As 
$$\langle k \rangle \rightarrow 0$$
,  $S_1 = 0$ , and  $\langle n \rangle \rightarrow 1$ .

- All nodes are isolated.
- As  $\langle k \rangle \to \infty$ ,  $S_1 \to 1$  and  $\langle n \rangle \to 0$ .
- No nodes are outside of the giant component.

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