

Overview of Complex Networks

Complex Networks, Course 295A, Spring, 2008

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- ▶ Office hours: Tuesday 10:45 am–12:30 pm
- ▶ Course outline
- ▶ Projects

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Complex: (Latin = with + fold/weave (com + plex))

Adjective

- ▶ Made up of multiple parts; intricate or detailed.
- ▶ Not simple or straightforward.



Complex System—Basic ingredients:

- ▶ Relationships are nonlinear
- ▶ Relationships contain feedback loops
- ▶ Complex systems are open (out of equilibrium)
- ▶ Memory
- ▶ Modular (nested)/multiscale structure
- ▶ Opaque boundaries
- ▶ May produce emergent phenomena



Basic definitions



Network: (net + work, 1500's)

Noun:

1. Any interconnected group or system
2. Multiple computers and other devices connected together to share information

Verb:

1. To interact socially for the purpose of getting connections or personal advancement
2. To connect two or more computers or other computerized devices

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- ▶ Many complex systems can be regarded as complex networks of physical or abstract interactions
- ▶ Opens door to mathematical and numerical analysis
- ▶ Dominant approach of last decade of a theoretical-physics/stat-mechish flavor.

Nodes = A collection of entities which have properties that are somehow related to each other

- ▶ e.g., people, forks in rivers, proteins, webpages, organisms,...

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Links = Connections between nodes

- ▶ **links**
 - ▶ may be real and fixed (rivers),
 - ▶ real and dynamic (airline routes),
 - ▶ abstract with physical impact (hyperlinks),
 - ▶ or purely abstract (semantic connections between concepts).
- ▶ **Links** may be directed or undirected.
- ▶ **Links** may be binary or weighted.

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Node degree = Number of links per node

- ▶ Notation: Node i 's degree = k_i .
- ▶ $k_i = 0, 1, 2, \dots$
- ▶ Notation: the average degree of a network = $\langle k \rangle$
(and sometimes as z)

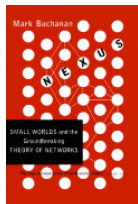
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Adjacency matrix:

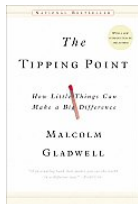
- ▶ We represent a graph or network by a matrix A with link weight a_{ij} for nodes i and j in entry (i, j) .
- ▶ e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan



The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell

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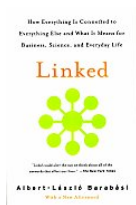
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Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási



Six Degrees: The Science of a Connected Age—Duncan Watts

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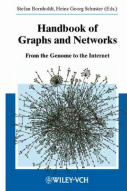
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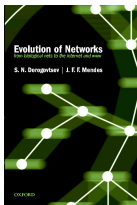
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Handbook of Graphs and Networks—editors:
Stefan Bornholdt and H. G. Schuster



Evolution of Networks—S. N. Dorogovtsev
and J. F. F. Mendes.

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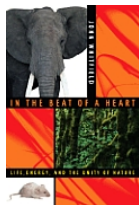
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Social Network Analysis—Stanley Wasserman and Kathleen Faust



In the Beat of a Heart: Life, Energy, and the Unity of Nature—John Whitfield

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Numerous others:

- ▶ **Complex Social Networks**—F. Vega-Redondo
- ▶ **Fractal River Basins: Chance and Self-Organization**—I. Rodríguez-Iturbe and A. Rinaldo
- ▶ **Random Graph Dynamics**—R. Durrette
- ▶ **Scale-Free Networks**—Guido Caldarelli
- ▶ **Evolution and Structure of the Internet: A Statistical Physics Approach**—Romu Pastor-Satorras and Alessandro Vespignani
- ▶ **Complex Graphs and Networks**—Fan Chung

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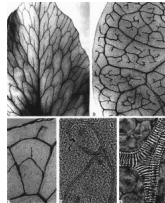
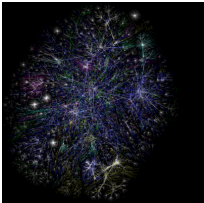
What passes for a complex network?

- ▶ Complex networks are **large** (in node number)
- ▶ Complex networks are **sparse** (low edge to node ratio)
- ▶ Complex networks are usually **dynamic** and **evolving**
- ▶ Complex networks can be social, economic, natural, informational, abstract, ...

Examples

Physical networks

- ▶ River networks
- ▶ Neural networks
- ▶ Trees and leaves
- ▶ Blood networks
- ▶ The Internet
- ▶ Road networks
- ▶ Power grids



- ▶ **Distribution** (branching) versus **redistribution** (cyclical)

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Examples

Interaction networks

- ▶ The Blogosphere
- ▶ Biochemical networks
- ▶ Gene-protein networks
- ▶ Food webs: who eats whom
- ▶ The World Wide Web (?)
- ▶ Airline networks
- ▶ Call networks (AT&T)
- ▶ The Media



datamining.typepad.com (田)

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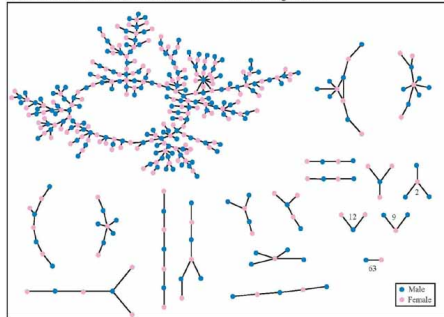
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Examples

Interaction networks: social networks

- ▶ Snogging
- ▶ Friendships
- ▶ Acquaintances
- ▶ Boards and directors
- ▶ Organizations
- ▶ myspace.com (☒),
- ▶ facebook.com (☒)

The Structure of Romantic and Sexual Relations at "Jefferson High School"



Each circle represents a student and lines connecting students represent romantic relations occurring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

(Bearman *et al.*, 2004)

- ▶ 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).

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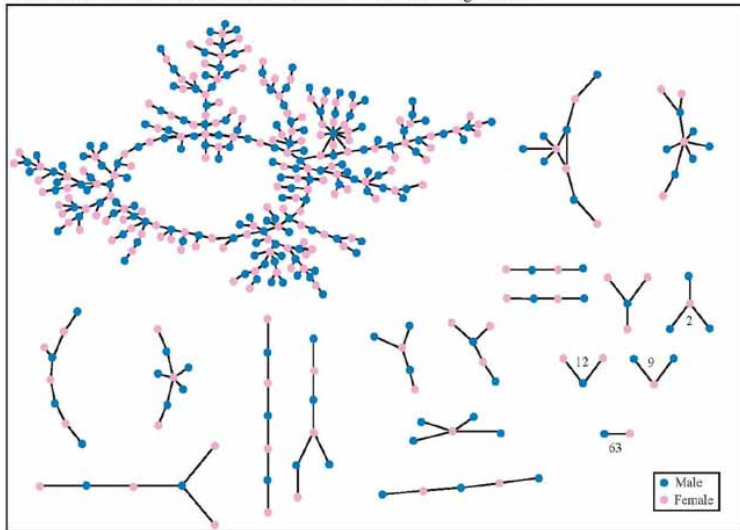
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The Structure of Romantic and Sexual Relations at "Jefferson High School"



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Relational networks

- ▶ Consumer purchases
(Wal-Mart: ≈ 1 petabyte = 10^{15} bytes)
- ▶ Thesauri: Networks of words generated by meanings
- ▶ Knowledge/Databases/Ideas
- ▶ Metadata—Tagging:
del.icio.us (田) <http://del.icio.us>, del.icio.us, [flickr](http://del.icio.us) (田)

common tags cloud | [list](#)

community daily dictionary education **encyclopedia**
 english free imported info information internet knowledge
 learning news **reference** research resource
 resources search tools useful web web2.0 **wiki**
wikipedia

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A notable features of large-scale networks:

- ▶ Graphical renderings of complex networks are often just a big mess.

Some key aspects of real complex networks:

- ▶ degree distribution
 - ▶ assortativity
 - ▶ homophily
 - ▶ clustering
 - ▶ motifs
 - ▶ modularity
 - ▶ concurrency
 - ▶ hierarchical scaling
 - ▶ network distances
 - ▶ centrality
 - ▶ efficiency
 - ▶ robustness
- ▶ + Coevolution of network structure and processes on networks.

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1. degree distribution P_k

- ▶ P_k is the probability that a randomly selected node has degree k
- ▶ k = node degree = number of connections
- ▶ **ex 1:** Erdős-Rényi random networks:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- ▶ Distribution is Poisson

1. degree distribution P_k

- ▶ **ex 2: “Scale-free” networks:** $P_k \propto k^{-\gamma} \Rightarrow$ ‘hubs’
- ▶ link cost controls skew
- ▶ hubs may facilitate or impede contagion

Note:

- ▶ Erdős-Rényi random networks are a *mathematical construct*.
- ▶ ‘Scale-free’ networks are **growing networks** that form according to a **plausible mechanism**.
- ▶ Randomness is out there, just not to the degree of a completely random network.

2. assortativity/3. homophily:

- ▶ Social networks: Homophily = birds of a feather
- ▶ e.g., degree is standard property for sorting: measure degree-degree correlations.
- ▶ **Assortative** network: ^[7] similar degree nodes connecting to each other.
*Often **social**: company directors, coauthors, actors.*
- ▶ **Disassortative** network: high degree nodes connecting to low degree nodes.
*Often **techological** or **biological**: Internet, WWW, protein interactions, neural networks, food webs.*

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4. clustering:

- ▶ Your friends tend to know each other.
- ▶ Two measures:
 1. Watts & Strogatz ^[12]

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

2. Newman ^[8]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

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First clustering measure:

- ▶ C_1 is the **average fraction of pairs of neighbors who are connected**.
- ▶ Fraction of pairs of neighbors who are connected is

$$\frac{\sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

where k_i is node i 's degree, and \mathcal{N}_i is the set of i 's neighbors.

- ▶ Averaging over all nodes, we have

$$C_1 = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} = \left\langle \frac{\sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

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- ▶ For sparse networks, C_1 tends to discount highly connected nodes.
- ▶ C_2 is a useful variant
- ▶ In general, $C_1 \neq C_2$.

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Properties

Triples and triangles

- ▶ Nodes i_1 , i_2 , and i_3 form a **triple** around i_1 if i_1 is connected to i_2 and i_3 .
- ▶ Nodes i_1 , i_2 , and i_3 form a **triangle** if each pair of nodes is connected
- ▶ The definition

$$C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$$

measures the fraction of **closed triples**

- ▶ Social Network Analysis (SNA): fraction of **transitive triples**.
- ▶ The '3' appears because for each triangle, we have 3 closed triples.

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Wait, there's more!

- ▶ Newman^[8]:

$$C_3 = \frac{6 \times \#\text{triangles}}{\#\text{ordered pairs}}$$

- ▶ Now count each triple twice
- ▶ Same as C_2 but interpretation is different
- ▶ Probability that a friend of i 's friend is also i 's friend.

Quick summary:

- ▶ C_1 = probability that two friends of a randomly chosen node are connected
- ▶ C_2 = probability that two nodes are connected given they have a friend in common.
- ▶ $C_3 (= C_2)$ = probability that a node's friend of a friend is also a friend of that node.

- ▶ For sparse networks, C_1 tends to discount highly connected nodes.
- ▶ While C_1 is a measure of clustering, it doesn't quite have as simple interpretation as C_2 .
- ▶ Some variability in which measure is used in the literature.
- ▶ Not always clear which one is being used...

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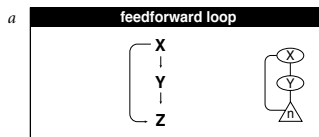
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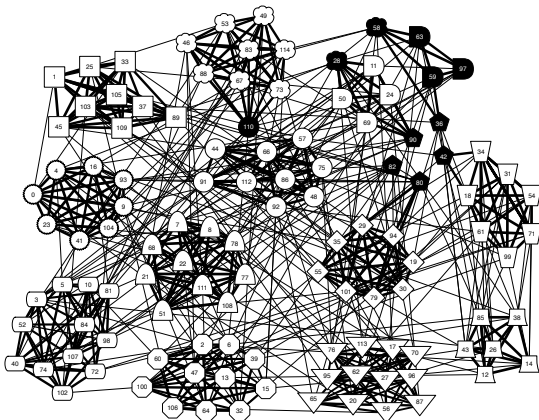
5. motifs:

- ▶ small, recurring functional subnetworks
- ▶ e.g., Feed Forward Loop:



Shen-Orr, Uri Alon, *et al.* [9]

6. modularity—community detection:



Clauset *et al.*, 2006^[4]: NCAA football

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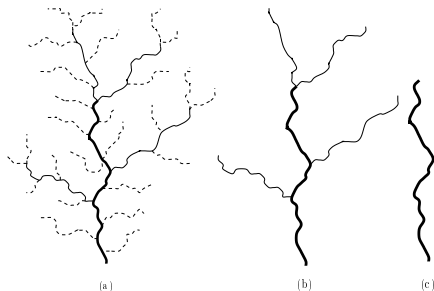
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7. concurrency:

- ▶ transmission of a contagious element only occurs during contact
- ▶ rather obvious but easily missed in a simple model
- ▶ dynamic property—static networks are not enough
- ▶ knowledge of previous contacts crucial
- ▶ beware cumulated network data
- ▶ Kretzschmar and Morris, 1996 ^[6]

8. Horton-Strahler ratios:

- ▶ Metrics for branching networks:
 - ▶ Method for ordering streams hierarchically
 - ▶ Number: $R_n = N_\omega / N_{\omega+1}$
 - ▶ Segment length: $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
 - ▶ Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$

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9. network distances:

(a) shortest path length d_{ij} :

- ▶ Fewest number of steps between nodes i and j .
- ▶ (Also called the chemical distance between i and j .)

(b) average path length $\langle d_{ij} \rangle$:

- ▶ Average shortest path length in whole network.
- ▶ Good algorithms exist for calculation.
- ▶ Weighted links can be accommodated.

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9. network distances:

- ▶ **network diameter d_{\max} :**
Maximum shortest path length between any two nodes.
- ▶ **closeness $d_{cl} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$:**
Average 'distance' between any two nodes.

10. centrality:

- ▶ Many such measures of a node's 'importance.'
- ▶ **ex 1:** Degree centrality: k_i .
- ▶ **ex 2:** Node i 's betweenness
= fraction of shortest paths that pass through i .
- ▶ **ex 3:** Recursive centrality: Hubs and Authorities
(Kleinberg^[5])

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Some important models:

1. generalized random networks
2. scale-free networks
3. small-world networks
4. statistical generative models (p^*)
5. generalized affiliation networks

1. generalized random networks:

- ▶ Arbitrary degree distribution P_k .
- ▶ Wire nodes together randomly.
- ▶ Create ensemble to test deviations from randomness.

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2. 'scale-free networks':

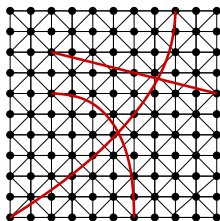
- ▶ Introduced by Barabasi and Albert^[1]
- ▶ Generative model
- ▶ Preferential attachment model with growth:
- ▶ $P[\text{attachment to node } i] \propto k_i^\alpha$.
- ▶ Produces $P_k \sim k^{-\gamma}$ when $\alpha = 1$.
- ▶ Trickiness: other models generate skewed degree distributions.

3. small-world networks

- ▶ Introduced by Watts and Strogatz ^[12]

Two scales:

- ▶ **local regularity** (an individual's friends know each other)
 - ▶ **global randomness** (shortcuts).
-
- ▶ Shortcuts allow disease to jump
 - ▶ Number of infectives increases exponentially in time



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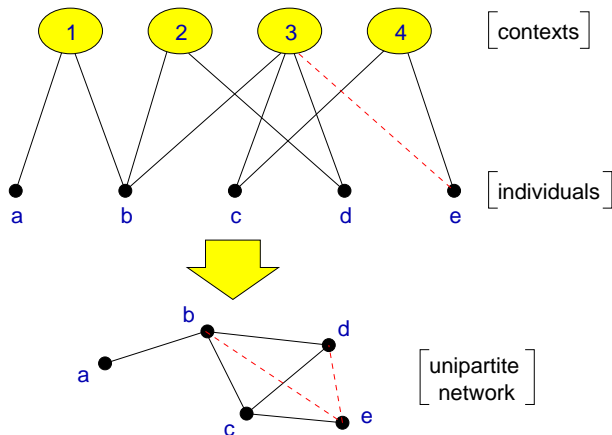
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5. generalized affiliation networks



Bipartite affiliation networks: boards and directors, movies and actors.

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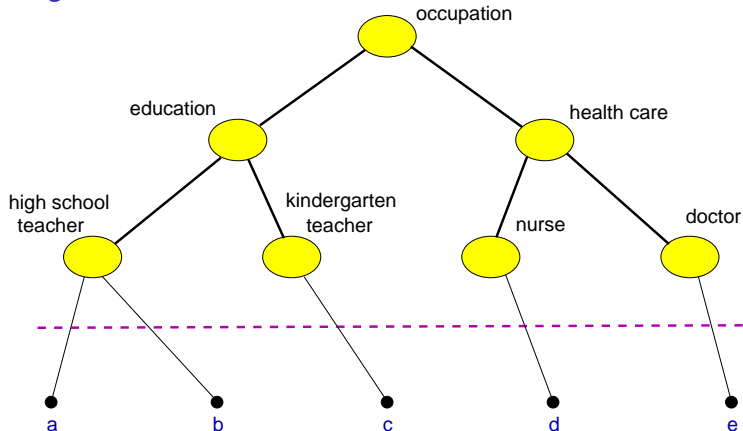
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5. generalized affiliation networks



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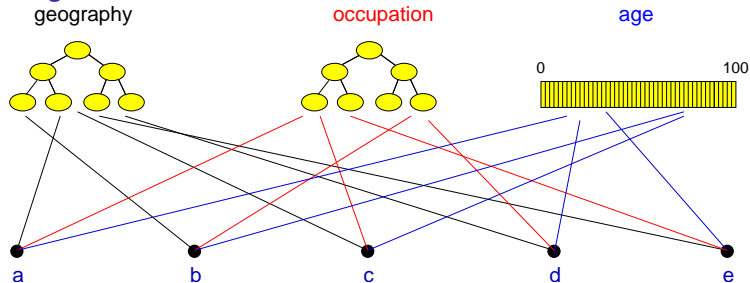
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- Blau & Schwartz^[2], Simmel^[10], Breiger^[3], Watts et al.^[11]

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“Collective dynamics of ‘small-world’ networks”^[12]

- ▶ Watts and Strogatz
Nature, 1998
- ▶ ≈ 2400 citations (as of Jan 14, 2008)

“Emergence of scaling in random networks”^[1]

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Science, 1999
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