# Branching Networks Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics University of Vermont



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0.License.

#### Introduction

River Networks

Allometr

Laws Stream Orderin

lorton's Laws

Horton ⇔ Toki Reducing Horto Scaling relation:

Fluctuations Models





### Introduction

#### Introduction

River Networks

Frame 2/121





### Introduction

#### River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

**Fluctuations** 

Models

#### Introduction

River Networks

Allometry Laws

.aws Stream Ord

Horton's Laws

Horton ⇔ Tok Reducing Horto

> caling relations luctuations





### Introduction

#### River Networks

**Definitions** 

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

**Fluctuations** 

Models

### References

#### Introduction

River Networks

References

Frame 2/121





# Branching networks are useful things:

Fundamental to material supply and collection

#### Introduction

River Networks

Allometry

aws stream Ord

Horton's Laws Tokunaga's Law

Reducing Horton scaling relations fluctuations

References

Frame 3/121



# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.

#### Introduction

River Networks

Allometry Laws

ws ream Orderi orton's Laws

orton's Laws kunaga's Law orton ⇔ Tokuna

caling relations uctuations





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.

#### Introduction

River Networks

Definitions

Allometry

llometry

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunag

educing Horton caling relations luctuations lodels





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

### **Examples:**

# Introduction River Networks

Definitions
Allometry
\_aws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Models





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

# **Examples:**

River networks (our focus)

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton \( \Delta \) Tokunaga
Reducing Horton





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

### **Examples:**

- River networks (our focus)
- Cardiovascular networks

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Horton 
Tokunaga's Law
Horton 
Tokunaga's Reducing Horton





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

### **Examples:**

- River networks (our focus)
- Cardiovascular networks
- Plants

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

### **Examples:**

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton





# Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

# **Examples:**

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

#### Introduction

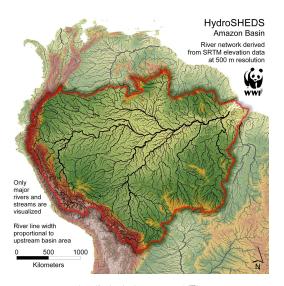
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga Reducing Horton

References

Frame 3/121



# Branching networks are everywhere...



 $\underline{\text{http://hydrosheds.cr.usgs.gov/}} \; (\boxplus)$ 

#### Introduction

River Networks

Allomet

Stream Orderin Horton's Laws

Tokunaga's Law Horton ⇔ Toku

Scaling relations

References

Frame 4/121



# Branching networks are everywhere...



http://en.wikipedia.org/wiki/Image:Applebox.JPG (⊞)

#### Introduction

River Networks

Allomet

Laws Stream Ordering

Horton's Laws

Reducing Horto

. .

References

Frame 5/121



#### Introduction

#### River Networks

#### **Definitions**

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

References

#### Introduction

River Networks

Definitions

Allometry Laws

\_aws Stream Ord

Horton's Laws Tokunaga's Law

Reducing Horto

luctuations Models





#### **Definitions**

▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.

#### Introduction

River Networks

Allometry Laws

aws ream Ordering orton's Laws

Reducing Horton scaling relations fluctuations

References

Frame 7/121



#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.

#### Introduction

River Networks

Allometry Laws

aws Stream Orderi Horton's Laws

Horton ⇔ Tokur Reducing Horton Scaling relations Fluctuations

References

Frame 7/121



#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.

#### Introduction

River Networks

Definitions Allometry Laws

.aws Stream Ord

Tokunaga's Law Horton ⇔ Toku

educing Hortor caling relations uctuations odels





#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.

Introduction

River Networks Definitions

Allometry Laws

Laws Stream Orc

Tokunaga's Law Horton ⇔ Tok Reducing Horto

caling relation luctuations lodels

References

Frame 7/121



#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.

Introduction

River Networks

Definitions

Allometry Laws

aws stream Orde

Horton's Laws Tokunaga's Law Horton ⇔ Tok

> Reducing Horto scaling relation fluctuations Models

References

Frame 7/121



#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.

Introduction

River Networks

Definitions Allometry

Laws Stream Ord

Horton's Laws Tokunaga's Law Horton ⇔ Tok

Reducing Horto scaling relation fluctuations Models





#### **Definitions**

- ▶ Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...

#### Introduction

River Networks
Definitions

Allometry

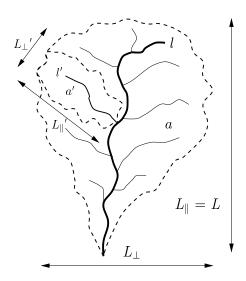
Horton's Laws
Tokunaga's Law
Horton ⇔ Toku

Reducing Horto Scaling relations Fluctuations Models

References

Frame 7/121





#### Introduction

River Networks

Definitions

Allometry

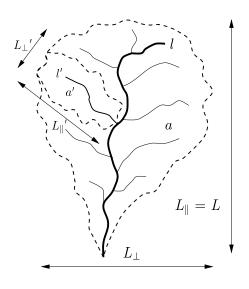
Stream Ordering

Tokunaga's Law

Scaling relati Fluctuations

References





a = drainage basin area Introduction

River Networks

Definitions

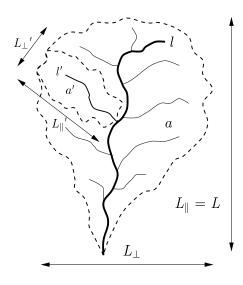
Allometry

Stream Ordering

Horton ⇔ Toku Reducing Horton Scaling relations

References





- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)

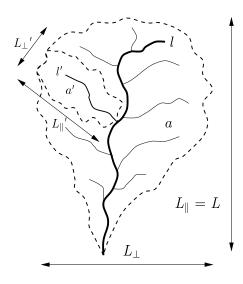
#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations

References





- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)
- ► L = L<sub>||</sub> = longitudinal length of basin

#### Introduction

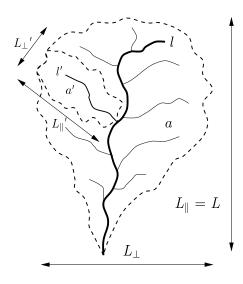
River Networks Definitions

Laws Stream Ordering

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations

References





- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)
- L = L<sub>||</sub> = longitudinal length of basin
- ►  $L = L_{\perp}$  = width of basin

#### Introduction

River Networks Definitions

\_aws Stream Ordering

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations

References



Introduction

### **River Networks**

Definitions

### Allometry

Laws

Stream Ordering

Tokungga'a Laws

Tokunaga's Law Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

**Fluctuations** 

Models

References

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

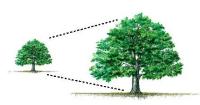
References

Frame 9/121



# **Allometry**

Isometry: dimensions scale linearly with each other.



#### Introduction

River Networks

Allometry

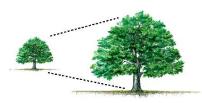
Frame 10/121

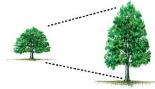




# **Allometry**

Isometry: dimensions scale linearly with each other.





Allometry: dimensions scale nonlinearly.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton 

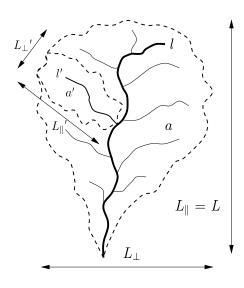
Tokun Horton 

Tokun Reducing Horton Scaling relations Fluctuations

References

Frame 10/121





**Allometric** relationships:

#### Introduction

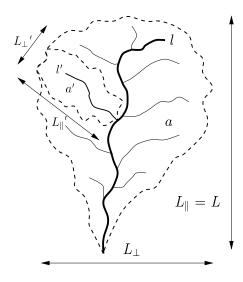
River Networks

Allometry









# Allometric relationships:

•

 $\ell \propto a^h$ 

#### Introduction

River Networks

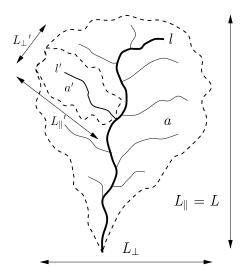
Allometry

Stream Ordering

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

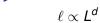






# Allometric relationships:





#### Introduction

River Networks

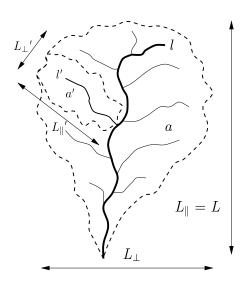
Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toke Reducing Horton Scaling relations Fluctuations







# Allometric relationships:

$$\ell \propto a^h$$

•

$$\ell \propto L^d$$

▶ Combine above:

$$a \propto L^{d/h} \equiv L^D$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton 
Tok
Reducing Horto
Scaling relation
Fluctuations





► Hack's law (1957) [6]:

$$\ell \propto a^h$$

reportedly 0.5 < h < 0.7

Allometry





► Hack's law (1957) [6]:

$$\ell \propto a^h$$

reportedly 0.5 < h < 0.7

Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^d$$

reportedly 1.0 < d < 1.1

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations





► Hack's law (1957) [6]:

$$\ell \propto a^h$$

reportedly 0.5 < h < 0.7

Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^d$$

reportedly 1.0 < d < 1.1

Basin allometry:

$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

 $D < 2 \rightarrow$  basins elongate.

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





## **Outline**

Introduction

### River Networks

Definitions Allometry

### Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models

References

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations







## There are a few more 'laws': [2]

Relation: Name or description:

### Introduction

River Networks

Allometry

Laws

tream Ordering

Tokunaga's Law

Scaling relation Fluctuations

References



## There are a few more 'laws': [2]

Relation: Name or description:

 $T_k = T_1(R_T)^k$  Tokunaga's law

#### Introduction

River Networks

Laws





$$T_k = T_1(R_T)^k$$
 Tokunaga's law  $\ell \sim L^d$  self-affinity of single channels

### Introduction

River Networks

Definitions

Allometry Laws

ws ream Orde

Horton's Laws
Tokunaga's Law

Reducing Horto Scaling relations Fluctuations

References



 $T_k = T_1(R_T)^k$  Tokunaga's law  $\ell \sim L^d$  self-affinity of single channels  $n_\omega/n_{\omega+1} = R_n$  Horton's law of stream numbers

#### Introduction

River Networks

Definitions

Allometry Laws

aws tream Ordering orton's Laws

Horton ⇔ Tok Reducing Horto Scaling relation

References



 $T_k = T_1(R_T)^k$  Tokunaga's law  $\ell \sim L^d$  self-affinity of single channels  $n_\omega/n_{\omega+1} = R_n$  Horton's law of stream numbers  $\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$  Horton's law of main stream lengths

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton
Scaling relations
Fluctuations
Models

References



$$T_k = T_1(R_T)^k$$
 Tokunaga's law  $\ell \sim L^d$  self-affinity of sing  $n_\omega/n_{\omega+1} = R_n$  Horton's law of str  $\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$  Horton's law of matrix  $\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$  Horton's law of ba

self-affinity of single channels Horton's law of stream numbers Horton's law of main stream lengths Horton's law of basin areas

#### Introduction

River Networks Laws

	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_{\omega}/n_{\omega+1}=R_n$	Horton's law of stream numbers
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{\emph{R}}_{\ell}$	Horton's law of main stream lengths
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	Horton's law of basin areas
$ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}$	Horton's law of stream segment lengths

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Horton ⇔ Toku
Reducing Hortor
Scaling relations
Fluctuations

 $T_k = T_1(R_T)^k$  Tokunaga's law  $\ell \sim L^d$  self-affinity of single channels  $n_\omega/n_{\omega+1} = R_n$  Horton's law of stream numbers  $\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$  Horton's law of main stream lengths  $\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$  Horton's law of basin areas  $\bar{s}_{\omega+1}/\bar{s}_\omega = R_s$  Horton's law of stream segment lengths  $L_\perp \sim L^H$  scaling of basin widths

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

 $T_k = T_1(R_T)^k$   $\ell \sim L^d$   $n_{\omega}/n_{\omega+1} = R_n$   $\bar{\ell}_{\omega+1}/\bar{\ell}_{\omega} = R_\ell$   $\bar{a}_{\omega+1}/\bar{a}_{\omega} = R_a$   $\bar{s}_{\omega+1}/\bar{s}_{\omega} = R_s$   $L_{\perp} \sim L^H$   $P(a) \sim a^{-\tau}$ 

Tokunaga's law
self-affinity of single channels
Horton's law of stream numbers
Horton's law of main stream lengths
Horton's law of basin areas
Horton's law of stream segment lengths
scaling of basin widths
probability of basin areas

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

$T_k = T_1(R_T)^k$	Tokunaga's law	
$\ell \sim {\sf L}^{\sf d}$	self-affinity of single channels	
$n_{\omega}/n_{\omega+1}=R_n$	Horton's law of stream numbers	
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{\emph{R}}_{\ell}$	Horton's law of main stream lengths	
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	Horton's law of basin areas	
$ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}$	Horton's law of stream segment lengths	
$L_{\perp} \sim L^{H}$	scaling of basin widths	
$P(a) \sim a^{- au}$	probability of basin areas	
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths	

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

 $T_{k} = T_{1}(R_{T})^{k}$   $\ell \sim L^{d}$   $n_{\omega}/n_{\omega+1} = R_{n}$   $\bar{\ell}_{\omega+1}/\bar{\ell}_{\omega} = R_{\ell}$   $\bar{a}_{\omega+1}/\bar{a}_{\omega} = R_{a}$   $\bar{s}_{\omega+1}/\bar{s}_{\omega} = R_{s}$   $L_{\perp} \sim L^{H}$   $P(a) \sim a^{-\tau}$   $P(\ell) \sim \ell^{-\gamma}$   $\ell \sim a^{h}$ 

 $R_1(R_T)^k$  Tokunaga's law  $\ell \sim L^d$  self-affinity of single channels  $\ell = R_n$  Horton's law of stream numbers  $\ell = R_\ell$  Horton's law of main stream lengths  $\ell = R_a$  Horton's law of basin areas  $\ell = R_a$  Horton's law of stream segment lengths  $\ell = L^H$  scaling of basin widths  $\ell = L^T$  probability of basin areas  $\ell = \ell = L^T$  probability of stream lengths  $\ell = L^T$  Hack's law

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

$T_k = T_1(R_T)^k$	Tokunaga's law	
$\ell \sim {\cal L}^{\sf d}$	self-affinity of single channels	
$n_{\omega}/n_{\omega+1}=R_n$	Horton's law of stream numbers	
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{\emph{R}}_{\ell}$	Horton's law of main stream lengths	
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	Horton's law of basin areas	
$ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}$	Horton's law of stream segment lengths	
${\it L}_{\perp} \sim {\it L}^{\it H}$	scaling of basin widths	
$P(a) \sim a^{- au}$	probability of basin areas	
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths	
$\ell \sim \pmb{a^h}$	Hack's law	
$a\sim L^D$	scaling of basin areas	
$\Lambda \sim \pmb{a}^eta$	Langbein's law	

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References



## There are a few more 'laws': [2]

Relation: Name	e or description:
----------------	-------------------

$T_k = T_1(R_T)^k$	Tokunaga's law	
$\ell \sim {\sf L}^{\sf d}$	self-affinity of single channels	
$n_{\omega}/n_{\omega+1}=R_n$	Horton's law of stream numbers	
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{ extit{R}}_{\ell}$	Horton's law of main stream lengths	
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	Horton's law of basin areas	
$ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}$	Horton's law of stream segment lengths	
$L_{\perp} \sim L^{H}$	scaling of basin widths	
$P(a) \sim a^{- au}$	probability of basin areas	
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths	
$\ell \sim \pmb{a^h}$	Hack's law	
$a\sim L^D$	scaling of basin areas	
$\Lambda \sim  extbf{ extit{a}}^eta$	Langbein's law	
$\lambda \sim \mathcal{L}^{arphi}$	variation of Langbein's law	

### Introduction

River Networks

Definitions

Allometry Laws

Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

References



## Reported parameter values: [2]

Parameter:	Real networks:
$R_n$	3.0-5.0
$R_a$	3.0-6.0
$R_\ell = R_T$	1.5–3.0
$T_1$	1.0-1.5
d	$1.1 \pm 0.01$
D	$1.8 \pm 0.1$
h	0.50-0.70
au	$\textbf{1.43} \pm \textbf{0.05}$
$\gamma$	$1.8 \pm 0.1$
Н	0.75-0.80
$\beta$	0.50-0.70
arphi	$\textbf{1.05} \pm \textbf{0.05}$

#### Introduction

River Networks

Definitions

Allometry Laws

stream Ordering forton's Laws

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations





### Order of business:

### Introduction

River Networks

Allometr

Laws

aws

Stream Orde Horton's Law

Tokunaga's Law Horton ⇔ Tokur

Scaling relation





### Order of business:

1. Find out how these relationships are connected.

#### Introduction

River Networks

Definitions

Allometry Laws

stream Ordering

Tokunaga's Law Horton ⇔ Toki Reducing Horto

Scaling relation
Fluctuations





## Order of business:

- 1. Find out how these relationships are connected.
- 2. Determine most fundamental description.

#### Introduction

River Networks

Definitions

Allometry Laws

stream Ordering forton's Laws

Horton ⇔ Tokul Reducing Horton

Scaling relation
Fluctuations
Models





## Order of business:

- 1. Find out how these relationships are connected.
- 2. Determine most fundamental description.
- 3. Explain origins of these parameter values

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toke Reducing Horton Scaling relations

Scaling relation Fluctuations Models





### Order of business:

- 1. Find out how these relationships are connected.
- 2. Determine most fundamental description.
- 3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...

#### Introduction

River Networks

Definitions

Allometry

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toke Reducing Horto Scaling relations

References





## Outline

### River Networks

Allometry

## Stream Ordering

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton Models

### Introduction

River Networks Stream Ordering







Method for describing network architecture:

#### Introduction

River Networks

Stream Ordering

References





## Method for describing network architecture:

▶ Introduced by Horton (1945) [7]

### Introduction

River Networks

Stream Ordering







## Method for describing network architecture:

- ► Introduced by Horton (1945) [7]
- ► Modified by Strahler (1957) [16]

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering

Horton ⇔ Tokur Reducing Horton

scaling relation Fluctuations Models

References





## Method for describing network architecture:

- ▶ Introduced by Horton (1945) [7]
- ► Modified by Strahler (1957) [16]
- Term: Horton-Strahler Stream Ordering [11]

### Introduction

River Networks

Stream Ordering

References



## Method for describing network architecture:

- ► Introduced by Horton (1945) [7]
- ► Modified by Strahler (1957) [16]
- ▶ Term: Horton-Strahler Stream Ordering [11]
- Can be seen as iterative trimming of a network.

#### Introduction

River Networks

Definitions

Allometry \_aws

Stream Ordering Horton's Laws

Reducing Hortor
Scaling relations

luctuations Models

References



### Some definitions:

A channel head is a point in landscape where flow becomes focused enough to form a stream.

#### Introduction

River Networks Stream Ordering







### Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga





### Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton





## Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- ▶ Use symbol  $\omega = 1, 2, 3, ...$  for stream order.

#### Introduction

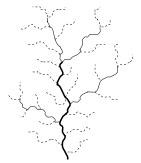
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Models

References

Frame 19/121





#### Introduction

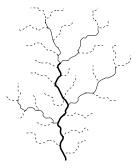
River Networks

Stream Ordering

Frame 20/121







1. Label all source streams as order  $\omega = 1$  and remove.

#### Introduction

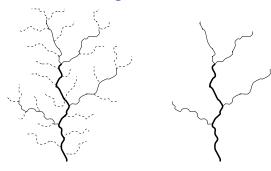
River Networks

Allometry aws

Stream Ordering

Reducing Horton Scaling relations Fluctuations

Models



1. Label all source streams as order  $\omega = 1$  and remove.

#### Introduction

River Networks

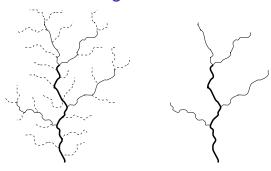
Allometry \_aws

Stream Ordering

Horton ⇔ Toku
Reducing Hortor
Scaling relations
Fluctuations

References

reierences



- 1. Label all source streams as order  $\omega = 1$  and remove.
- 2. Label all new source streams as order  $\omega = 2$  and remove.

#### Introduction

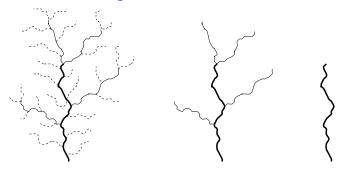
River Networks

llometry aws

Stream Ordering Horton's Laws

Horton ⇔ Toke Reducing Horton Scaling relations Fluctuations





- 1. Label all source streams as order  $\omega = 1$  and remove.
- 2. Label all new source streams as order  $\omega = 2$  and remove.

#### Introduction

River Networks

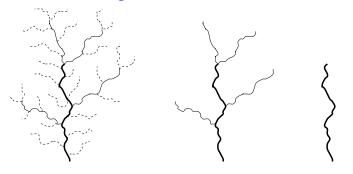
Definitions

illometry aws stream Orderi

Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Hortor Scaling relations Fluctuations Models



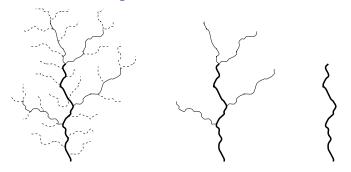


- 1. Label all source streams as order  $\omega = 1$  and remove.
- 2. Label all new source streams as order  $\omega = 2$  and remove.
- 3. Repeat until one stream is left (order =  $\Omega$ )

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga





- 1. Label all source streams as order  $\omega = 1$  and remove.
- 2. Label all new source streams as order  $\omega = 2$  and remove.
- 3. Repeat until one stream is left (order =  $\Omega$ )
- 4. Basin is said to be of the order of the last stream removed.

#### Introduction

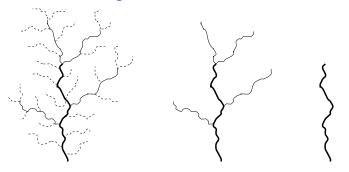
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Fluctuations Models

References

Frame 20/121





- 1. Label all source streams as order  $\omega = 1$  and remove.
- 2. Label all new source streams as order  $\omega = 2$  and remove.
- 3. Repeat until one stream is left (order =  $\Omega$ )
- 4. Basin is said to be of the order of the last stream removed.
- 5. Example above is a basin of order  $\Omega = 3$ .

#### Introduction

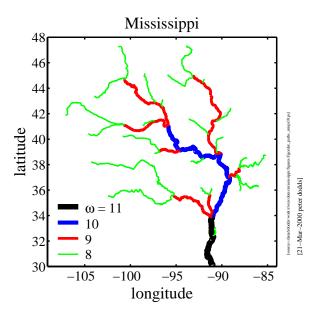
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

References

Frame 20/121



# Stream Ordering—A large example:



Introduction

River Networks

Allometry

Stream Ordering

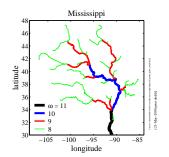
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

References

Frame 21/121



## Another way to define ordering:



### Introduction

River Networks

Allometry

Stream Ordering

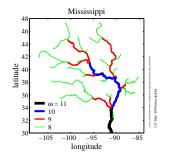
Horton ⇔ Toke Reducing Horto Scaling relations

References



## Another way to define ordering:

▶ As before, label all source streams as order  $\omega = 1$ .



#### Introduction

River Networks

Allometry

Stream Ordering

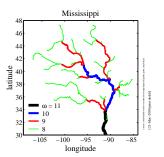
Tokunaga's Law Horton ⇔ Tokuna Reducing Horton Scaling relations

References



## Another way to define ordering:

- ▶ As before, label all source streams as order  $\omega = 1$ .
- ► Follow all labelled streams downstream



### Introduction

River Networks

Definitions

llometry aws

Stream Ordering

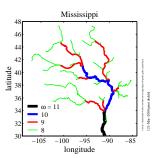
Horton ⇔ Toke Reducing Horto Scaling relations Fluctuations

References



## Another way to define ordering:

- ▶ As before, label all source streams as order  $\omega = 1$ .
- Follow all labelled streams downstream
- Whenever two streams of the same order  $(\omega)$  meet, the resulting stream has order incremented by 1  $(\omega + 1)$ .



### Introduction

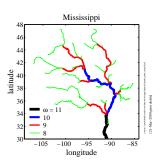
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References



## Another way to define ordering:

- ▶ As before, label all source streams as order  $\omega = 1$ .
- Follow all labelled streams downstream
- ▶ Whenever two streams of the same order  $(\omega)$  meet, the resulting stream has order incremented by 1  $(\omega + 1)$ .
- If streams of different orders  $\omega_1$  and  $\omega_2$  meet, then the resultant stream has order equal to the largest of the two.



#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References

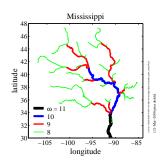


## Another way to define ordering:

- ▶ As before, label all source streams as order  $\omega = 1$ .
- Follow all labelled streams downstream
- Whenever two streams of the same order  $(\omega)$  meet, the resulting stream has order incremented by 1  $(\omega + 1)$ .
- ▶ If streams of different orders  $\omega_1$  and  $\omega_2$  meet, then the resultant stream has order equal to the largest of the two.
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where  $\delta$  is the Kronecker delta.



Introduction

River Networks

Definitions

Allometry

Stream Ordering

Tokunaga's Law
Horton ⇔ Tokuna
Reducing Horton
Scaling relations
Fluctuations
Models

References



## One problem:

Resolution of data messes with ordering

#### Introduction

River Networks

Stream Ordering







## One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)

#### Introduction

River Networks

Definitions

Allometry

Laws Stream Ordering

Horton's Laws
Tokunaga's Law

Reducing Horto Scaling relations

iviodolo





## One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.

#### Introduction

River Networks

Definitions

Allometry

Stream Ordering

Tokunaga's Law Horton ⇔ Toku Reducing Hortor Scaling relations





## **Utility:**

### Introduction

River Networks

Stream Ordering







## **Utility:**

Stream ordering helpfully discretizes a network.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toke Reducing Horto Scaling relations

.....





## **Utility:**

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Hortor Scaling relations Fluctuations





### Resultant definitions:

 $\triangleright$  A basin of order  $\Omega$  has  $n_{\omega}$  streams (or sub-basins) of order  $\omega$ .

#### Introduction

River Networks

Stream Ordering

References

Frame 25/121





### Resultant definitions:

- $\triangleright$  A basin of order  $\Omega$  has  $n_{\omega}$  streams (or sub-basins) of order  $\omega$ .
  - ho  $n_{\omega} > n_{\omega+1}$

#### Introduction

River Networks

Stream Ordering

References

Frame 25/121





### Resultant definitions:

- A basin of order  $\Omega$  has  $n_{\omega}$  streams (or sub-basins) of order  $\omega$ .
  - ho  $n_{\omega} > n_{\omega+1}$
- ▶ An order  $\omega$  basin has area  $a_{\omega}$ .

#### Introduction

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations

lodels





### Resultant definitions:

- A basin of order  $\Omega$  has  $n_{\omega}$  streams (or sub-basins) of order  $\omega$ .
  - ho  $n_{\omega} > n_{\omega+1}$
- ▶ An order  $\omega$  basin has area  $a_{\omega}$ .
- ▶ An order  $\omega$  basin has a main stream length  $\ell_{\omega}$ .

### Introduction

River Networks

Definitions

definitions allometry aws

Stream Ordering Horton's Laws

orton ⇔ Toku educing Horton caling relations luctuations





### Resultant definitions:

- A basin of order Ω has n<sub>ω</sub> streams (or sub-basins) of order ω.
  - ho  $n_{\omega} > n_{\omega+1}$
- ▶ An order  $\omega$  basin has area  $a_{\omega}$ .
- ▶ An order  $\omega$  basin has a main stream length  $\ell_{\omega}$ .
- An order  $\omega$  basin has a stream segment length  $s_{\omega}$

### Introduction

River Networks

Definitions

definitions allometry aws

Stream Ordering Horton's Laws

> orton ⇔ Toku educing Horton caling relations luctuations

> lodels





### Resultant definitions:

- A basin of order Ω has n<sub>ω</sub> streams (or sub-basins) of order ω.
  - ho  $n_{\omega} > n_{\omega+1}$
- ▶ An order  $\omega$  basin has area  $a_{\omega}$ .
- ▶ An order  $\omega$  basin has a main stream length  $\ell_{\omega}$ .
- An order  $\omega$  basin has a stream segment length  $s_{\omega}$ 
  - 1. an order  $\omega$  stream segment is only that part of the stream which is actually of order  $\omega$

#### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws

> orton ⇔ Toku educing Hortor caling relations uctuations





## Resultant definitions:

- A basin of order Ω has n<sub>ω</sub> streams (or sub-basins) of order ω.
  - ho  $n_{\omega} > n_{\omega+1}$
- ▶ An order  $\omega$  basin has area  $a_{\omega}$ .
- ▶ An order  $\omega$  basin has a main stream length  $\ell_{\omega}$ .
- ▶ An order  $\omega$  basin has a stream segment length  $s_{\omega}$ 
  - 1. an order  $\omega$  stream segment is only that part of the stream which is actually of order  $\omega$
  - 2. an order  $\omega$  stream segment runs from the basin outlet up to the junction of two order  $\omega-1$  streams

River Networks

Definitions

Laws
Stream Ordering

ream Ordering
orton's Laws
kunaga's Law
orton ⇔ Tokunaga
educing Horton
caling relations





## **Outline**

Introduction

## River Networks

Allometry

Laws

Stream Ordering

### Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton Scaling relations

Mariala

Models

References

# Introduction River Networks

Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations

References

Frame 26/121





## Self-similarity of river networks

#### Introduction

River Networks

Horton's Laws







## Self-similarity of river networks

► First quantified by Horton (1945) [7], expanded by Schumm (1956) [14]

Introduction

River Networks

Allometry

aws tream Orde

Horton's Laws Tokunaga's Law

Reducing Hortor Scaling relations

uctuations odels





## Self-similarity of river networks

► First quantified by Horton (1945) [7], expanded by Schumm (1956) [14]

## Three laws:

#### Introduction

River Networks

Allometry aws

Laws Stream Ord

Horton's Laws Tokunaga's Lav

> Reducing Hortor Scaling relations Fluctuations





## Self-similarity of river networks

► First quantified by Horton (1945) [7], expanded by Schumm (1956) [14]

### Three laws:

Horton's law of stream numbers:

$$n_{\omega}/n_{\omega+1}=R_n>1$$

llometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokuna Reducing Horton

Scaling relation Fluctuations Models





## Self-similarity of river networks

► First quantified by Horton (1945) [7], expanded by Schumm (1956) [14]

### Three laws:

Horton's law of stream numbers:

$$n_{\omega}/n_{\omega+1}=R_n>1$$

Horton's law of stream lengths:

$$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}>1$$

#### Introduction

River Networks

Definitions

Allometry

Horton's Laws

Reducing Horte Scaling relation Fluctuations





## Self-similarity of river networks

► First quantified by Horton (1945) <sup>[7]</sup>, expanded by Schumm (1956) <sup>[14]</sup>

### Three laws:

Horton's law of stream numbers:

$$n_{\omega}/n_{\omega+1}=R_n>1$$

Horton's law of stream lengths:

$$oxed{ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}>1}$$

Horton's law of basin areas:

$$\bar{a}_{\omega+1}/\bar{a}_{\omega}=R_a>1$$

#### Introduction

River Networks

Definitions

aws

Horton's Laws Tokunaga's Law

Reducing Hort Scaling relation Fluctuations





### Horton's Ratios:

▶ So... Horton's laws are defined by three ratios:

 $R_n$ ,  $R_\ell$ , and  $R_a$ .

#### Introduction

River Networks

Allometry Laws

Laws Stream Ordering

Horton's Laws

Reducing Horton Scaling relations Fluctuations





### Horton's Ratios:

▶ So... Horton's laws are defined by three ratios:

$$R_n$$
,  $R_\ell$ , and  $R_a$ .

Horton's laws describe exponential decay or growth:

$$n_{\omega} = n_{\omega-1}/R_n$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokul Reducing Horton

caling relations uctuations odels





### Horton's Ratios:

So... Horton's laws are defined by three ratios:

$$R_n$$
,  $R_\ell$ , and  $R_a$ .

Horton's laws describe exponential decay or growth:

$$n_{\omega} = n_{\omega - 1}/R_n$$
$$= n_{\omega - 2}/R_n^2$$

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokun Reducing Horton

caling relations uctuations odels





### Horton's Ratios:

So... Horton's laws are defined by three ratios:

$$R_n$$
,  $R_\ell$ , and  $R_a$ .

Horton's laws describe exponential decay or growth:

$$n_{\omega} = n_{\omega-1}/R_n$$

$$= n_{\omega-2}/R_n^2$$

$$\vdots$$

$$= n_1/R_n^{\omega-1}$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations

lodels





## Horton's Ratios:

So... Horton's laws are defined by three ratios:

$$R_n$$
,  $R_\ell$ , and  $R_a$ .

Horton's laws describe exponential decay or growth:

$$n_{\omega} = n_{\omega-1}/R_n$$

$$= n_{\omega-2}/R_n^2$$

$$\vdots$$

$$= n_1/R_n^{\omega-1}$$

$$= n_1 e^{-(\omega-1)\ln R_n}$$

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

lodels





Frame 29/121





# Similar story for area and length:

$$\bar{a}_{\omega} = \bar{a}_1 e^{(\omega-1) \ln R_a}$$

$$ar{\ell}_{\omega} = ar{\ell}_1 e^{(\omega-1) \ln R_{\ell}}$$

## Introduction

River Networks

Horton's Laws





# Similar story for area and length:

$$\bar{a}_{\omega} = \bar{a}_1 e^{(\omega-1) \ln R_a}$$

•

$$\bar{\ell}_{\omega} = \bar{\ell}_1 e^{(\omega-1) \ln R_{\ell}}$$

► As stream order increases, number drops and area and length increase.

## Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunag Reducing Horton Scaling relations

References

Frame 29/121



# A few more things:

## Introduction

River Networks

Horton's Laws







# A few more things:

► Horton's laws are laws of averages.

#### Introduction

River Networks

Horton's Laws







# A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law
Horton ⇔ Tokunag
Beducing Horton

Scaling relation
Fluctuations
Models





# A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga





# A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.
- Horton's ratios go a long way to defining a branching network...

## Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Frame 30/121



# A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.
- Horton's ratios go a long way to defining a branching network...
- But we need one other piece of information...

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton & Tokunaga's Reducing Horton





## A bonus law:

Horton's law of stream segment lengths:

$$ar{ar{s}_{\omega+1}}/ar{ar{s}_{\omega}}=R_{ar{s}}>1$$

#### Introduction

River Networks

Horton's Laws





## A bonus law:

► Horton's law of stream segment lengths:

$$ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}>1$$

▶ Can show that  $R_s = R_\ell$ .

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tok Reducing Horto Scaling relation

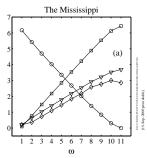
Models

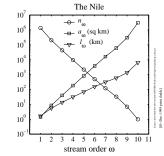
References

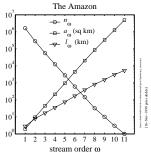
Frame 31/121



## Horton's laws in the real world:







#### Introduction

River Networks

Definitions

Allometry

Laws

Horton's Laws

Horton ⇔ Tokuna Reducing Horton Scaling relations Fluctuations

References

Frame 32/121



## Introduction

River Networks

Horton's Laws

Frame 33/121





# Horton's laws-at-large

## Blood networks:

 Horton's laws hold for sections of cardiovascular networks

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunag

Reducing Horton Scaling relation Fluctuations





## Blood networks:

- Horton's laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...

## Introduction

River Networks

Horton's Laws







## Blood networks:

- Horton's laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton's law.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Toku

Reducing Horton Scaling relation Fluctuations

References

Frame 33/121



## Observations:

Horton's ratios vary:

```
R_n 3.0-5.0 R_a 3.0-6.0 R_\ell 1.5-3.0
```

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Orderi

Tokunaga's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

Models

References

Frame 34/121



## Observations:

Horton's ratios vary:

$$R_n$$
 3.0-5.0  $R_a$  3.0-6.0  $R_\ell$  1.5-3.0

No accepted explanation for these values.

## Introduction

River Networks

Horton's Laws

References

Frame 34/121



## Observations:

Horton's ratios vary:

$$R_n$$
 3.0-5.0  $R_a$  3.0-6.0  $R_\ell$  1.5-3.0

- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...

## Introduction

River Networks

Horton's Laws





## Observations:

Horton's ratios vary:

$$R_n$$
 3.0-5.0  $R_a$  3.0-6.0  $R_\ell$  1.5-3.0

- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...
- ... but they don't explain how networks are structured.

#### Introduction

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tok

Reducing Horto Scaling relation Fluctuations Models





# Outline

## River Networks

Allometry

# Tokunaga's Law

Horton ⇔ Tokunaga Reducing Horton Models

## Introduction

River Networks Tokunaga's Law







Allometry Laws

tream Ordering

Tokunaga's Law

Reducing Horto Scaling relation Fluctuations

References

Frame 36/121



# Tokunaga's law

# Delving deeper into network architecture:

► Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Laws

Reducing Horton Scaling relations Fluctuations





- Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]
- As per Horton-Strahler, use stream ordering.

## Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law

Reducing Horte Scaling relation Fluctuations

References

Frame 36/121



- Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law

Reducing Horto Scaling relation Fluctuations





- Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
- Tokunaga's law is also a law of averages.

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws Tokunaga's Law

Horton ⇔ Toku Reducing Hortor Scaling relations

caling relations uctuations





## Definition:

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton







## Definition:

- $ightharpoonup T_{\mu,\nu}$  = the average number of side streams of order  $\nu$  that enter as tributaries to streams of order  $\mu$
- $\mu$ ,  $\nu$  = 1, 2, 3, ...

## Introduction

River Networks

Tokunaga's Law







## **Definition:**

- ►  $T_{\mu,\nu}$  = the average number of side streams of order  $\nu$  that enter as tributaries to streams of order  $\mu$
- $\mu$ ,  $\nu$  = 1, 2, 3, ...
- $\mu \geq \nu + 1$

#### Introduction

River Networks

Definitions

allometry \_aws Stream Orderii

Horton's Laws Tokunaga's Law

Reducing Horto
Scaling relation:





## Definition:

- ►  $T_{\mu,\nu}$  = the average number of side streams of order  $\nu$  that enter as tributaries to streams of order  $\mu$
- $\mu$ ,  $\nu$  = 1, 2, 3, ...
- $\mu \geq \nu + 1$
- ▶ Recall each stream segment of order  $\mu$  is 'generated' by two streams of order  $\mu$  1

#### Introduction

River Networks

Definitions

Allometry \_aws Stream Ordering

Horton's Laws Tokunaga's Law

Reducing Horton Scaling relation Fluctuations





## Definition:

- ►  $T_{\mu,\nu}$  = the average number of side streams of order  $\nu$  that enter as tributaries to streams of order  $\mu$
- $\mu, \nu = 1, 2, 3, \dots$
- $\mu \geq \nu + 1$
- ▶ Recall each stream segment of order  $\mu$  is 'generated' by two streams of order  $\mu$  − 1
- These generating streams are not considered side streams.

## Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Hortor Scaling relations Fluctuations



## Introduction

#### River Networks

## Tokunaga's Law

Frame 38/121





# Tokunaga's law

Property 1: Scale independence—depends only on difference between orders:

#### Introduction

River Networks

Tokunaga's Law







## Tokunaga's law

Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

aws

lorton's Laws

Tokunaga's Law Horton ⇔ Toku

> caling relations uctuations





# Tokunaga's law

Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

Property 2: Number of side streams grows exponentially with difference in orders:

#### Introduction

River Networks

Allometry

tream Order

Tokunaga's Law

Reducing Hort Scaling relation Fluctuations





### **Network Architecture**

### Tokunaga's law

Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,
u} = T_{\mu-
u}$$

Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1 (R_T)^{\mu-\nu-1}$$

#### Introduction

River Networks

llometry

ream Ordering orton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

caling relation





### **Network Architecture**

### Tokunaga's law

Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1 (R_T)^{\mu-\nu-1}$$

We usually write Tokunaga's law as:

$$\left| T_k = T_1 (R_T)^{k-1} \right|$$
 where  $R_T \simeq 2$ 

#### Introduction

River Networks

Definitions Allometry .aws

Horton's Laws
Tokunaga's Law

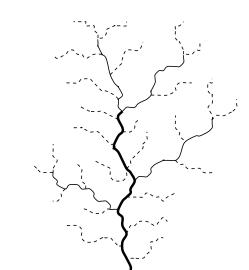
Reducing Horto Scaling relation Fluctuations





## Tokunaga's law—an example:

 $T_1 \simeq 2$   $R_T \simeq 4$ 



Introduction

River Networks

llometry

stream Ordering

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton

Scaling relations
Fluctuations
Models

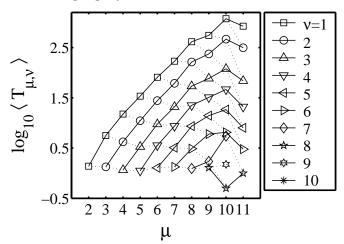
References

Frame 39/121



## The Mississippi

### A Tokunaga graph:



#### Introduction

River Networks

Definitions Allometry Laws

aws tream Orderin

Tokunaga's Law Horton ⇔ Tokuna

Reducing Horte Scaling relation Fluctuations





### Outline

### River Networks

Allometry

Tokunaga's Law

### Horton ⇔ Tokunaga

Reducing Horton

Models

#### Introduction

River Networks

Horton ⇔ Tokunaga







### Introduction

River Networks

Allometry

Laws Stream Oi

Horton's Laws

Horton ⇔ Tokunaga Reducing Horton

> caling relations uctuations odels

References

Frame 42/121



### Horton and Tokunaga seem different:

Horton's laws appear to contain less detailed information than Tokunaga's law.

#### Introduction

River Networks

Horton ⇔ Tokunaga







## Horton and Tokunaga seem different:

- Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's law has three parameters and Tokunaga has two parameters.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Horton ⇔ Tokunaga Reducing Horton Scaling relations Fluctuations





## Horton and Tokunaga seem different:

- Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's law has three parameters and Tokunaga has two parameters.
- $ightharpoonup R_n$ ,  $R_\ell$ , and  $R_s$  versus  $T_1$  and  $R_T$ .

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Frame 42/121



## Horton and Tokunaga seem different:

- Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's law has three parameters and Tokunaga has two parameters.
- ▶  $R_n$ ,  $R_\ell$ , and  $R_s$  versus  $T_1$  and  $R_T$ .
- ➤ To make a connection, clearest approach is to start with Tokunaga's law...

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Frame 42/121



## Horton and Tokunaga seem different:

- Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's law has three parameters and Tokunaga has two parameters.
- $ightharpoonup R_n$ ,  $R_\ell$ , and  $R_s$  versus  $T_1$  and  $R_T$ .
- ➤ To make a connection, clearest approach is to start with Tokunaga's law...
- ► Known result: Tokunaga → Horton [21, 22, 23, 10, 2]

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton





### We need one more ingredient:

#### Introduction

River Networks

Horton ⇔ Tokunaga

Frame 43/121





We need one more ingredient:

Space-fillingness

#### Introduction

River Networks

Horton ⇔ Tokunaga







### We need one more ingredient:

### Space-fillingness

A network is space-filling if the average distance between adjacent streams is roughly constant.

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

caling relations uctuations odels





### We need one more ingredient:

### Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- ▶ Reasonable for river and cardiovascular networks

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





### We need one more ingredient:

### Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:

  Drainage density  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton

caling relation luctuations lodels





### We need one more ingredient:

### Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:

  Drainage density  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.
- ▶ In terms of basin characteristics:

$$ho_{
m dd} \simeq rac{\sum {
m stream \ segment \ lengths}}{{
m basin \ area}}$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

Models

References

Frame 43/121



### We need one more ingredient:

### Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:

  Drainage density  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

$$ho_{
m dd} \simeq rac{\sum {
m stream \ segment \ lengths}}{{
m basin \ area}} = rac{\sum_{\omega=1}^{\Omega} n_{\omega} s_{\omega}}{a_{\Omega}}$$

#### Introduction

River Networks

Definitions

Allometry Laws Stream Ordering

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations





Start with Tokunaga's law:  $T_k = T_1 R_T^{k-1}$ 

### Introduction

River Networks

Horton ⇔ Tokunaga







## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

Start looking for Horton's stream number law:

$$n_{\omega}/n_{\omega+1}=R_n$$

#### Introduction

River Networks

Horton ⇔ Tokunaga







## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .

Definitions Allometry Laws

Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations





## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .
- ▶ Observe that each stream of order  $\omega$  terminates by either:

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horto
Scaling relation:
Fluctuations
Models





## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ► Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .
- ▶ Observe that each stream of order  $\omega$  terminates by either:

$$\omega=3$$
 $\omega=3$ 
 $\omega=4$ 

1. Running into another stream of order  $\omega$  and generating a stream of order  $\omega+1...$ 

#### Introduction

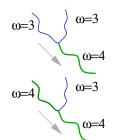
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton





## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ► Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .
- ▶ Observe that each stream of order  $\omega$  terminates by either:



- 1. Running into another stream of order  $\omega$  and generating a stream of order  $\omega+1...$
- 2. Running into and being absorbed by a stream of higher order  $\omega' > \omega...$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

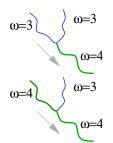
References

Frame 44/121



## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .
- ▶ Observe that each stream of order  $\omega$  terminates by either:



- 1. Running into another stream of order  $\omega$  and generating a stream of order  $\omega+1...$ 
  - ▶  $2n_{\omega+1}$  streams of order  $\omega$  do this
- 2. Running into and being absorbed by a stream of higher order  $\omega' > \omega...$

#### Introduction

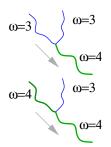
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga





## Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law:  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Estimate  $n_{\omega}$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}$ ,  $\omega' > \omega$ .
- ▶ Observe that each stream of order  $\omega$  terminates by either:



- 1. Running into another stream of order  $\omega$  and generating a stream of order  $\omega+1...$ 
  - ▶  $2n_{\omega+1}$  streams of order  $\omega$  do this
- 2. Running into and being absorbed by a stream of higher order  $\omega' > \omega...$ 
  - $n'_{\omega}T_{\omega'-\omega}$  streams of order  $\omega$  do this

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References

Frame 44/121



### Putting things together:



$$n_{\omega} = 2n_{\omega+1} +$$
generation

#### Introduction

River Networks

Horton ⇔ Tokunaga







### Putting things together:



$$n_{\omega} = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{M} \frac{T_{\omega'-\omega}n_{\omega}}{\text{absorption}}$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





### Putting things together:

$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$

▶ Substitute in  $T_{\omega'-\omega} = T_1(R_T)^{\omega'-\omega-1}$ :

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Scaling relations
Fluctuations
Models





### Putting things together:

$$n_{\omega} = \underbrace{\frac{2n_{\omega+1}}{\text{generation}}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{\frac{\mathcal{T}_{\omega'-\omega}n_{\omega'}}{\text{absorption}}}_{\text{absorption}}$$

▶ Substitute in  $T_{\omega'-\omega} = T_1(R_T)^{\omega'-\omega-1}$ :

$$n_{\omega} = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1(R_T)^{\omega'-\omega-1} n_{\omega'}$$

#### Introduction

River Networks

Definitions

Allometry

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations





### Putting things together:

$$n_{\omega} = \underbrace{\frac{2n_{\omega+1}}{\text{generation}}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{\frac{T_{\omega'-\omega}n_{\omega'}}{\text{absorption}}}_{\text{absorption}}$$

▶ Substitute in  $T_{\omega'-\omega} = T_1(R_T)^{\omega'-\omega-1}$ :

$$n_{\omega} = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1(R_T)^{\omega'-\omega-1} n_{\omega'}$$

▶ Shift index to  $k = \omega' - \omega$ :

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Horton Scaling relations Fluctuations





### Putting things together:

•

$$n_{\omega} = \underbrace{\frac{2n_{\omega+1}}{\text{generation}}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{\frac{T_{\omega'-\omega}n_{\omega'}}{\text{absorption}}}_{\text{absorption}}$$

▶ Substitute in  $T_{\omega'-\omega} = T_1(R_T)^{\omega'-\omega-1}$ :

$$n_{\omega} = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1(R_T)^{\omega'-\omega-1} n_{\omega'}$$

▶ Shift index to  $k = \omega' - \omega$ :

$$n_{\omega} = 2n_{\omega+1} + \sum_{k=1}^{M-\omega} T_1(R_T)^{k-1} n_{\omega+k}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Fluctuations Models





### Introduction

River Networks

Horton ⇔ Tokunaga

References

Frame 46/121





### **Create Horton ratios:**

▶ Divide through by  $n_{\omega+1}$ :

#### Introduction

River Networks

Horton ⇔ Tokunaga







### Create Horton ratios:

▶ Divide through by  $n_{\omega+1}$ :

$$\frac{\textit{n}_{\omega}}{\textit{n}_{\omega+1}} = \frac{2\textit{n}_{\omega+1}}{\textit{n}_{\omega+1}} + \sum_{k=1}^{\Omega-\omega} \textit{T}_{1}(\textit{R}_{T})^{k-1} \frac{\textit{n}_{\omega+k}}{\textit{n}_{\omega+1}}$$

Stream Ordering

Horton ⇔ Tokunaga Reducing Horton

icaling relations fluctuations





### Create Horton ratios:

▶ Divide through by  $n_{\omega+1}$ :

$$\frac{n_{\omega}}{n_{\omega+1}} = \frac{2n_{\omega+1}}{n_{\omega+1}} + \sum_{k=1}^{\Omega-\omega} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$

Allometry

Stream Ordering

Horton ⇔ Tokunaga Reducing Horton Scaling relations

caling relations luctuations lodels

### Create Horton ratios:

▶ Divide through by  $n_{\omega+1}$ :

$$\frac{n_{\omega}}{n_{\omega+1}} = \frac{2n_{\omega+1}}{n_{\omega+1}} + \sum_{k=1}^{\Omega-\omega} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$

▶ Left hand side looks good but we have  $n_{\omega+k}/n_{\omega+1}$ 's hanging around on the right.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





### Create Horton ratios:

▶ Divide through by  $n_{\omega+1}$ :

$$\frac{n_{\omega}}{n_{\omega+1}} = \frac{2\underline{n_{\omega+1}}}{\underline{n_{\omega+1}}} + \sum_{k=1}^{\Omega-\omega} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{\underline{n_{\omega+1}}}$$

- ▶ Left hand side looks good but we have  $n_{\omega+k}/n_{\omega+1}$ 's hanging around on the right.
- ▶ Recall, we want to show  $R_n = n_{\omega}/n_{\omega+1}$  is a constant, independent of  $\omega$ ...

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations





### Finding Horton ratios:

▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

lefinitions .llometry aws

ws ream Ordering orton's Laws

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

educing Horto aling relations uctuations





### Finding Horton ratios:

▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

The ratio  $n_{\omega+k}/n_{\omega+1}$  can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).

#### Introduction

River Networks

Definitions Allometry aws

ream Ordering orton's Laws

Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton

Scaling relatio Fluctuations Models





▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

- ► The ratio  $n_{\omega+k}/n_{\omega+1}$  can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).
- ▶ The ratio  $n_{\omega}/n_{\omega+1}$  is independent of  $\omega$  and depends only on  $T_1$  and  $R_T$ .

#### Introduction

River Networks

efinitions llometry aws

tream Ordering orton's Laws

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton Scaling relations

Models

References

▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

- ► The ratio  $n_{\omega+k}/n_{\omega+1}$  can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).
- ▶ The ratio  $n_{\omega}/n_{\omega+1}$  is independent of  $\omega$  and depends only on  $T_1$  and  $R_T$ .
- ▶ Can now call  $n_{\omega}/n_{\omega+1} = R_n$ .

#### Introduction

River Networks

efinitions llometry aws

tream Ordering orton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Models

▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

- ► The ratio  $n_{\omega+k}/n_{\omega+1}$  can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).
- ▶ The ratio  $n_{\omega}/n_{\omega+1}$  is independent of  $\omega$  and depends only on  $T_1$  and  $R_T$ .
- ▶ Can now call  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Immediately have  $n_{\omega+k}/n_{\omega+1} = R_n^{-(k-1)}$ .

#### Introduction

River Networks

efinitions ometry ws

iws ream Ordering orton's Laws

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

caling relations uctuations odels



▶ Letting  $\Omega \to \infty$ , we have

$$\frac{n_{\omega}}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$
 (1)

- ► The ratio  $n_{\omega+k}/n_{\omega+1}$  can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).
- ▶ The ratio  $n_{\omega}/n_{\omega+1}$  is independent of  $\omega$  and depends only on  $T_1$  and  $R_T$ .
- ▶ Can now call  $n_{\omega}/n_{\omega+1} = R_n$ .
- ▶ Immediately have  $n_{\omega+k}/n_{\omega+1} = R_n^{-(k-1)}$ .
- ▶ Plug into Eq. (1)...

#### Introduction

River Networks

efinitions ometry ws

aws tream Ordering orton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

caling relations uctuations odels





### Finding Horton ratios:

Now have:

$$R_n = 2 + \sum_{k=1}^{\infty} T_1(R_T)^{k-1} R_n^{-(k-1)}$$

#### Introduction

River Networks

llometry

aws Stream Ordering

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horto Scaling relations Fluctuations





### Finding Horton ratios:

Now have:

$$R_n = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)}$$
$$= 2 + T_1 \sum_{k=1}^{\infty} (R_T / R_n)^{k-1}$$

#### Introduction

River Networks

llometry

tream Ordering

Tokunaga's Law Horton ⇔ Tokunaga

Scaling relations
Fluctuations
Models

References

Frame 48/121



### Finding Horton ratios:

Now have:

$$R_n = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)}$$

$$= 2 + T_1 \sum_{k=1}^{\infty} (R_T / R_n)^{k-1}$$

$$= 2 + T_1 \frac{1}{1 - R_T / R_n}$$

#### Introduction

River Networks

Definitions

aws

tream Ordering lorton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Fluctuations
Models





### Finding Horton ratios:

Now have:

$$R_n = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)}$$

$$= 2 + T_1 \sum_{k=1}^{\infty} (R_T / R_n)^{k-1}$$

$$= 2 + T_1 \frac{1}{1 - R_T / R_n}$$

Rearrange to find:

$$(R_n-2)(1-R_T/R_n)=T_1$$

#### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws

Tokunaga's Law
Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





Finding  $R_n$  in terms of  $T_1$  and  $R_T$ :

#### Introduction

River Networks

Allometr

Allometry Laws

Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Reducing Hort Scaling relatio

uctuations odels





## Finding $R_n$ in terms of $T_1$ and $R_T$ :

• We are here:  $(R_n - 2)(1 - R_T/R_n) = T_1$ 

#### Introduction

River Networks

Horton ⇔ Tokunaga







### Finding $R_n$ in terms of $T_1$ and $R_T$ :

- ▶ We are here:  $(R_n 2)(1 R_T/R_n) = T_1$
- $\triangleright$   $\times R_n$  to find quadratic in  $R_n$ :

$$(R_n-2)(R_n-R_T)=T_1R_n$$

#### Introduction

River Networks

efinitions Illometry aws

Stream Ordering

Tokunaga's Law Horton ⇔ Tokunaga

Reducing Horton Scaling relation Fluctuations





### Finding $R_n$ in terms of $T_1$ and $R_T$ :

- ▶ We are here:  $(R_n 2)(1 R_T/R_n) = T_1$
- $\triangleright$   $\times R_n$  to find quadratic in  $R_n$ :

$$(R_n-2)(R_n-R_T)=T_1R_n$$

•

$$R_n^2 - (2 + R_T + T_1)R_n + 2R_T = 0$$

#### Introduction

River Networks

llometry

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Reducing Horton Scaling relation Fluctuations





## Finding $R_n$ in terms of $T_1$ and $R_T$ :

- ▶ We are here:  $(R_n 2)(1 R_T/R_n) = T_1$
- $\triangleright$   $\times R_n$  to find quadratic in  $R_n$ :

$$(R_n-2)(R_n-R_T)=T_1R_n$$

$$R_0^2 - (2 + R_T + T_1)R_0 + 2R_T = 0$$

Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

#### Introduction

River Networks

llometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

References

Frame 49/121



## Finding other Horton ratios

## Connect Tokunaga to $R_s$

▶ Now use uniform drainage density  $\rho_{dd}$ .

#### Introduction

River Networks

Allometry

Laws Stream Or

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Scaling relations





## Finding other Horton ratios

## Connect Tokunaga to R<sub>s</sub>

- Now use uniform drainage density  $\rho_{\mathrm{dd}}$ .
- Assume side streams are roughly separated by distance  $1/\rho_{\rm dd}$ .

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

uctuations





## Connect Tokunaga to R<sub>s</sub>

- Now use uniform drainage density  $\rho_{\mathrm{dd}}$ .
- Assume side streams are roughly separated by distance 1/ρ<sub>dd</sub>.
- $\blacktriangleright$  For an order  $\omega$  stream segment, expected length is

$$\bar{s}_{\omega} \simeq \rho_{\mathrm{dd}}^{-1} \left( 1 + \sum_{k=1}^{\omega - 1} T_k \right)$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Horton ⇔ Tokunaga Reducing Horton Scaling relations Fluctuations Models





## Finding other Horton ratios

## Connect Tokunaga to R<sub>s</sub>

- ▶ Now use uniform drainage density  $\rho_{dd}$ .
- Assume side streams are roughly separated by distance 1/ρ<sub>dd</sub>.
- $\blacktriangleright$  For an order  $\omega$  stream segment, expected length is

$$\bar{s}_{\omega} \simeq \rho_{\mathrm{dd}}^{-1} \left( 1 + \sum_{k=1}^{\omega - 1} T_k \right)$$

Substitute in Tokunaga's law  $T_k = T_1 R_T^{k-1}$ :

$$\bar{\mathbf{s}}_{\omega} \simeq 
ho_{\mathrm{dd}}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right)$$

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

References

Frame 50/121



## Finding other Horton ratios

## Connect Tokunaga to R<sub>s</sub>

- ▶ Now use uniform drainage density  $\rho_{dd}$ .
- Assume side streams are roughly separated by distance 1/ρ<sub>dd</sub>.
- $\blacktriangleright$  For an order  $\omega$  stream segment, expected length is

$$\bar{s}_{\omega} \simeq 
ho_{\mathrm{dd}}^{-1} \left( 1 + \sum_{k=1}^{\omega - 1} T_k \right)$$

Substitute in Tokunaga's law  $T_k = T_1 R_T^{k-1}$ :

$$ar{\mathbf{s}}_{\omega} \simeq 
ho_{\mathrm{dd}}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^{\omega}$$

#### Introduction

River Networks

Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations Fluctuations





### Altogether then:



$$\Rightarrow ar{s}_{\omega}/ar{s}_{\omega-1}=R_T$$

#### Introduction

River Networks

Horton ⇔ Tokunaga







## Altogether then:



$$\Rightarrow \bar{s}_{\omega}/\bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

#### Introduction

River Networks

Horton ⇔ Tokunaga







### Altogether then:

Þ

$$ightarrow ar{s}_{\omega}/ar{s}_{\omega-1}=R_T 
ightarrow R_s=R_T$$

▶ Recall  $R_{\ell} = R_s$  so

$$R_{\ell} = R_T$$

#### Introduction

### River Networks

Allometry

Stream Ordering

Horton's Laws Tokunaga's Law

Horton ⇔ Tokunaga Reducing Horton Scaling relations

luctuations lodels





### Altogether then:

$$ightarrow ar{s}_{\omega}/ar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

▶ Recall  $R_{\ell} = R_s$  so

$$R_{\ell} = R_T$$

And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

#### Introduction

River Networks

efinitions lometry ws

Stream Ordering

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Fluctuations Models





### Some observations:

▶  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .

#### Introduction

River Networks

Allometry

Laws Stream Ore

Horton's Laws

Horton ⇔ Tokunaga

Scaling relations





### Some observations:

- ▶  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
- Seems that R<sub>a</sub> must as well...

#### Introduction

River Networks

Allometry

Laws Stream Orde

Horton's Laws

Horton ⇔ Tokunaga Reducing Horton

> caling relation luctuations lodels





### Some observations:

- ▶  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
- Seems that R<sub>a</sub> must as well...
- Suggests Horton's laws must contain some redundancy

#### Introduction

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton

Scaling relation
Fluctuations





### Some observations:

- ▶  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
- Seems that R<sub>a</sub> must as well...
- Suggests Horton's laws must contain some redundancy
- ▶ We'll in fact see that  $R_a = R_n$ .

#### Introduction

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

Scaling relation Fluctuations Models





### Some observations:

- ▶  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
- Seems that R<sub>a</sub> must as well...
- Suggests Horton's laws must contain some redundancy
- ▶ We'll in fact see that  $R_a = R_n$ .
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between statistical distributions. [3, 4]

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

lodels





### The other way round

▶ Note: We can invert the expresssions for  $R_n$  and  $R_\ell$  to find Tokunaga's parameters in terms of Horton's parameters.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga Reducing Horton





### The other way round

Note: We can invert the expresssions for  $R_n$  and  $R_\ell$  to find Tokunaga's parameters in terms of Horton's parameters.

$$R_T = R_\ell$$

$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n$$
.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations





### The other way round

Note: We can invert the expresssions for  $R_n$  and  $R_\ell$  to find Tokunaga's parameters in terms of Horton's parameters.

$$R_{\tau} = R_{\ell}$$

**•** 

$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n$$
.

Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform)...

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

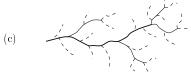
References

Frame 53/121

# Horton and Tokunaga are friends

## From Horton to Tokunaga [2]

(a)  $\downarrow$   $(\times R_l)$ 



### Introduction

River Networks

Allometry

Laws

Horton's Laws

Horton ⇔ Tokunaga

Scaling relations





# Horton and Tokunaga are friends

## From Horton to Tokunaga [2]

(a) (b) (c)

Assume Horton's laws hold for number and length

#### Introduction

River Networks

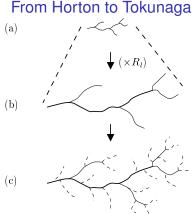
Horton ⇔ Tokunaga

References

Frame 54/121



## From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with an order ω stream

#### Introduction

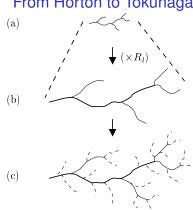
River Networks Horton ⇔ Tokunaga







### From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with an order ω stream
- Scale up by a factor of R<sub>ℓ</sub>, orders increment

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

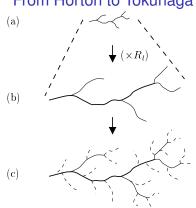
Horton ⇔ Tokunaga Reducing Horton Scaling relations Fluctuations

References

Frame 54/121



## From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with an order ω stream
- Scale up by a factor of  $R_{\ell}$ , orders increment
- Maintain drainage density by adding new order 1 streams

#### Introduction River Networks

Horton ⇔ Tokunaga

References

Frame 54/121





### ... and in detail:

Must retain same drainage density.

#### Introduction

River Networks

Horton ⇔ Tokunaga

References

Frame 55/121





### ... and in detail:

- Must retain same drainage density.
- Add an extra  $(R_{\ell} 1)$  first order streams for each original tributary.

#### Introduction

River Networks

Allometry Laws

Stream Ordering

Horton ⇔ Tokunaga Reducing Horton Scaling relations

ctuations dels





### ... and in detail:

- Must retain same drainage density.
- Add an extra  $(R_{\ell} 1)$  first order streams for each original tributary.
- Since number of first order streams is now given by  $T_{k+1}$  we have:

#### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

odels





### ... and in detail:

- Must retain same drainage density.
- Add an extra  $(R_{\ell} 1)$  first order streams for each original tributary.
- Since number of first order streams is now given by  $T_{k+1}$  we have:

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right).$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations

dels





### ... and in detail:

- Must retain same drainage density.
- Add an extra  $(R_{\ell} 1)$  first order streams for each original tributary.
- Since number of first order streams is now given by  $T_{k+1}$  we have:

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right).$$

▶ For large  $\omega$ , Tokunaga's law is the solution—let's check...

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga Reducing Horton Scaling relations Fluctuations





### Just checking:

Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

#### Introduction

River Networks

aws

aws tream Ordering orton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





### Just checking:

Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

•

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_1 R_{\ell}^{i-1} + 1 \right)$$

#### Introduction

River Networks

aws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunaga

Scaling relation
Fluctuations





## Just checking:

Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

**•** 

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_1 R_{\ell}^{i-1} + 1 \right)$$

$$= (R_{\ell} - 1) T_1 \left( \frac{R_{\ell}^{k} - 1}{R_{\ell} - 1} + 1 \right)$$

#### Introduction

River Networks

llometry

Stream Ordering

Tokunaga's Law
Horton ⇔ Tokunaga

Scaling relation
Fluctuations
Models





### Just checking:

Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

Þ

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_{1} R_{\ell}^{i-1} + 1 \right)$$

$$= (R_{\ell} - 1) T_{1} \left( \frac{R_{\ell}^{k} - 1}{R_{\ell} - 1} + 1 \right)$$

$$\simeq (R_{\ell} - 1) T_{1} \frac{R_{\ell}^{k}}{R_{\ell} - 1}$$

### Introduction

River Networks

efinitions llometry aws

tream Ordering

Tokunaga's Law
Horton ⇔ Tokunaga

Scaling relation

References

Frame 56/121



### Just checking:

Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

Þ

$$T_{k+1} = (R_{\ell} - 1) \left( \sum_{i=1}^{k} T_1 R_{\ell}^{i-1} + 1 \right)$$

$$= (R_{\ell} - 1) T_1 \left( \frac{R_{\ell}^{k} - 1}{R_{\ell} - 1} + 1 \right)$$

$$\simeq (R_{\ell} - 1) T_1 \frac{R_{\ell}^{k}}{R_{\ell} - 1} = T_1 R_{\ell}^{k} \quad ... \text{ yep.}$$

#### Introduction

River Networks

efinitions llometry aws

tream Ordering

Tokunaga's Law Horton ⇔ Tokunaga

Scaling relation

References

Frame 56/121



### **Outline**

Introduction

### **River Networks**

Definitions Allometry

Laws

Stream Ordering Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

### Reducing Horton

Scaling relations

Models

References

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga Reducing Horton
Scaling relations
Fluctuations

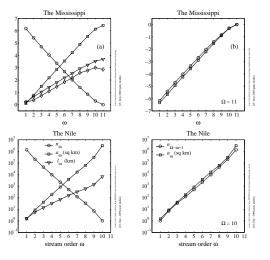
References

Frame 57/121





### Horton's laws of area and number:



► In right plots, stream number graph has been flipped vertically.

#### Introduction

River Networks

Definitions

Allometry

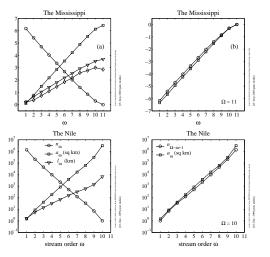
Stream Ordering Horton's Laws Tokunaga's Law Horton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations Models





### Horton's laws of area and number:



- In right plots, stream number graph has been flipped vertically.
- ▶ Highly suggestive that  $R_n \equiv R_a$ ...

Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokui Reducing Horton

Scaling relations
Fluctuations

References

Frame 58/121



# Measuring Horton ratios is tricky:

How robust are our estimates of ratios?

### Introduction

River Networks

Reducing Horton

References

Frame 59/121





# Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

#### Introduction

River Networks

References

Reducing Horton

Frame 59/121





# Mississippi:

$\omega$ range	$R_n$	$R_a$	$R_\ell$	$R_s$	$R_a/R_n$
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3,8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7,8]	4.16	4.67	2.41	2.56	1.12
mean $\mu$	4.69	4.85	2.40	2.33	1.04
std dev $\sigma$	0.21	0.13	0.04	0.07	0.03
$\sigma/\mu$	0.045	0.027	0.015	0.031	0.024

Introduction

River Networks

Allometry \_aws

Stream Order Horton's Law

Tokunaga's Law
Horton ⇔ Tokun
Reducing Horton
Scaling relations

Scaling relations Fluctuations Models

References

Frame 60/121



### Amazon:

$\omega$ range	$R_n$	$R_a$	$R_\ell$	$R_s$	$R_a/R_n$
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean $\mu$	4.42	4.53	2.25	2.10	1.02
std dev $\sigma$	0.17	0.10	0.10	0.09	0.02
$\sigma/\mu$	0.038	0.023	0.045	0.042	0.019

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton & Tokunaga
Reducing Horton
Scaling relations

References

Frame 61/121



Rough first effort to show  $R_n \equiv R_a$ :

#### Introduction

River Networks

Reducing Horton

References

Frame 62/121





### Rough first effort to show $R_n \equiv R_a$ :

▶  $a_{\Omega}$  ∝ sum of all stream lengths in a order Ω basin (assuming uniform drainage density)

#### Introduction

River Networks

Reducing Horton

References

Frame 62/121





### Rough first effort to show $R_n \equiv R_a$ :

- $ightharpoonup a_{\Omega} \propto$  sum of all stream lengths in a order  $\Omega$  basin (assuming uniform drainage density)
- So:

$$a_{\Omega}\simeq\sum_{\omega=1}^{\Omega}n_{\omega}ar{s}_{\omega}/
ho_{
m dd}$$

Definitions Illometry aws

stream Ordering Horton's Laws

Reducing Horton
Scaling relations
Fluctuations

odels

## Rough first effort to show $R_n \equiv R_a$ :

- ▶  $a_{\Omega}$   $\propto$  sum of all stream lengths in a order  $\Omega$  basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^\Omega n_\omega ar{s}_\omega/
ho_{
m dd}$$

$$\propto \sum_{\omega=1}^{\Omega}$$

#### Introduction

River Networks

Definitions Allometry .aws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations

References

Frame 62/121



### Rough first effort to show $R_n \equiv R_a$ :

- $ightharpoonup a_{\Omega} \propto$  sum of all stream lengths in a order  $\Omega$  basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^\Omega n_\omega ar{s}_\omega/
ho_{
m dd}$$

$$\propto \sum_{\omega=1}^{\Omega} R_n^{\Omega-\omega} \cdot 1$$

#### Introduction

River Networks

Definitions Illometry aws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations

uctuations odels





### Rough first effort to show $R_n \equiv R_a$ :

- ▶  $a_{\Omega}$   $\propto$  sum of all stream lengths in a order  $\Omega$  basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^\Omega n_\omega ar{s}_\omega/
ho_{
m dd}$$

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \underbrace{1}_{n_{\omega}}}_{\underline{s}_{\omega}} \underline{\bar{s}_1 \cdot R_s^{\omega-1}}_{\underline{\bar{s}}_{\omega}}$$

#### Introduction

River Networks

efinitions llometry aws

stream Ordering forton's Laws

Reducing Horton Scaling relations Fluctuations

Models





### Rough first effort to show $R_n \equiv R_a$ :

- ▶  $a_{\Omega}$   $\propto$  sum of all stream lengths in a order  $\Omega$  basin (assuming uniform drainage density)
- ► So:

$$a_{\Omega} \simeq \sum_{\omega=1}^{\Omega} n_{\omega} \bar{s}_{\omega} / \rho_{\rm dd}$$

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_{n}^{\Omega-\omega} \cdot 1}_{n_{\omega}} \underbrace{\bar{s}_{1} \cdot R_{s}^{\omega-1}}_{\bar{s}_{\omega}}$$

$$= \frac{R_{n}^{\Omega}}{R_{s}} \bar{s}_{1} \sum_{\omega=1}^{\Omega} \left(\frac{R_{s}}{R_{n}}\right)^{\omega}$$

#### Introduction

River Networks

Definitions Allometry aws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations

References

Frame 62/121



### Continued ...

$${f a}_\Omega \propto {R_n^\Omega \over R_s} ar s_1 \sum_{\omega=1}^\Omega \left( {R_s \over R_n} 
ight)^\omega$$

#### Introduction

River Networks

Reducing Horton

References

Frame 63/121





### Continued ...

$$egin{aligned} \mathbf{a}_{\Omega} & \propto rac{R_n^{\Omega}}{R_s} ar{\mathbf{s}}_1 \sum_{\omega=1}^{\Omega} \left(rac{R_s}{R_n}
ight)^{\omega} \ & = rac{R_n^{\Omega}}{R_s} ar{\mathbf{s}}_1 rac{R_s}{R_n} rac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)} \end{aligned}$$

llometry

Stream Ordering Horton's Laws

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Scaling relations
Fluctuations

References

Frame 63/121



### Continued ...

$$egin{aligned} oldsymbol{a}_{\Omega} & \propto rac{R_n^{\Omega}}{R_s} ar{s}_1 \sum_{\omega=1}^{\Omega} \left(rac{R_s}{R_n}
ight)^{\omega} \ & = rac{R_n^{\Omega}}{R_s} ar{s}_1 rac{R_s}{R_n} rac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)} \ & \sim R_n^{\Omega-1} ar{s}_1 rac{1}{1 - (R_s/R_n)} ext{ as } \Omega 
endaligned$$

#### Introduction

River Networks

Allometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations Models





### Continued ...

•

$$\begin{aligned} & \boldsymbol{a}_{\Omega} \propto \frac{R_{n}^{\Omega}}{R_{s}} \bar{\boldsymbol{s}}_{1} \sum_{\omega=1}^{\Omega} \left(\frac{R_{s}}{R_{n}}\right)^{\omega} \\ & = \frac{R_{n}^{\Omega}}{R_{s}} \bar{\boldsymbol{s}}_{1} \frac{R_{s}}{R_{n}} \frac{1 - (R_{s}/R_{n})^{\Omega}}{1 - (R_{s}/R_{n})} \\ & \sim R_{n}^{\Omega-1} \bar{\boldsymbol{s}}_{1} \frac{1}{1 - (R_{s}/R_{n})} \text{ as } \Omega \nearrow \end{aligned}$$

▶ So,  $a_{\Omega}$  is growing like  $R_n^{\Omega}$  and therefore:

$$R_n \equiv R_a$$

#### Introduction

River Networks

Definitions

definitions allometry aws

Stream Ordering Horton's Laws

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

Models





### Not quite:

... But this only a rough argument as Horton's laws do not imply a strict hierarchy

#### Introduction

River Networks

Reducing Horton







### Not quite:

- ... But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations

luctuations Models

References

Frame 64/121





### Not quite:

- But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Problem set 1 question....

#### Introduction

River Networks

Reducing Horton

References

Frame 64/121





# Equipartitioning:

### Intriguing division of area:

Observe: Combined area of basins of order ω independent of  $\omega$ .

#### Introduction

River Networks

Reducing Horton







## **Equipartitioning:**

### Intriguing division of area:

- Observe: Combined area of basins of order ω independent of  $\omega$ .
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.

#### Introduction

River Networks

Reducing Horton







## Equipartitioning:

## Intriguing division of area:

- ▶ Observe: Combined area of basins of order  $\omega$  independent of  $\omega$ .
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

### Introduction

River Networks

Definitions

Allometry

llometry aws tream Order

ordering Horton's Laws Okunaga's Law

Reducing Horton Scaling relations Fluctuations

odels





## Equipartitioning:

## Intriguing division of area:

- ▶ Observe: Combined area of basins of order  $\omega$  independent of  $\omega$ .
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

► Reason:

$$n_{\omega} \propto (R_n)^{-\omega}$$
 $\bar{a}_{\omega} \propto (R_a)^{\omega} \propto n_{\omega}^{-1}$ 

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

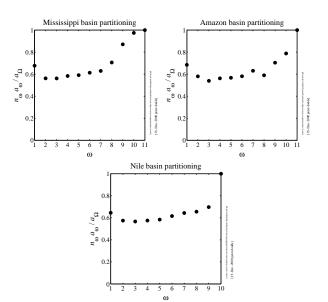
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





# Equipartitioning:

## Some examples:



#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton & Tokunaga
Reducing Horton

References

Frame 66/121



## **Outline**

### Introduction

### **River Networks**

Definitions Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

References

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton \( \infty\) Tokunaga
Reducing Horton
Scaling relations

References

Frame 67/121





## The story so far:

### Introduction

River Networks

Scaling relations







## The story so far:

 Natural branching networks are hierarchical, self-similar structures

### Introduction

River Networks

Scaling relations







## The story so far:

- Natural branching networks are hierarchical, self-similar structures
- ► Hierarchy is mixed

### Introduction

River Networks

Scaling relations

References

Frame 68/121





## The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture:  $T_k = T_1 R_{\tau}^{k-1}$ .

Scaling relations

References

Frame 68/121





## The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture:  $T_k = T_1 R_T^{k-1}$ .
- ▶ We have connected Tokunaga's and Horton's laws

### Introduction

River Networks

Definitions

Allometry Laws Stream Ordering

forton's Laws

Reducing Hortor Scaling relations Fluctuations

vioueis

## The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- ► Tokunaga's law describes detailed architecture:  $T_k = T_1 R_T^{k-1}$ .
- We have connected Tokunaga's and Horton's laws
- ▶ Only two Horton laws are independent  $(R_n = R_a)$

### Introduction

River Networks

Definitions

Allometry \_aws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations





## The story so far:

- Natural branching networks are hierarchical, self-similar structures
- ▶ Hierarchy is mixed
- ► Tokunaga's law describes detailed architecture:  $T_k = T_1 R_T^{k-1}$ .
- We have connected Tokunaga's and Horton's laws
- ▶ Only two Horton laws are independent  $(R_n = R_a)$
- ▶ Only two parameters are independent:  $(T_1, R_T) \Leftrightarrow (R_n, R_s)$

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Reducing Hortor
Scaling relations
Fluctuations

References

Frame 68/121



A little further...

### Introduction

River Networks

Scaling relations

Frame 69/121





### A little further...

Ignore stream ordering for the moment

#### Introduction

River Networks

Scaling relations

References

Frame 69/121





## A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network *p*.

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations Fluctuations





### A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network *p*.
- ► Each point *p* is associated with a basin and a longest stream length

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Hortor Scaling relations Fluctuations

lodels





### A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a?

### Introduction

River Networks

Scaling relations

References

Frame 69/121





### A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network p.
- ► Each point *p* is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a?
- ▶ Q: What is probability that the longest stream from p has length ℓ?

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations

References

Frame 69/121



### A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network *p*.
- ► Each point *p* is associated with a basin and a longest stream length
- ▶ Q: What is probability that the *p*'s drainage basin has area *a*?  $P(a) \propto a^{-\tau}$  for large *a*
- ▶ Q: What is probability that the longest stream from p has length ℓ?

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Hortor
Scaling relations
Fluctuations





### A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network *p*.
- ► Each point *p* is associated with a basin and a longest stream length
- ▶ Q: What is probability that the p's drainage basin has area a?  $P(a) \propto a^{-\tau}$  for large a
- ▶ Q: What is probability that the longest stream from p has length  $\ell$ ?  $P(\ell) \propto \ell^{-\gamma}$  for large  $\ell$

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations

.





### A little further...

- Ignore stream ordering for the moment
- ▶ Pick a random location on a branching network *p*.
- Each point p is associated with a basin and a longest stream length
- ▶ Q: What is probability that the p's drainage basin has area a?  $P(a) \propto a^{-\tau}$  for large a
- ▶ Q: What is probability that the longest stream from p has length  $\ell$ ?  $P(\ell) \propto \ell^{-\gamma}$  for large  $\ell$
- ▶ Roughly observed: 1.3  $\lesssim \tau \lesssim$  1.5 and 1.7  $\lesssim \gamma \lesssim$  2.0

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations Fluctuations





Probability distributions with power-law decays

### Introduction

River Networks

Scaling relations

References

Frame 70/121





## Probability distributions with power-law decays

▶ We see them everywhere:

#### Introduction

River Networks

Allometry Laws

aws tream Orde

Horton's Laws Tokunaga's Law

Reducing Horton Scaling relations

uctuations odels







## Probability distributions with power-law decays

- ▶ We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)

### Introduction

River Networks

Scaling relations







## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)

### Introduction

River Networks

Definitions

Allometry

stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

caling relatior uctuations odels





## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - ► Word frequency (Zipf's law) [24]

#### Introduction

River Networks

Definitions

Allometry \_aws

Stream Ordering Horton's Laws

Reducing Hortor

Scaling relations

Fluctuations

luctuations lodels





## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)

#### Introduction

River Networks

Scaling relations







## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - ► Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]

### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws

Horton ⇔ Toku
Reducing Hortor
Scaling relations

luctuations Models





## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems

#### Introduction

River Networks

Definitions

Allometry aws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations

Models





## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...

#### Introduction

River Networks

Definitions

llometry aws

tream Ordering orton's Laws okunaga's Law

Reducing Horton Scaling relations Fluctuations





## Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
- Our task is always to illuminate the mechanism...

Introduction

River Networks

Definitions

aws

Stream Ordering Horton's Laws

Reducing Hortor
Scaling relations
Fluctuations

References

Frame 70/121



## Connecting exponents

### Introduction

River Networks

Scaling relations

Frame 71/121





## Connecting exponents

▶ We have the detailed picture of branching networks (Tokunaga and Horton)

#### Introduction

River Networks

Scaling relations







## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton 
Toku
Reducing Hortor
Scaling relations
Fluctuations

Models

References

Frame 71/121



## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on  $P(\ell)$ ...

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Horton
Scaling relations
Fluctuations





## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on  $P(\ell)$ ...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.

### Introduction

River Networks

Definitions

Definitions Allometry Laws

Stream Ordering Horton's Laws Tokunaga's Law

Reducing Horton Scaling relations Fluctuations Models





## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on P(ℓ)...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low  $\omega$  and high  $\omega$  but not too much.)

### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on  $P(\ell)$ ...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low  $\omega$  and high  $\omega$  but not too much.)
- Next: place stick between teeth.

#### Introduction

River Networks

Definitions

Allometry

Laws Stream Ordering Horton's Laws

Reducing Horton
Scaling relations
Fluctuations
Models





## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on P(ℓ)...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low  $\omega$  and high  $\omega$  but not too much.)
- Next: place stick between teeth. Bite stick.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations





## Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story [20, 1, 2]
- ▶ Let's work on  $P(\ell)$ ...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low  $\omega$  and high  $\omega$  but not too much.)
- ▶ Next: place stick between teeth. Bite stick. Proceed.

#### Introduction

River Networks

Definitions

Allometry

Laws

Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





# Finding $\gamma$ :

### Introduction

River Networks

Scaling relations







## Finding $\gamma$ :

 Often useful to work with cumulative distributions, especially when dealing with power-law distributions.

#### Introduction

River Networks

Scaling relations







## Finding $\gamma$ :

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell = \ell_*}^{\ell_{\mathsf{max}}} P(\ell) \mathrm{d}\ell$$

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





## Finding $\gamma$ :

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell = \ell_*}^{\ell_{\mathsf{max}}} P(\ell) \mathrm{d}\ell$$

•

$$P_{>}(\ell_*)=1-P(\ell<\ell_*)$$

#### Introduction

River Networks

Definitions

Allometry

Laws Stream Ordering Horton's Laws

Horton  $\Leftrightarrow$  Tokun Reducing Horton Scaling relations Fluctuations





## Finding $\gamma$ :

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell = \ell_*}^{\ell_{\mathsf{max}}} P(\ell) \mathrm{d}\ell$$

$$P_{>}(\ell_{*}) = 1 - P(\ell < \ell_{*})$$

Also known as the exceedance probability.

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References

Scaling relations





## Finding $\gamma$ :

► The connection between P(x) and  $P_>(x)$  when P(x) has a power law tail is simple:

Introduction

River Networks

letinitions

aws

lorton's Laws

orton ⇔ Toku educing Horton

Scaling relations Fluctuations Models





### Finding $\gamma$ :

- ► The connection between P(x) and  $P_{>}(x)$  when P(x) has a power law tail is simple:
- ▶ Given  $P(\ell) \sim \ell^{-\gamma}$  large  $\ell$  then for large enough  $\ell_*$

$$P_{>}(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \,\mathrm{d}\ell$$

etinitions lometry

tream Ordering orton's Laws

Horton 

Tokur
Reducing Horton
Scaling relations
Fluctuations

References

. .0.0.0.0..000



### Finding $\gamma$ :

- ► The connection between P(x) and P>(x) when P(x) has a power law tail is simple:
- ▶ Given  $P(\ell) \sim \ell^{-\gamma}$  large  $\ell$  then for large enough  $\ell_*$

$$P_{>}(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \,\mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_{\rm m}}^{\ell_{\rm max}} \frac{\ell^{-\gamma}}{\ell} d\ell$$

### Introduction

River Networks

Definitions

lometry

stream Ordering forton's Laws

Horton  $\Leftrightarrow$  Tokur Reducing Horton Scaling relations Fluctuations



### Finding $\gamma$ :

- ► The connection between P(x) and  $P_{>}(x)$  when P(x) has a power law tail is simple:
- ▶ Given  $P(\ell) \sim \ell^{-\gamma}$  large  $\ell$  then for large enough  $\ell_*$

$$egin{aligned} P_{>}(\ell_*) &= \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \, \mathrm{d}\ell \ &\sim \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} rac{\ell^{-\gamma}}{\mathrm{d}\ell} \mathrm{d}\ell \ &= rac{\ell^{-\gamma+1}}{-\gamma+1} igg|_{\ell=\ell_*}^{\ell_{\mathsf{max}}} \end{aligned}$$

### Introduction

River Networks

Definitions

llometry

Stream Ordering Horton's Laws

Horton ⇔ Tokur Reducing Horton Scaling relations Fluctuations





### Finding $\gamma$ :

- ► The connection between P(x) and  $P_{>}(x)$  when P(x) has a power law tail is simple:
- ▶ Given  $P(\ell) \sim \ell^{-\gamma}$  large  $\ell$  then for large enough  $\ell_*$

$$P_{>}(\ell_{*}) = \int_{\ell=\ell_{*}}^{\ell_{\mathsf{max}}} P(\ell) \, \mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_{*}}^{\ell_{\mathsf{max}}} \frac{\ell^{-\gamma}}{\mathrm{d}\ell} \, \mathrm{d}\ell$$

$$= \frac{\ell^{-\gamma+1}}{-\gamma+1} \Big|_{\ell=\ell_{*}}^{\ell_{\mathsf{max}}}$$

$$\propto \ell_{*}^{-\gamma+1} \quad \text{for } \ell_{\mathsf{max}} \gg \ell_{*}$$

### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws

Horton ⇔ Tokur Reducing Horton Scaling relations Fluctuations

References

Frame 73/121



## Finding $\gamma$ :

▶ Aim: determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$ 

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Bedusing Horton

References

Scaling relations





## Finding $\gamma$ :

- ▶ Aim: determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$
- Assume some spatial sampling resolution Δ

### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws Tokunaga's Law

Reducing Hortor
Scaling relations
Fluctuations

References

Frame 74/121



## Finding $\gamma$ :

- ▶ Aim: determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$
- Assume some spatial sampling resolution Δ
- ▶ Landscape is broken up into grid of  $\Delta \times \Delta$  sites

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations





### Finding $\gamma$ :

- ▶ Aim: determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$
- Assume some spatial sampling resolution Δ
- ▶ Landscape is broken up into grid of  $\Delta \times \Delta$  sites
- ▶ Approximate  $P_>(\ell_*)$  as

$$P_{>}(\ell_*) = \frac{N_{>}(\ell_*; \Delta)}{N_{>}(0; \Delta)}.$$

where  $N_{>}(\ell_*; \Delta)$  is the number of sites with main stream length  $> \ell_*$ .

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Scaling relations





## Finding $\gamma$ :

- ▶ Aim: determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$
- Assume some spatial sampling resolution Δ
- ▶ Landscape is broken up into grid of  $\Delta \times \Delta$  sites
- ▶ Approximate  $P_>(\ell_*)$  as

$$P_{>}(\ell_*) = \frac{N_{>}(\ell_*; \Delta)}{N_{>}(0; \Delta)}.$$

where  $N_{>}(\ell_*; \Delta)$  is the number of sites with main stream length  $> \ell_*$ .

▶ Use Horton's law of stream segments:  $s_{\omega}/s_{\omega-1} = R_s...$ 

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton
Scaling relations
Fluctuations
Models





## Finding $\gamma$ :

▶ Set  $\ell_* = \ell_\omega$  for some 1  $\ll \omega \ll \Omega$ .

### Introduction

River Networks

Scaling relations





### Finding $\gamma$ :

▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = \frac{N_{>}(\ell_{\omega}; \Delta)}{N_{>}(0; \Delta)}$$

### Introduction

River Networks

Allometry Laws

Stream Ordering

Horton's Laws
Tokunaga's Law

Reducing Horton
Scaling relations

uctuations odels





### Finding $\gamma$ :

▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = \frac{\textit{N}_{>}(\ell_{\omega};\Delta)}{\textit{N}_{>}(0;\Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}$$

### Introduction

River Networks

Allometry Laws

Stream Ordering Horton's Laws

Tokunaga's Law Horton ⇔ Tokunag

Scaling relations

▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$\textit{P}_{>}(\ell_{\omega}) = \frac{\textit{N}_{>}(\ell_{\omega};\Delta)}{\textit{N}_{>}(0;\Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}$$

Δ's cancel

### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Scaling relations Fluctuations Models





▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = \frac{\textit{N}_{>}(\ell_{\omega};\Delta)}{\textit{N}_{>}(0;\Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is  $a_{\Omega}\rho_{\rm dd}$ , a constant.

### Introduction

River Networks

Allometry Laws

> tream Ordering orton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

luctuations lodels





▶ Set  $\ell_* = \ell_\omega$  for some 1  $\ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = \frac{\textit{N}_{>}(\ell_{\omega};\Delta)}{\textit{N}_{>}(0;\Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} \textit{n}_{\omega'} \textit{s}_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is  $a_{\Omega} \rho_{\rm dd}$ , a constant.
- ► So...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}$$

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Scaling relations





## Finding $\gamma$ :

▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = rac{N_{>}(\ell_{\omega};\Delta)}{N_{>}(0;\Delta)} \simeq rac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is  $a_{\Omega} \rho_{\rm dd}$ , a constant.
- ► So...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega}$$

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Scaling relations





▶ Set  $\ell_* = \ell_\omega$  for some 1  $\ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = rac{N_{>}(\ell_{\omega};\Delta)}{N_{>}(0;\Delta)} \simeq rac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is  $a_{\Omega} \rho_{\rm dd}$ , a constant.
- ► So... using Horton's laws...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'})$$

### Introduction

River Networks

Definitions

Allometry Laws

> stream Ordering forton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations

References

Frame 75/121

### Finding $\gamma$ :

▶ Set  $\ell_* = \ell_\omega$  for some  $1 \ll \omega \ll \Omega$ .

$$P_{>}(\ell_{\omega}) = rac{N_{>}(\ell_{\omega};\Delta)}{N_{>}(0;\Delta)} \simeq rac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is  $a_{\Omega}\rho_{\rm dd}$ , a constant.
- ► So... using Horton's laws...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations



▶ We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

aws tream Order

lorton's Laws okunaga's Law

Reducing Horton Scaling relations Fluctuations





▶ We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

Cleaning up irrelevant constants:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_{s}}{R_{n}}\right)^{\omega'}$$

### Introduction

River Networks

Definitions

llometry

ream Ordering orton's Laws kunaga's Law

Reducing Hortor Scaling relations Fluctuations

. .





## Finding $\gamma$ :

▶ We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$

Cleaning up irrelevant constants:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_{s}}{R_{n}}\right)^{\omega'}$$

• Change summation order by substituting  $\omega'' = \Omega - \omega'$ .

### Introduction

River Networks

Definitions

aws

tream Ordering orton's Laws okunaga's Law

Reducing Hortor Scaling relations Fluctuations





## Finding $\gamma$ :

▶ We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

Cleaning up irrelevant constants:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_{s}}{R_{n}}\right)^{\omega'}$$

- ► Change summation order by substituting  $\omega'' = \Omega \omega'$ .
- ▶ Sum is now from  $\omega'' = 0$  to  $\omega'' = \Omega \omega 1$

#### Introduction

River Networks

Definitions

efinitions illometry aws

tream Ordering lorton's Laws

Reducing Hortor Scaling relations Fluctuations





## Finding $\gamma$ :

▶ We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

Cleaning up irrelevant constants:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega'}$$

- ► Change summation order by substituting  $\omega'' = \Omega \omega'$ .
- Sum is now from  $\omega'' = 0$  to  $\omega'' = \Omega \omega 1$  (equivalent to  $\omega' = \Omega$  down to  $\omega' = \omega + 1$ )

### Introduction

River Networks

Definitions

efinitions Illometry aws

tream Ordering lorton's Laws bkunaga's Law lorton ⇔ Tokunaga

Reducing Horton Scaling relations Fluctuations Models

References

Frame 76/121



## Finding $\gamma$ :

•

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(rac{R_s}{R_n}
ight)^{\Omega-\omega''}$$

### Introduction

River Networks

Definitions

Allometry Laws

aws tream Orde

Horton's Laws Tokunaga's Law Horton ⇔ Tokunan

Scaling relations Fluctuations

luctuations Iodels





## Finding $\gamma$ :

•

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton 
Tokunaga's Law

Scaling relations
Fluctuations
Models

References

Frame 77/121



$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

▶ Since  $R_n < R_s$  and  $1 \ll \omega \ll \Omega$ ,

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Tokunaga's Law
Horton ⇔ Tokunar
Reducing Horton
Scaling relations
Fluctuations





## Finding $\gamma$ :

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

▶ Since  $R_n < R_s$  and  $1 \ll \omega \ll \Omega$ ,

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{\Omega-\omega}$$

again using 
$$\sum_{i=0}^{n} a^{n} = (a^{i+1} - 1)/(a-1)$$

#### Introduction

River Networks

Definitions

aws

awa dream Ordering dorton's Laws okunaga's Law dorton ⇔ Tokunaga deducing Horton

Scaling relations Fluctuations Models



### Finding $\gamma$ :

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

▶ Since  $R_n < R_s$  and  $1 \ll \omega \ll \Omega$ ,

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{\Omega-\omega} \propto \left(rac{R_n}{R_s}
ight)^{-\omega}$$

again using 
$$\sum_{i=0}^{n} a^{n} = (a^{i+1} - 1)/(a-1)$$

#### Introduction

River Networks

diometry aws

tream Ordering
orton's Laws
okunaga's Law
orton ⇔ Tokunaga

Scaling relations Fluctuations Models

References

Frame 77/121



### Finding $\gamma$ :

▶ Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega}$$

#### Introduction

River Networks

Scaling relations





### Finding $\gamma$ :

▶ Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega} \, = e^{-\omega \ln(R_n/R_s)}$$

Scaling relations

### Finding $\gamma$ :

► Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

▶ Need to express right hand side in terms of  $\ell_{\omega}$ .

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





### Finding $\gamma$ :

► Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

- ▶ Need to express right hand side in terms of  $\ell_{\omega}$ .
- ▶ Recall that  $\ell_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$ .

#### Introduction

River Networks

Definitions

Allometry Laws

ream Ordering orton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

luctuations Models

References

Frame 78/121



### Finding $\gamma$ :

► Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega} \, = e^{-\omega \ln(R_n/R_s)}$$

- ▶ Need to express right hand side in terms of  $\ell_{\omega}$ .
- ▶ Recall that  $\ell_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$ .

$$\ell_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$

#### Introduction

River Networks

Allometry aws

tream Ordering orton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations

Models





### Finding $\gamma$ :

▶ Therefore:

$$P_{>}(\ell_{\omega}) \propto e^{-\omega \ln(R_n/R_s)}$$

#### Introduction

River Networks

Definitions

Allometry Laws

Laws Stream Or

Horton's Laws Tokunaga's Law

Reducing Horton
Scaling relations

uctuations odels





### Finding $\gamma$ :

▶ Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
ight)^{-\ln(R_n/R_s)/\ln(R_s)}$$

efinitions llometry

aws

tream Ordering lorton's Laws

Tokunagas Lav Horton ⇔ Tol Beducing Horti

Scaling relations Fluctuations

Deference



### Finding $\gamma$ :

▶ Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
ight)^{-\ln(R_n/R_s)/\ln(R_s)}$$

$$\propto \ell_{\omega} - \ln(R_n/R_s) / \ln R_s$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





### Finding $\gamma$ :

▶ Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
ight)^{-\ln(R_n/R_s)/\ln(R_s)}$$

$$\propto \ell_{\omega} - \ln(R_n/R_s) / \ln R_s$$

$$=\ell_{\omega}^{-(\ln R_n-\ln R_s)/\ln R_s}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Horton
Scaling relations
Fluctuations
Models

References

Frame 79/121



### Finding $\gamma$ :

▶ Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
ight)^{-\ln(R_n/R_s)/\ln(R_s)}$$

$$\propto \ell_{\omega} - \ln(R_n/R_s) / \ln R_s$$

•

$$=\ell_{\odot}^{-(\ln R_n-\ln R_s)/\ln R_s}$$

•

$$=\ell_{\omega}^{-\ln R_n/\ln R_s+1}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Scaling relations





### Finding $\gamma$ :

▶ Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
ight)^{-\ln(R_n/R_s)/\ln(R_s)}$$

$$\propto \ell_{\omega} - \ln(R_n/R_s) / \ln R_s$$

•

$$=\ell_{\cdot,\cdot}^{-(\ln R_n-\ln R_s)/\ln R_s}$$

$$=\ell_{c}^{-\ln R_n/\ln R_s+1}$$

$$=\ell_{\omega}^{-\gamma+1}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton Scaling relations Fluctuations

References

Frame 79/121



### Finding $\gamma$ :

And so we have:

$$\gamma = \ln R_n / \ln R_s$$

#### Introduction

River Networks

Definitions

Allometr Laws

Laws Stream Orc

Horton's Laws Tokunaga's Law

Horton ⇔ Toki Reducing Horto

Scaling relations Fluctuations





### Finding $\gamma$ :

And so we have:

$$\gamma = \ln R_n / \ln R_s$$

Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

#### Introduction

River Networks

Definitions

Allometry Laws

stream Ordering Horton's Laws

Reducing Hortor
Scaling relations
Fluctuations

Models

References

Frame 80/121



### Finding $\gamma$ :

And so we have:

$$\gamma = \ln R_{\rm n}/\ln R_{\rm s}$$

Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

 Such connections between exponents are called scaling relations

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





### Finding $\gamma$ :

And so we have:

$$\gamma = \ln R_n / \ln R_s$$

Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws Tokunaga's Law

Reducing Hortor
Scaling relations
Fluctuations
Models





### Hack's law: [6]





### Introduction

River Networks

Allometry Laws

Stream Ordering

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton

Scaling relations
Fluctuations





### Hack's law: [6]

$$\ell \propto a^h$$

▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Horton
Scaling relations

lodels





### Hack's law: [6]

Þ

$$\ell \propto \textit{a}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- Use Horton laws to connect h to Horton ratios:

$$\ell_\omega \propto R_s^\omega$$
 and  $a_\omega \propto R_n^\omega$ 

# Introduction River Networks

Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton Scaling relations Fluctuations

. .





### Hack's law: [6]

Þ

$$\ell \propto \textbf{\textit{a}}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- ▶ Use Horton laws to connect *h* to Horton ratios:

$$\ell_\omega \propto R_s^\omega$$
 and  $a_\omega \propto R_n^\omega$ 

Observe:

$$\ell_\omega \propto e^{\omega \ln R_s}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton Scaling relations Fluctuations

References

Frame 81/121



### Hack's law: [6]

Þ

$$\ell \propto \textbf{\textit{a}}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- ▶ Use Horton laws to connect *h* to Horton ratios:

$$\ell_\omega \propto R_s^\omega$$
 and  $a_\omega \propto R_n^\omega$ 

Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}
ight)^{\ln R_s/\ln R_n}$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Tokuna
Reducing Horton
Scaling relations

luctuations Models





### Hack's law: [6]

$$\ell \propto \textit{a}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- ▶ Use Horton laws to connect *h* to Horton ratios:

$$\ell_{\omega} \propto R_{s}^{\omega}$$
 and  $a_{\omega} \propto R_{n}^{\omega}$ 

Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}
ight)^{\ln R_s/\ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s/\ln R_n}$$

#### Introduction

River Networks

Definitions

Allometry

Stream Ordering Horton's Laws

Horton ⇔ Toku
Reducing Horton
Scaling relations
Fluctuations





### Hack's law: [6]

Þ

$$\ell \propto \textbf{\textit{a}}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- ▶ Use Horton laws to connect *h* to Horton ratios:

$$\ell_{\omega} \propto R_{s}^{\omega}$$
 and  $a_{\omega} \propto R_{n}^{\omega}$ 

Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}
ight)^{\ln R_s/\ln R_n}$$

$$\propto (R_n^{\omega})^{\ln R_s/\ln R_n} = a_{\omega}^{\ln R_s/\ln R_n}$$

#### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Reducing Horton
Scaling relations
Fluctuations





### Hack's law: [6]

Þ

$$\ell \propto \textit{a}^{\textit{h}}$$

- ▶ Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .
- ▶ Use Horton laws to connect *h* to Horton ratios:

$$\ell_{\omega} \propto R_{s}^{\omega}$$
 and  $a_{\omega} \propto R_{n}^{\omega}$ 

Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}
ight)^{\ln R_s/\ln R_n}$$

$$\propto (R_n^{\omega})^{\ln R_s/\ln R_n} = a_{\omega}^{\ln R_s/\ln R_n} \Rightarrow h = \ln R_s/\ln R_n$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Reducing Horton
Scaling relations
Fluctuations
Models





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

#### Introduction

River Networks

Scaling relations

References





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

> relation: scaling relation/parameter: [2]

#### Introduction

River Networks

Scaling relations

References





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d

#### Introduction

River Networks

Scaling relations

References





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\cal L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$

Scaling relations

References





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$

Scaling relations

References





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\sf L}^{\sf d}$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = R_n$

llometry

ream Ordering orton's Laws

Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton

Scaling relations
Fluctuations
Medicle

Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

scaling relation/parameter: [2]
d
$T_1 = R_n - R_s - 2 + 2R_s/R_n$
$R_T = R_s$
$R_n$
$R_a = \frac{R_n}{R_n}$
$R_\ell =  extstyle{R_{ extstyle S}}$

llometry

ream Ordering orton's Laws

Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

caing relations luctuations lodels





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\sf L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = R_n$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{ extit{R}}_{\ell}$	$R_\ell = R_s$
$\ell \sim \pmb{a^h}$	$h = \log \frac{R_s}{\log R_n}$

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\sf L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extbf{ extit{R}}_{\ell}$	$R_\ell = R_s$
$\ell \sim \pmb{a^h}$	$h = \log R_s / \log R_n$
$oldsymbol{a} \sim oldsymbol{\mathcal{L}}^{oldsymbol{D}}$	D = d/h

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extcolor{R}_{\ell}$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log \frac{R_s}{\log R_n}$
$a\sim L^D$	D = d/h
${\it L}_{\perp} \sim {\it L}^{\it H}$	H = d/h - 1

ometry ws

eam Ordering rton's Laws

Horton ⇔ Tokunag Reducing Horton Scaling relations

References

. .0.0.0.000



Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}= extcolor{R}_{\ell}$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log \frac{R_s}{\log R_n}$
$a\sim L^D$	D = d/h
$L_{\perp} \sim L^{H}$	H = d/h - 1
$P(a) \sim a^{- au}$	au = 2 - h

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Scaling relations





Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\cal L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log R_s / \log R_n$
$a\sim L^D$	D = d/h
$L_{\perp} \sim L^{H}$	H = d/h - 1
$P(a) \sim a^{- au}$	au = 2 - h
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$

llometry aws

ream Ordering orton's Laws

Reducing Horton Scaling relations

References



## Connecting exponents

Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

relation:	scaling relation/parameter: [2]
$\ell \sim {\sf L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	$R_n$
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log \frac{R_s}{\log R_n}$
$a\sim L^D$	D = d/h
$L_{\perp} \sim L^{H}$	H = d/h - 1
$P(a) \sim a^{- au}$	au = 2 - h
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim  extbf{ extit{a}}^eta$	$\beta = 1 + h$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga Reducing Horton

References

Scaling relations

Frame 82/121



## Connecting exponents

Only 3 parameters are independent: e.g., take d,  $R_n$ , and  $R_s$ 

scaling relation/parameter: [2]
d
$T_1 = R_n - R_s - 2 + 2R_s/R_n$
$R_T = R_s$
$R_n$
$R_a = \frac{R_n}{R_n}$
$R_\ell = R_s$
$h = \log \frac{R_s}{l} \log \frac{R_n}{l}$
D = d/h
H = d/h - 1
au = 2 - h
$\gamma = 1/h$
$\beta = 1 + h$
$arphi= extsf{d}$

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

References

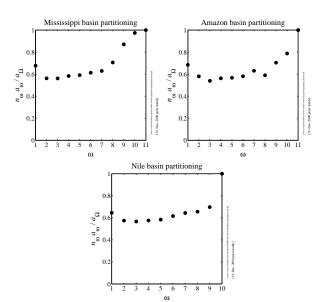
Scaling relations

Frame 82/121



## Equipartitioning reexamined:

### Recall this story:



#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

References

Frame 83/121



# Equipartitioning

What about

$$P(a) \sim a^{-\tau}$$
 ?

#### Introduction

River Networks

Scaling relations

References





What about

$$P(a) \sim a^{-\tau}$$
 ?

▶ Since  $\tau$  > 1, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

### Introduction

River Networks

Definitions

aws

Stream Orderi Horton's Laws

Tokunaga's Law Horton ⇔ Toku

Scaling relations
Fluctuations

lodels

References



# Equipartitioning

▶ What about

$$P(a) \sim a^{-\tau}$$
 ?

▶ Since  $\tau$  > 1, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

▶ *P*(*a*) overcounts basins within basins...

### Introduction

River Networks

Definitions

Allometry \_aws

Stream Orderii Horton's Laws

Horton ⇔ Toku Reducing Hortor

Scaling relations
Fluctuations

viodeis

References



What about

$$P(a) \sim a^{-\tau}$$
 ?

▶ Since  $\tau > 1$ , suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

- ▶ *P*(*a*) overcounts basins within basins...
- while stream ordering separates basins...

### Introduction

River Networks

Definitions

Allometry \_aws

Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

Models

References



### **Outline**

Introduction

### River Networks

Definitions

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

**Fluctuations** 

Models

References

### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton & Tokunaga
Reducing Horton
Scaling relations
Fluctuations

References

Frame 85/121





## Moving beyond the mean:

### Introduction

River Networks

Fluctuations

Frame 86/121





### Moving beyond the mean:

▶ Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_{\omega}/\bar{s}_{\omega-1}=R_{s}$$

#### Introduction

River Networks

Definitions

Allometry Laws

stream Ordering forton's Laws

Horton ⇔ Tokuna Reducing Horton Scaling relations

Scaling relations Fluctuations Models





### Moving beyond the mean:

 Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$ar{s}_{\omega}/ar{s}_{\omega-1}=R_{s}$$

 Natural generalization to consideration relationships between probability distributions

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations

Fluctuations Models





## Moving beyond the mean:

▶ Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$ar{s}_{\omega}/ar{s}_{\omega-1}=R_s$$

- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Tokul Reducing Horton Scaling relations Fluctuations





### Moving beyond the mean:

 Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_{\omega}/\bar{s}_{\omega-1}=R_{s}$$

- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness...

### Introduction

River Networks

Definitions

Allometry Laws

Stream Ordering Horton's Laws

Horton ⇔ Toku
Reducing Horton
Scaling relations

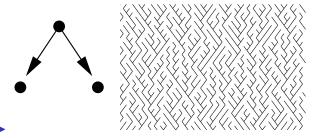
lodels





# A toy model—Scheidegger's model

### Directed random networks [12, 13]



•

$$P(\searrow) = P(\swarrow) = 1/2$$

- Flow is directed downwards
- Useful and interesting test case—more later...

### Introduction

River Networks

Definitions Allometry Laws

Stream Ordering Horton's Laws Tokunaga's Law

Reducing Horton Scaling relations Fluctuations





$$\blacktriangleright \ \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega}F_{\ell}(\ell/R_{\ell}^{\omega})$$

### Introduction

River Networks

Allometry

aws

Horton's Laws

Reducing Hortor Scaling relations

Fluctuations Models





- $\blacktriangleright \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega} F_{\ell}(\ell/R_{\ell}^{\omega})$
- $lack ar a_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$

### Introduction

River Networks

Allometry Laws

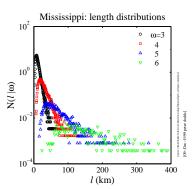
Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Horton Scaling relations Fluctuations





- $\blacktriangleright \ \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow \textit{N}(\ell|\omega) = (R_{\textit{n}}R_{\ell})^{-\omega}F_{\ell}(\ell/R_{\ell}^{\omega})$
- $lack ar a_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$



### Introduction

River Networks

Allometry aws

Stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations

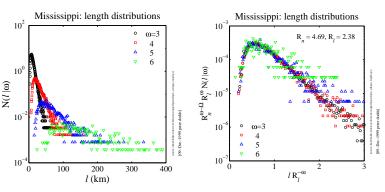
References

Frame 88/121



$$\blacktriangleright \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega} F_{\ell}(\ell/R_{\ell}^{\omega})$$

$$lack ar a_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$$



Scaling collapse works well for intermediate orders

#### Introduction

River Networks

Definitions

Allometry

Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Fluctuations

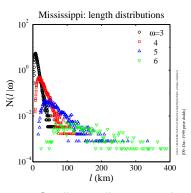
Models

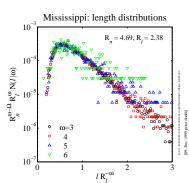
References





- $\blacktriangleright \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega} F_{\ell}(\ell/R_{\ell}^{\omega})$
- ullet  $ar{a}_{\omega} \propto (R_a)^{\omega} \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^{\omega})$





- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

#### Introduction

River Networks

Definitions

Allometry

aws tream Ordering orton's Laws

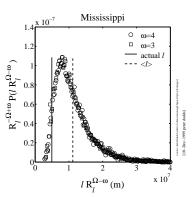
Horton ⇔ Tokuna Reducing Horton Scaling relations Fluctuations

References

Frame 88/121



How well does overall basin fit internal pattern?



#### Introduction

River Networks

efinitions llometry aws

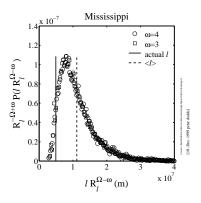
Stream Ordering Horton's Laws

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations

References



How well does overall basin fit internal pattern?



➤ Actual length = 4920 km (at 1 km res)

### Introduction

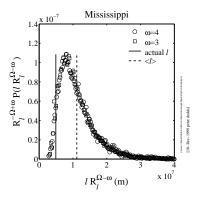
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Horton ⇔ Tokunag Reducing Horton Scaling relations Fluctuations

References



How well does overall basin fit internal pattern?



- ► Actual length = 4920 km (at 1 km res)
- ► Predicted Mean length = 11100 km

#### Introduction

River Networks

Definitions

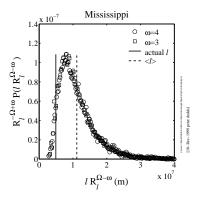
aws stream Ordering

Tokunaga's Law
Horton ⇔ Tokunag
Reducing Horton
Scaling relations
Fluctuations

References



How well does overall basin fit internal pattern?



- ► Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km

# Introduction River Networks

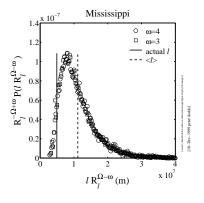
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Fluctuations



How well does overall basin fit internal pattern?



- ► Actual length = 4920 km (at 1 km res)
- Predicted Mean length= 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %

# Introduction River Networks

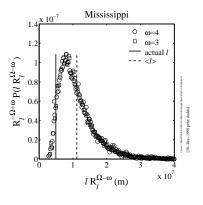
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

References

Fluctuations



How well does overall basin fit internal pattern?



- ► Actual length = 4920 km (at 1 km res)
- Predicted Mean length= 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.

# Introduction River Networks

Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Fluctuations

Models

References





Comparison of predicted versus measured main stream lengths for large scale river networks (in 10<sup>3</sup> km):

basin:	$\ell_{\Omega}$	$ar{\ell}_{\Omega}$	$\sigma_\ell$	$\ell/ar{\ell}_{\Omega}$	$\sigma_\ell/ar\ell_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	а	$ar{a}_\Omega$	$\sigma_{a}$	$a/ar{a}_\Omega$	$\sigma_{a}/ar{a}_{\Omega}$
Mississippi	а 2.74	$\bar{a}_{\Omega}$ 7.55	<i>σ</i> <sub>a</sub> 5.58	$a/\bar{a}_{\Omega}$ 0.36	$\sigma_a/\bar{a}_\Omega$ 0.74
Mississippi Amazon			u	,	
	2.74	7.55	5.58	0.36	0.74
Amazon	2.74 5.40	7.55 9.07	5.58 8.04	0.36 0.60	0.74
Amazon Nile	2.74 5.40 3.08	7.55 9.07 0.96	5.58 8.04 0.79	0.36 0.60 3.19	0.74 0.89 0.82

### Introduction

River Networks

Fluctuations

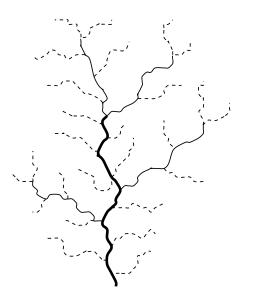
References

Frame 90/121





## Combining stream segments distributions:



 Stream segments sum to give main stream lengths

$$\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$

### Introduction

River Networks

Definitions

Allometry

ometry ws ream Ordering

ream Ordering orton's Laws kunaga's Law

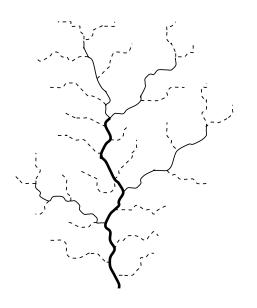
Reducing Hortor
Scaling relations
Fluctuations
Models

References

Frame 91/121



# Combining stream segments distributions:



 Stream segments sum to give main stream lengths

 $\ell_{\omega} = \sum_{n=1}^{\mu=\omega} s_n$ 

 $ightharpoonup P(\ell_{\omega})$  is a convolution of distributions for the  $s_{\omega}$ 

#### Introduction

River Networks

Definitions

Allometry

stream Ordering Horton's Laws okunaga's Law

Reducing Horto Scaling relations Fluctuations Models

References

Frame 91/121



Sum of variables  $\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$  leads to convolution of distributions:

$$\textit{N}(\ell|\omega) = \textit{N}(\textit{s}|1) * \textit{N}(\textit{s}|2) * \cdots * \textit{N}(\textit{s}|\omega)$$

### Introduction

River Networks

llometry

tream Ordering orton's Laws

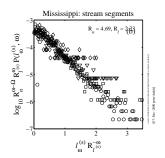
Horton ⇔ Tokun: Reducing Horton Scaling relations Fluctuations





Sum of variables  $\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$  leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



$$N(s|\omega) = rac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega
ight)$$

$$F(x) = e^{-x/\xi}$$

Mississippi:  $\xi \simeq 900$  m.

### Introduction

River Networks

efinitions llometry

stream Ordering

Horton ⇔ Tokul Reducing Horton Scaling relations

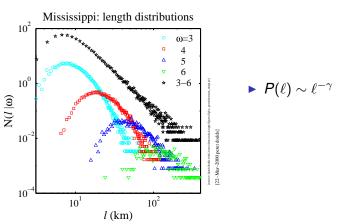
Models

References

Frame 92/121



Next level up: Main stream length distributions must combine to give overall distribution for stream length



#### Introduction

River Networks

efinitions

aws tream Ordering lorton's Laws

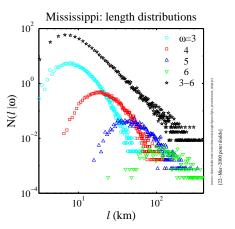
Horton ⇔ Tokur Reducing Horton Scaling relations Fluctuations

References

Frame 93/121



Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $ightharpoonup P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting...

#### Introduction

River Networks

llometry

tream Ordering lorton's Laws

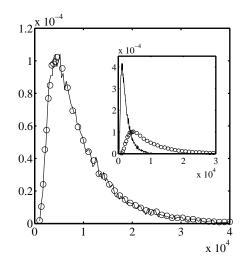
Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations

References

Frame 93/121



Number and area distributions for the Scheidegger model  $P(n_{1.6})$  versus  $P(a_6)$ .



#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Scaling relations Fluctuations Models

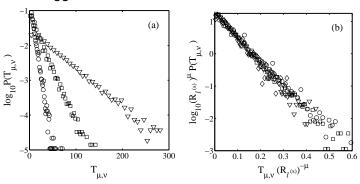
References

Frame 94/121



# Generalizing Tokunaga's law

### Scheidegger:



- lackbox Observe exponential distributions for  $T_{\mu,
  u}$
- Scaling collapse works using R<sub>s</sub>

#### Introduction

River Networks

Definitions

Allometry

Laws

Eaws Stream Ordering Horton's Laws Tokunaga's Law Horton ⇔ Tokunaga

Fluctuations

Models

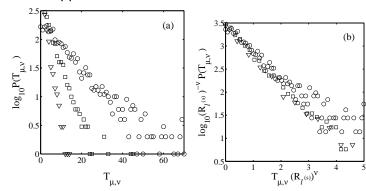
References

Frame 95/121



# Generalizing Tokunaga's law

### Mississippi:



Same data collapse for Mississippi...

#### Introduction

River Networks

Definitions

ometry ws

tream Ordering orton's Laws okunaga's Law

Reducing Hortor Scaling relations Fluctuations





# Generalizing Tokunaga's law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[ T_{\mu,\nu}/(R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$P(s_{\mu}) \Leftrightarrow P(T_{\mu,\nu})$$

- Exponentials arise from randomness.
- ▶ Look at joint probability  $P(s_{\mu}, T_{\mu,\nu})$ .

### Introduction

River Networks

llometry

Stream Ordering Horton's Laws

Horton ⇔ Tokun Reducing Horton

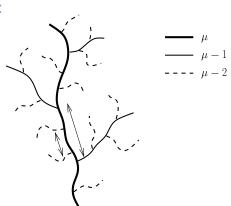
Scaling relations Fluctuations Models





## Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Fluctuations

Models

References

Frame 98/121



Follow streams segments down stream from their beginning

#### Introduction

River Networks

Fluctuations

References

Frame 99/121





- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1} \xi_s$$

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Fluctuations

Models

References





- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$ilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1} \xi_s$$

Probability decays exponentially with stream order

#### Introduction

River Networks

Definitions

Allometry

Laws

.aws
Stream Ordering
Horton's Laws
Fokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations

Fluctuations

Models

References





- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$ilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1} \xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed

#### Introduction

River Networks

Definitions

Allometry

Laws

Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations





- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$ilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1} \xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- ➤ ⇒ random spatial distribution of stream segments

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Reducing Horton Scaling relations Fluctuations Models





Joint distribution for generalized version of Tokunaga's law:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, \nu}} 
ho_{
u}^{T_{\mu, 
u}} (1 - p_{
u} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, 
u} - 1}$$

#### where

•  $p_{\nu} =$  probability of absorbing an order  $\nu$  side stream

#### Introduction

River Networks

lometry

ream Ordering

Tokunaga's Law Horton ⇔ Tokuna

Scaling relation
Fluctuations
Models





Joint distribution for generalized version of Tokunaga's law:

$$P(s_{\mu}, T_{\mu, 
u}) = ilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, 
u}} 
ho_{
u}^{T_{\mu, 
u}} (1 - p_{
u} - ilde{p}_{\mu})^{s_{\mu} - T_{\mu, 
u} - 1}$$

#### where

- $p_{\nu} =$  probability of absorbing an order  $\nu$  side stream
- $m{ ilde{
  ho}}_{\mu}=$  probability of an order  $\mu$  stream terminating

#### Introduction

River Networks

Definitions

ometry

ream Ordering

Tokunaga's Law Horton ⇔ Tokuna

Scaling relations
Fluctuations
Models





Joint distribution for generalized version of Tokunaga's law:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, \nu}} 
ho_{
u}^{T_{\mu, 
u}} (1 - p_{
u} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, 
u} - 1}$$

#### where

- $p_{\nu} =$  probability of absorbing an order  $\nu$  side stream
- $ilde{p}_{\mu} = ext{probability of an order } \mu ext{ stream terminating }$
- Approximation: depends on distance units of  $s_{\mu}$
- ► In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

#### Introduction

River Networks

Definitions

lometry

tream Ordering orton's Laws

Horton ⇔ Tokur Reducing Horton Scaling relations Fluctuations



Now deal with thing:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} {s_{\mu} - 1 \choose T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$

River Networks

llometry

Stream Ordering

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations

References

Frame 101/121



Now deal with thing:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, \nu}} 
ho_{
u}^{T_{\mu, 
u}} (1 - p_{
u} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, 
u} - 1}$$

Set  $(x, y) = (s_{\mu}, T_{\mu, \nu})$  and  $q = 1 - p_{\nu} - \tilde{p}_{\mu}$ , approximate liberally.

#### Introduction

River Networks

Definitions

efinitions lometry iws

stream Ordering Horton's Laws

Reducing Horton Scaling relations Fluctuations





Now deal with thing:

$$P(s_{\mu}, T_{\mu, 
u}) = ilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, 
u}} 
ho_{
u}^{T_{\mu, 
u}} (1 - 
ho_{
u} - ilde{p}_{\mu})^{s_{\mu} - T_{\mu, 
u} - 1}$$

- ▶ Set  $(x, y) = (s_{\mu}, T_{\mu, \nu})$  and  $q = 1 p_{\nu} \tilde{p}_{\mu}$ , approximate liberally.
- Obtain

$$P(x, y) = Nx^{-1/2} [F(y/x)]^{x}$$

where

$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

#### Introduction

River Networks

efinitions lometry ws

tream Ordering

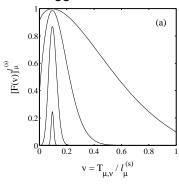
Horton  $\Leftrightarrow$  Tokur Reducing Horton Scaling relations Fluctuations

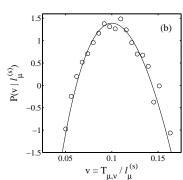




▶ Checking form of  $P(s_{\mu}, T_{\mu,\nu})$  works:

## Scheidegger:





#### Introduction

River Networks

Definitions

llometry aws tream Ordering

Stream Ordering Horton's Laws Tokunaga's Law Horton ⇔ Tokunaga Reducing Horton

References

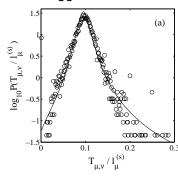
Fluctuations

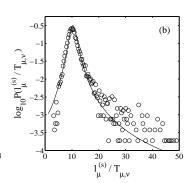
Frame 102/121



▶ Checking form of  $P(s_{\mu}, T_{\mu,\nu})$  works:

## Scheidegger:





#### Introduction

River Networks

Definitions

llometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokuna Reducing Horton Scaling relations Fluctuations

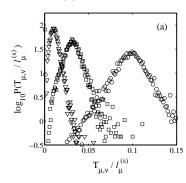
References

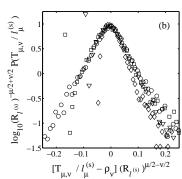
Frame 103/121



▶ Checking form of  $P(s_{\mu}, T_{\mu,\nu})$  works:

## Scheidegger:





#### Introduction

River Networks

Definitions

llometry aws

Stream Ordering Horton's Laws

Horton ⇔ Tokuna Reducing Horton Scaling relations Fluctuations

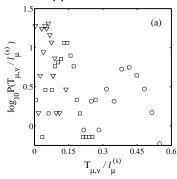
References

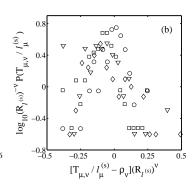
Frame 104/121



▶ Checking form of  $P(s_{\mu}, T_{\mu,\nu})$  works:

## Mississippi:





#### Introduction

River Networks

Definitions

llometry

Stream Ordering Horton's Laws

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations

References

Frame 105/121



## **Outline**

#### Introduction

### River Networks

Definitions Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

References

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Models

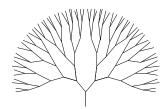
References

Frame 106/121





## Random subnetworks on a Bethe lattice [15]



#### Introduction

River Networks

Allometry

tream Ordering

Tokunaga's Law Horton ⇔ Tokunag

Scaling relat

Models

References

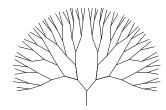
Frame 107/121





## Random subnetworks on a Bethe lattice [15]

Dominant theoretical concept for several decades.



#### Introduction

River Networks

lometry

ws eam Orderin

Horton's Laws Tokunaga's Law

Reducing Horte Scaling relation

Models

References

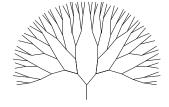
Frame 107/121





## Random subnetworks on a Bethe lattice [15]

- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.



## Introduction River Networks

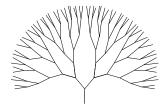
Models







## Random subnetworks on a Bethe lattice [15]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [8]

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering

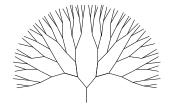
Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Hortor Scaling relations Fluctuations





## Random subnetworks on a Bethe lattice [15]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [8]
- But Bethe lattices unconnected with surfaces.

#### Introduction

River Networks

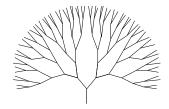
Models







## Random subnetworks on a Bethe lattice [15]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [8]
- But Bethe lattices unconnected with surfaces.
- ► In fact, Bethe lattices ~ infinite dimensional spaces (oops).

#### Introduction

River Networks

Definitions

Allometry

Laws

Stream Ordering

Horton's Laws

Tokunaga's Law

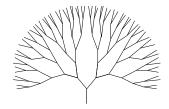
Horton ⇔ Tokunaga

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations Models





## Random subnetworks on a Bethe lattice [15]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [8]
- But Bethe lattices unconnected with surfaces.
- ► In fact, Bethe lattices ~ infinite dimensional spaces (oops).
- So let's move on...

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Horton ⇔ Tokun Reducing Horton Scaling relations Fluctuations Models

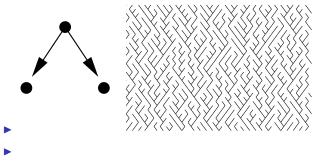
References

Frame 107/121



# Scheidegger's model

## Directed random networks [12, 13]



$$P(\searrow) = P(\swarrow) = 1/2$$

#### Introduction

River Networks

Allometry

aws

Horton's Laws

Tokunaga's Law Horton ⇔ Toku

Reducing Horto Scaling relation

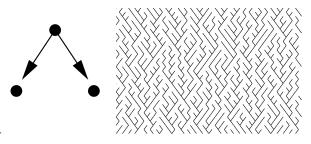
Models





# Scheidegger's model

## Directed random networks [12, 13]



•

$$P(\searrow) = P(\swarrow) = 1/2$$

► Functional form of all scaling laws exhibited but exponents differ from real world [18, 19, 17]

#### Introduction

River Networks

llometry

Stream Ordering Horton's Laws Tokunaga's Law

Reducing Horto

Models

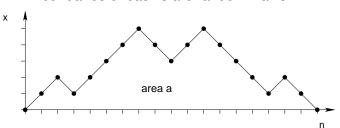




# A toy model—Scheidegger's model

### Random walk basins:

Boundaries of basins are random walks



#### Introduction

River Networks

Allometry

Stream Ordering

Tokunaga's Law Horton ⇔ Tokunag

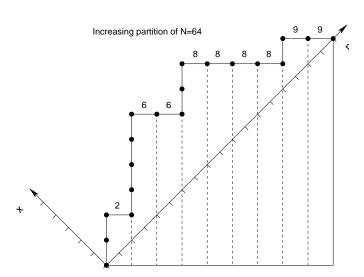
Scaling relation

Models





# Scheidegger's model



#### Introduction

River Networks

Definitions Allometry Laws

aws Stream Ordering Horton's Laws

Horton ⇔ Toku Reducing Horton Scaling relations

Scaling relations Fluctuations Models

Models

References

Frame 110/121



# Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions:

Introduction

River Networks

Models

References

Frame 111/121





$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .

#### Introduction

River Networks

Models





$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .

▶ Typical area for a walk of length n is  $\propto n^{3/2}$ :

$$\ell \propto a^{2/3}$$
.

#### Introduction

River Networks

llometry

aws

tream Ordering forton's Laws

Reducing Horto Scaling relations

Models

References

Frame 111/121



$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .

▶ Typical area for a walk of length n is  $\propto n^{3/2}$ :

$$\ell \propto a^{2/3}$$
.

Find  $\tau = 4/3$ , h = 2/3,  $\gamma = 3/2$ , d = 1.

#### Introduction

River Networks

Allometry

aws stream Orderin

lorton's Laws okunaga's Law lorton ⇔ Tokunaga

Scaling relations
Fluctuations
Models

100013

$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .

▶ Typical area for a walk of length n is  $\propto n^{3/2}$ :

$$\ell \propto a^{2/3}$$
.

- Find  $\tau = 4/3$ , h = 2/3,  $\gamma = 3/2$ , d = 1.
- Note  $\tau = 2 h$  and  $\gamma = 1/h$ .

#### Introduction

River Networks

Allometry

aws tream Orderin torton's Laws

orton's Laws okunaga's Law orton ⇔ Tokunaga

Reducing Horto Scaling relations Fluctuations

$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .

▶ Typical area for a walk of length n is  $\propto n^{3/2}$ :

$$\ell \propto a^{2/3}$$
.

- Find  $\tau = 4/3$ , h = 2/3,  $\gamma = 3/2$ , d = 1.
- Note  $\tau = 2 h$  and  $\gamma = 1/h$ .
- ▶  $R_n$  and  $R_\ell$  have not been derived analytically.

#### Introduction

River Networks

Allometry

Stream Ordering Horton's Laws

Reducing Horto
Scaling relations
Fluctuations

Models

# Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [11]

#### Introduction

River Networks

Models

References

Frame 112/121





# Optimal channel networks

## Rodríguez-Iturbe, Rinaldo, et al. [11]

Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

#### Introduction

River Networks

Models

References

Frame 112/121



## Rodríguez-Iturbe, Rinaldo, et al. [11]

Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathrm{d} \vec{r} \ (\mathrm{flux}) \times (\mathrm{force})$$

Models





## Rodríguez-Iturbe, Rinaldo, et al. [11]

Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d} ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i 
abla h_i$$

lometry

aws Stream Ordering Horton's Laws

lorton ⇔ Tokur Reducing Horton

Fluctuations

Models

. .

## Rodríguez-Iturbe, Rinaldo, et al. [11]

Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d} ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i 
abla h_i \sim \sum_i a_i^{\gamma}$$

efinitions

tream Ordering orton's Laws

okunaga's Law lorton ⇔ Tokuna leducing Horton

caling relations

Models

# Rodríguez-Iturbe, Rinaldo, et al. [11]

▶ Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d}ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i 
abla h_i \sim \sum_i a_i^{\gamma}$$

 Landscapes obtained numerically give exponents near that of real networks.

#### Introduction

River Networks

elinitions

aws tream Ordering lorton's Laws

orton ⇔ Tok educing Horto caling relation

Scaling relations Fluctuations Models





## Rodríguez-Iturbe, Rinaldo, et al. [11]

▶ Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d} ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i 
abla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.

#### Introduction

River Networks

Definitions

lometry

tream Ordering lorton's Laws

lorton 

Tokun
leducing Horton
caling relations

Models





## Rodríguez-Iturbe, Rinaldo, et al. [11]

Landscapes  $h(\vec{x})$  evolve such that energy dissipation  $\dot{\varepsilon}$  is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d} ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i 
abla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [9]

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law

Models References

Frame 112/121



### Theoretical networks

# Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

 $h \Rightarrow \ell \propto a^h$  (Hack's law).  $d \Rightarrow \ell \propto L_{||}^d$  (stream self-affinity).

#### Introduction

River Networks

efinitions llometry

aws Stream Ordering

Horton ⇔ Toke Reducing Horton

Fluctuations Models





### References I

- H. de Vries, T. Becker, and B. Eckhardt.

  Power law distribution of discharge in ideal networks.

  Water Resources Research, 30(12):3541–3543,

  December 1994.
- P. S. Dodds and D. H. Rothman.
  Unified view of scaling laws for river networks.

  Physical Review E, 59(5):4865–4877, 1999. pdf (⊞)
- P. S. Dodds and D. H. Rothman.
  Geometry of river networks. II. Distributions of component size and number.

  Physical Review E, 63(1):016116, 2001. pdf (⊞)
- P. S. Dodds and D. H. Rothman. Geometry of river networks. III. Characterization of component connectivity. Physical Review E, 63(1):016117, 2001. pdf (⊞)

Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

References

Frame 114/121



### References II



N. Goldenfeld.

Lectures on Phase Transitions and the Renormalization Group, volume 85 of Frontiers in Physics.

Addison-Wesley, Reading, Massachusetts, 1992.



J. T. Hack.

Studies of longitudinal stream profiles in Virginia and Maryland.

*United States Geological Survey Professional Paper*, 294-B:45–97, 1957.



R. E. Horton.

Erosional development of streams and their drainage basins; hydrophysical approach to quatitative morphology.

Bulletin of the Geological Society of America, 56(3):275–370, 1945.

#### Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton & Tokunaga's Reducing Horton
Scaling relations

References

Frame 115/121



### References III



Statistical inevitability of Horton's laws and the apparent randomness of stream channel networks. *Geology*, 21:591–594, July 1993.

A. Maritan, F. Colaiori, A. Flammini, M. Cieplak, and J. R. Banavar.

Universality classes of optimal channel networks. *Science* 272:984–986 1996 pdf (⊞)

Science, 272:984–986, 1996. pdf (⊞)

S. D. Peckham.

New results for self-similar trees with applications to river networks.

Water Resources Research, 31(4):1023–1029, April 1995.

Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton 🖘 Tokunaga

References

Frame 116/121



### References IV

I. Rodríguez-Iturbe and A. Rinaldo. Fractal River Basins: Chance and Self-Organization. Cambridge University Press, Cambrigde, UK, 1997.

A. E. Scheidegger.

A stochastic model for drainage patterns into an intramontane trench.

Bull. Int. Assoc. Sci. Hydrol., 12(1):15-20, 1967.

í

A. E. Scheidegger. Theoretical Geomorphology.

Springer-Verlag, New York, third edition, 1991.

÷

Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws

Tokunaga's Law
Horton ⇔ Tokuna
Reducing Horton
Scaling relations
Fluctuations





### References V

S. A. Schumm.

Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersey. Bulletin of the Geological Society of America, 67:597–646, May 1956.

R. L. Shreve.
Infinite topologically random channel networks. *Journal of Geology*, 75:178–186, 1967.

A. N. Strahler.

Hypsometric (area altitude) analysis of erosional topography.

Bulletin of the Geological Society of America, 63:1117–1142, 1952.

Introduction

River Networks
Definitions
Allometry
Laws

Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton

Models

References

Frame 118/121



### References VI



H. Takayasu.

Steady-state distribution of generalized aggregation system with injection.

Physcial Review Letters, 63(23):2563–2565, December 1989.

H. Takayasu, I. Nishikawa, and H. Tasaki.

Power-law mass distribution of aggregation systems with injection.

Physical Review A, 37(8):3110-3117, April 1988.

M. Takayasu and H. Takayasu. Apparent independency of an aggregation system with injection.

Physical Review A, 39(8):4345-4347, April 1989.

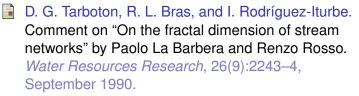
Introduction

River Networks





### References VII



E. Tokunaga.

The composition of drainage network in Toyohira River Basin and the valuation of Horton's first law. *Geophysical Bulletin of Hokkaido University*, 15:1–19, 1966

E. Tokunaga.

Consideration on the composition of drainage networks and their evolution.

Geographical Reports of Tokyo Metropolitan University, 13:1–27, 1978.

Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga

Models

References

Frame 120/121



### References VIII



E. Tokunaga.

Ordering of divide segments and law of divide segment numbers.

Transactions of the Japanese Geomorphological Union, 5(2):71-77, 1984.



G. K. Zipf.

Human Behaviour and the Principle of Least-Effort. Addison-Wesley, Cambridge, MA, 1949.

#### Introduction

River Networks



