295A Complex Networks—Assignment 3
University of Vermont, Spring 2008

Dispersed: Wednesday, April 2, 2008.
Due: By start of lecture, 9:30 am, Tuesday, April 15, 2008.
Some useful reminders:
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Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2008-01UVM-295/`

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

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1. Generating functions and giant components: In this question, you will use generating functions to obtain a number of results we found in class for standard random networks.

   (a) For an infinite standard random network (Erdös-Rényi/ER network) with average degree $\langle k \rangle$, compute the generating function F_P for the degree distribution P_k.
   (Recall the degree distribution is Poisson: $P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$, $k \geq 0$.)

   (b) Show that $F'(1) = \langle k \rangle$ (as it should).

   (c) Using the joyous properties of generating functions, show that the second moment of the degree distribution is $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.

   (d) Find the generating function for the degree distribution P_k of a finite random network with N nodes and an edge probability of p.

   (e) Show that the generating function for the finite ER network tends to the generating function for the infinite one. Do this by taking the limit $N \to \infty$ and $p \to 0$ such that $p(N-1) = \langle k \rangle$ remains constant.

2. (a) Continuing on from Q1, find the generating function $F_R(x)$ for the $\{r_k\}$, where r_k is the probability that a node arrived at by following a random direction on a randomly chosen edge has k outgoing edges.

   (b) Now determine the average number of outgoing edges from a randomly-arrived-at-along-a-random-edge node.

(c) Given your findings above, what is the condition on $\langle k \rangle$ for a standard random network to have a giant component?

(Hint: you need to find for what values of $\langle k \rangle$, a randomly chosen neighbor will, on average, have at least one other neighbor.)

3. Consider the simple spreading mechanism on generalized random networks for which each link has a probability $\beta \leq 1$ of successfully transmitting a disease.

We assume that this transmission probability is tested only once: either a link will or will not be able to send an infection one way or the other (this is a bond percolation problem). We'll call these edges 'active.'

Denote the degree distribution of the network as P_k and the corresponding generating function as F_P. In class, we wrote down the probability that a node has k active edges as

$$P'_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$

(a) Given a random network with degree distribution P_k, find $F_{P'}$, the generating function for P'_k, in terms of F_P.

(b) Find the generating function for R'_k, the analogous version of R_k, the probability that a random friend has k other friends.

4. (a) For standard random networks, use your results for Q3 to find an expression connecting β, the average degree $\langle k \rangle$, and the size of the giant component S_1.

(b) What is slope of the S_1 curve near the critical point for ER networks?

(c) Using whichever method you find most exciting, plot how S_1 depends on $\langle k \rangle$ for $\beta = 1$, $\beta = 0.8$, and $\beta = 0.5$.

5. Consider a network with a degree distribution that obeys a power law and is otherwise random.

Assume that the network is drawn from an ensemble of networks which have N nodes whose degrees are drawn from the probability distribution $P_k = ck^{-\gamma}$ where $k \geq 1$ and $2 < \gamma < 3$.

(a) Estimate $\min k_{\max}$, the approximate minimum of the largest degree in the network, finding how it depends on N. (Hint: we expect on the order of 1 of the N nodes to have a degree of $\min k_{\max}$ or greater.)

(b) Determine the average degree of nodes with degree $k \geq \min k_{\max}$ to find how the expected value of $k_{\max}$ scales with N.