

Optimal Supply Networks I: Branching

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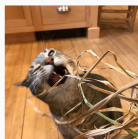
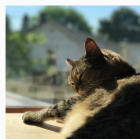
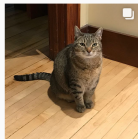
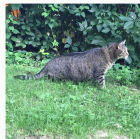
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Optimal supply networks

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What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...

 **Some** fundamental network problems:

1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks

 Supply and Collection are equivalent problems



Single source optimal supply

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Basic question for distribution/supply networks:

 How does flow behave given cost:

$$C = \sum_j I_j^\gamma Z_j$$

where

I_j = current on link j

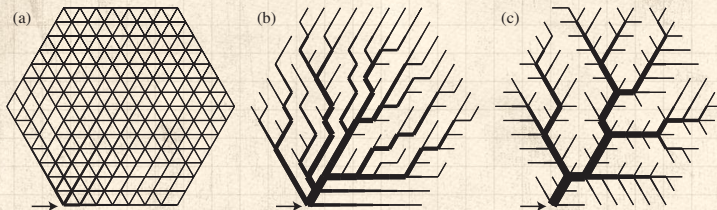
and

Z_j = link j 's impedance.

 Example: $\gamma = 2$ for electrical networks.



Single source optimal supply



(a) $\gamma > 1$: **Braided** (bulk) flow

(b) $\gamma < 1$: Local minimum: **Branching** flow

(c) $\gamma < 1$: Global minimum: **Branching** flow



Note: This is a single source supplying a region.

From Bohn and Magnasco ^[3]

See also Banavar *et al.* ^[1]: “Topology of the Fittest Transportation Network”; focus is on presence or absence of loops—same story



Single source optimal supply

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Optimal paths related to transport (Monge) problems

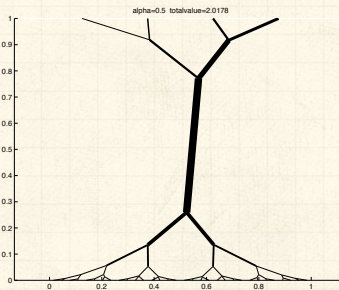
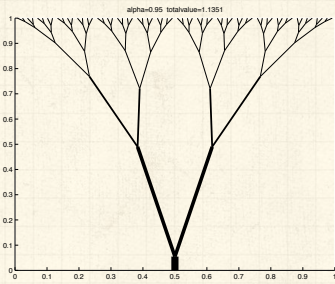
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“Optimal paths related to transport problems” 

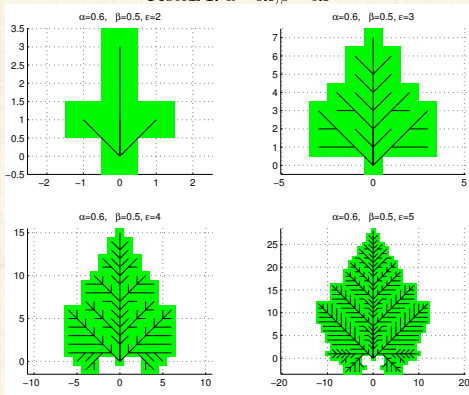
Qinglan Xia,

Communications in Contemporary Mathematics, **5**,
251–279, 2003. ^[20]



Growing networks—two parameter model: [21]

FIGURE 1. $\alpha = 0.6, \beta = 0.5$



Parameters control impedance ($0 \leq \alpha < 1$) and angles of junctions ($0 < \beta$)

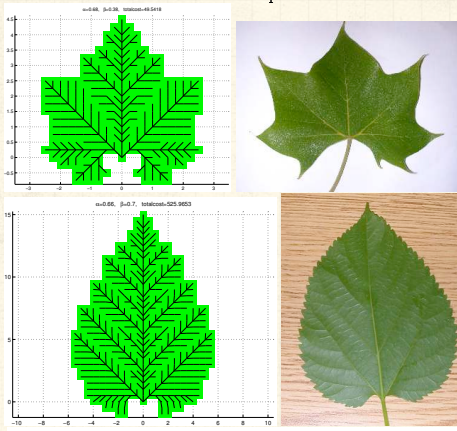


For this example: $\alpha = 0.6$ and $\beta = 0.5$



Growing networks: [21]

FIGURE 3. A maple leaf



Top: $\alpha = 0.66, \beta = 0.38$; Bottom: $\alpha = 0.66, \beta = 0.70$



Single source optimal supply

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An immensely controversial issue ...

-  The form of natural branching networks:
Random, optimal, or some combination? [6, 19, 2, 5, 4]
-  River networks, blood networks, trees, ...

Two observations:

-  Self-similar networks appear everywhere in nature for single source supply/single sink collection.
-  Real networks **differ** in **details of scaling** but reasonably agree in **scaling relations**.



River network models

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Optimality:

 Optimal channel networks ^[13]

 Thermodynamic analogy ^[14]

versus ...

Randomness:

 Scheidegger's directed random networks

 Undirected random networks



Optimization—Murray's law

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Murray's law (1926) connects
branch radii at
forks: [11, 10, 12, 7, 17]

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$

where r_{parent} = radius of
'parent' branch, and $r_{\text{offspring1}}$
and $r_{\text{offspring2}}$ are radii of the
two 'offspring' sub-branches.



Holds up well for outer branchings of blood networks [15].



Also found to hold for trees [12, 8] when xylem is not a
supporting structure [9].



See D'Arcy Thompson's "On Growth and Form" for
background and general inspiration [16, 17].

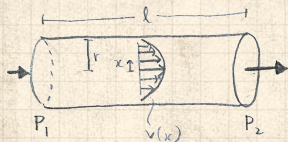




Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



Fluid mechanics: Poiseuille impedance  for smooth Poiseuille flow  in a tube of radius r and length ℓ :

$$Z = \frac{8\eta\ell}{\pi r^4}$$



η = dynamic viscosity  (units: $ML^{-1}T^{-1}$).



Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$



Also have rate of energy expenditure in maintaining blood given metabolic constant c :

$$P_{\text{metabolic}} = cr^2\ell$$



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Aside on P_{drag}

 Work done = $F \cdot d$ = energy transferred by force F

 Power = P = rate work is done = $F \cdot v$

 Δp = Pressure differential = Force per unit area

 Φ = Volume flow per unit time (current)
= cross-sectional area $\cdot$ velocity

 So $\Phi \Delta p$ = Force $\cdot$ velocity



Optimization—Murray's law

Murray's law:

 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

 Observe power increases linearly with ℓ

 But r 's effect is nonlinear:

-  increasing r makes flow easier **but increases metabolic cost** (as r^2)
-  decreasing r decrease metabolic cost **but impedance goes up** (as r^{-4})



Optimization—Murray's law

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Murray's law:



Minimize P with respect to r :

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$



Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches



Optimization—Murray's law

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Murray's law:



Find:

$$\Phi = kr^3$$



Insert assignment question 



All of this means we have a groovy cube-law:

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$



Murray meets Tokunaga:

 Φ_ω = volume rate of flow into an order ω vessel segment

 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

 Using $\phi_\omega = kr_\omega^3$

$$(r_\omega)^3 = 2(r_{\omega-1})^3 + \sum_{k=1}^{\omega-1} T_k (r_{\omega-k})^3$$

 Same form as:

$$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$



Murray meets Tokunaga:

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1}$.
- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$R_r^3 = R_n = R_v$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?



Murray meets Tokunaga:

 Isometry: $V_\omega \propto \ell_\omega^3$

 Gives

$$R_\ell^3 = R_r^3 = R_n = R_v$$

 We need one more constraint ...

 West *et al.* (1997) ^[19] achieve similar results following Horton's laws (but this work is a disaster).

 So does Turcotte *et al.* (1998) ^[18] using Tokunaga (sort of).



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