

Optimal Supply Networks I: Branching

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Outline

Optimal transportation

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Optimal supply networks

What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...

 **Some** fundamental network problems:

1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks

 Supply and Collection are equivalent problems

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Single source optimal supply

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Basic question for distribution/supply networks:

 How does flow behave given cost:

$$C = \sum_j I_j^\gamma Z_j$$

where

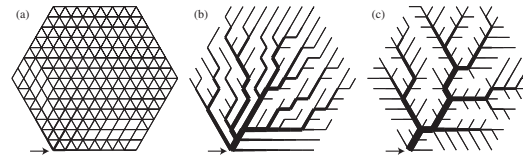
I_j = current on link j

and

Z_j = link j 's impedance.

 Example: $\gamma = 2$ for electrical networks.

Single source optimal supply



- (a) $\gamma > 1$: **Braided** (bulk) flow
(b) $\gamma < 1$: Local minimum: **Branching** flow
(c) $\gamma < 1$: Global minimum: **Branching** flow

 Note: This is a single source supplying a region.

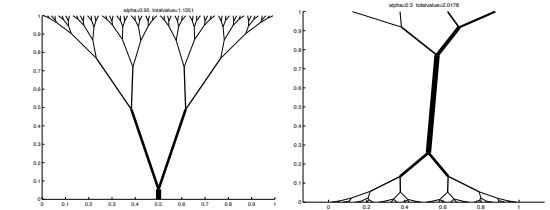
From Bohn and Magnasco [3]

See also Banavar *et al.* [1]: "Topology of the Fittest Transportation Network"; focus is on presence or absence of loops—same story

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Optimal paths related to transport (Monge) problems 

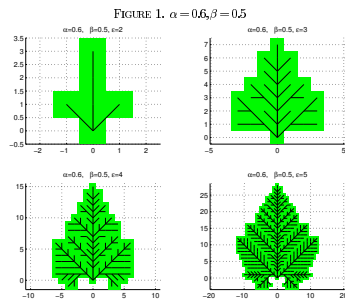


"Optimal paths related to transport problems" 
Qinglan Xia,
Communications in Contemporary Mathematics, 5,
251–279, 2003. [20]

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Growing networks—two parameter model: [21]

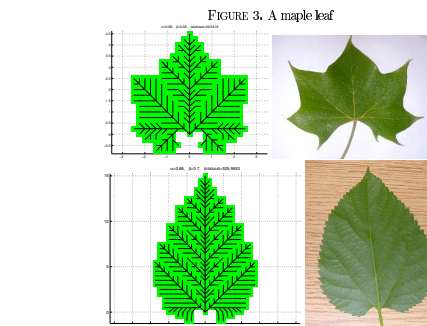
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 Parameters control impedance ($0 \leq \alpha < 1$) and angles of junctions ($0 < \beta$)

 For this example: $\alpha = 0.6$ and $\beta = 0.5$

Growing networks: [21]



 Top: $\alpha = 0.66, \beta = 0.38$; Bottom: $\alpha = 0.66, \beta = 0.70$

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Single source optimal supply

An immensely controversial issue ...

-  The form of natural branching networks:
Random, optimal, or some combination? [6, 19, 2, 5, 4]
 River networks, blood networks, trees, ...

Two observations:

-  Self-similar networks appear everywhere in nature for single source supply/single sink collection.
 Real networks **differ** in **details of scaling** but reasonably agree in **scaling relations**.

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River network models

Optimality:

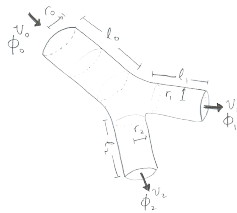
- Optimal channel networks [13]
- Thermodynamic analogy [14]

versus ...

Randomness:

- Scheidegger’s directed random networks
- Undirected random networks

Optimization—Murray’s law



Murray’s law (1926) connects branch radii at forks: [11, 10, 12, 7, 17]

$r^3_{\text{parent}} = r^3_{\text{offspring1}} + r^3_{\text{offspring2}}$

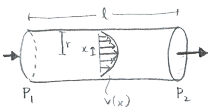
where r_{parent} = radius of ‘parent’ branch, and $r_{\text{offspring1}}$ and $r_{\text{offspring2}}$ are radii of the two ‘offspring’ sub-branches.

- Holds up well for outer branchings of blood networks [15].
- Also found to hold for trees [12, 8] when xylem is not a supporting structure [9].
- See D’Arcy Thompson’s “On Growth and Form” for background and general inspiration [16, 17].

Use hydraulic equivalent of Ohm’s law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ :

$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c :

$$P_{\text{metabolic}} = cr^2\ell$$

Optimization—Murray’s law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Pressure differential = Force per unit area
- Φ = Volume flow per unit time (current)
= cross-sectional area $\cdot$ velocity
- So $\Phi \Delta p$ = Force $\cdot$ velocity

Optimization—Murray’s law

Murray’s law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

- Observe power increases linearly with ℓ
- But r ’s effect is nonlinear:
 - increasing r makes flow easier **but increases metabolic cost** (as r^2)
 - decreasing r decrease metabolic cost **but impedance goes up** (as r^{-4})

Optimization—Murray’s law

Murray’s law:

- Minimize P with respect to r :

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$

- Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

Optimization—Murray’s law

Murray’s law:

- Find:

$$\Phi = kr^3$$

Insert assignment question

- All of this means we have a groovy cube-law:

$r^3_{\text{parent}} = r^3_{\text{offspring1}} + r^3_{\text{offspring2}}$

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = kr^3_\omega$

$$(r_\omega)^3 = 2(r_{\omega-1})^3 + \sum_{k=1}^{\omega-1} T_k (r_{\omega-k})^3$$

- Same form as:

$$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$

Optimization

Murray meets Tokunaga:

- Find Horton ratio for vessel radius $R_r = r_\omega/r_{\omega-1}$.
- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$R_r^3 = R_n = R_v$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Optimization

Murray meets Tokunaga:

Isometry: $V_\omega \propto \ell_\omega^3$

Gives

$$R_\ell^3 = R_r^3 = R_n = R_v$$

We need one more constraint ...

West *et al.* (1997)^[19] achieve similar results following Horton’s laws (but this work is a disaster).

So does Turcotte *et al.* (1998)^[18] using Tokunaga (sort of).

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