

Scaling—a Plenitude of Power Laws

Last updated: 2026/08/19, 12:31:01 EDT

DOI:

Principles of Complex Systems,
Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

Deliverator: Calla Beauregard
Showrunner: Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
University of Vermont | Santa Fe Institute



Licensed under the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/)



The PoCVerse

Scaling

1 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



These slides are brought to you by:

Sealie & Lambie Productions

The PoCverse

Scaling

2 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

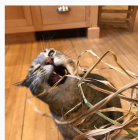
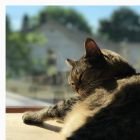
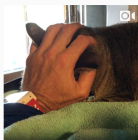
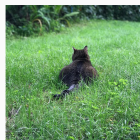
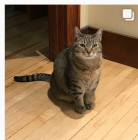
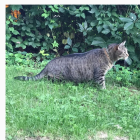
Technology

References



These slides are also brought to you by:

Special Guest Executive Producer



 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

The PoCverse

Scaling

3 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Outline

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

The PoCSverse

Scaling

4 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

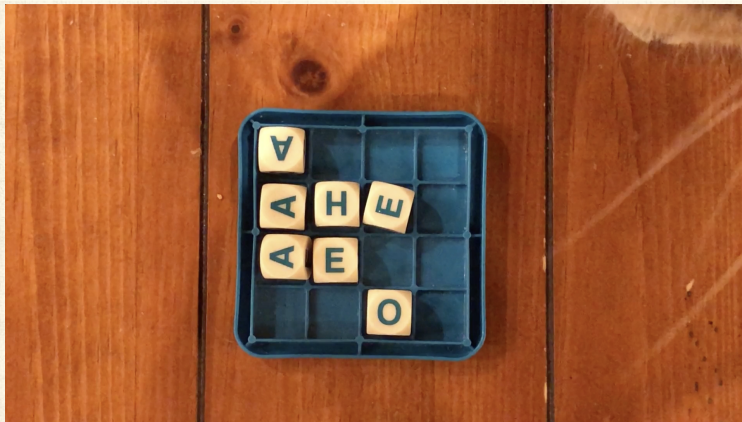
Language

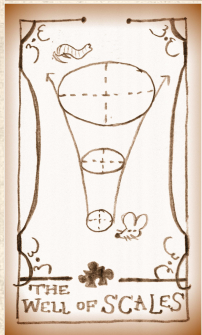
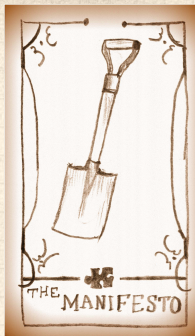
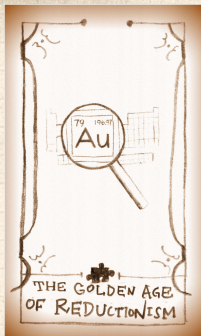
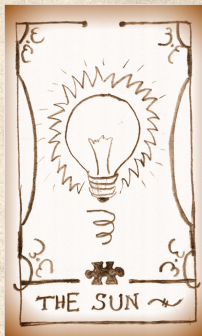
Technology

References



The Boggoracle Speaks:





Archival object:

The PoCverse

Scaling

7 of 114

Scaling-at-large

Allometry

Biology

Physics

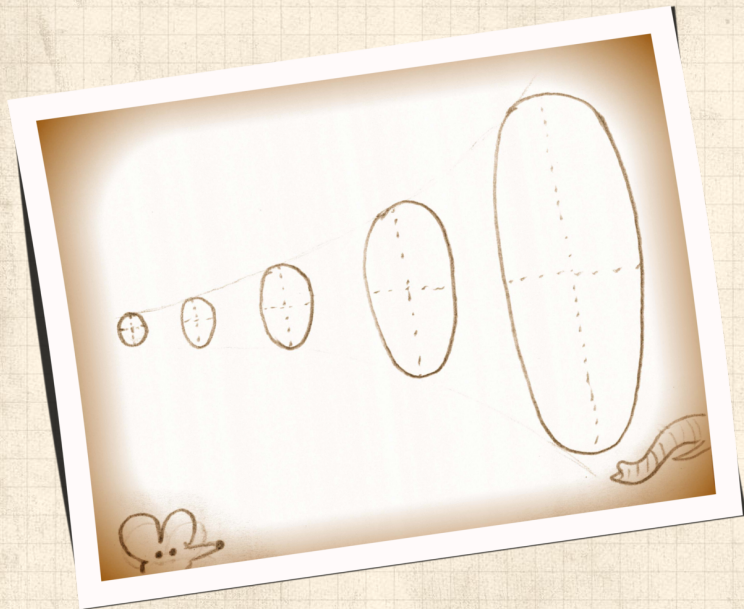
People

Money

Language

Technology

References



Scalingarama

The PoCSverse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.



Scalingarama

The PoCVerse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:



Scalingarama

The PoCVerse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:



Basic definitions.



Scalingarama

The PoCverse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

 Basic definitions.

 Examples.



Scalingarama

The PoCverse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

 Basic definitions.

 Examples.

Possibly later:



Scalingarama

The PoCverse

Scaling

8 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

 Basic definitions.

 Examples.

Possibly later:

 Advances in measuring your power-law relationships.



General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

 Basic definitions.

 Examples.

Possibly later:

 Advances in measuring your power-law relationships.

 Scaling in blood and river networks.



General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

 Basic definitions.

 Examples.

Possibly later:

 Advances in measuring your power-law relationships.

 Scaling in blood and river networks.

 The Unsolved Allometry Theoricides.



A **power law** relates two variables x and y as follows:

$$y = cx^{\alpha}$$



α is the **scaling exponent** (or just exponent)



α can be any number in principle but we will find various restrictions.



c is the **prefactor** (which can be important!)



Definitions

The PoCSverse

Scaling

10 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The prefactor c must balance dimensions.



Definitions

The PoCSverse

Scaling

10 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 The prefactor c must balance dimensions.

 Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$



Definitions

The PoCverse

Scaling

10 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 The prefactor c must balance dimensions.

 Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$

 Using $[\cdot]$ to indicate dimension, then

$$[c] = [\ell]/[v^{1/4}] = L/L^{3/4} = L^{1/4}.$$



Definitions

The PoCverse

Scaling

10 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 The prefactor c must balance dimensions.

 Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$

 Using $[\cdot]$ to indicate dimension, then

$$[c] = [\ell]/[v^{1/4}] = L/L^{3/4} = L^{1/4}.$$

 More on this later with the Buckingham π theorem.





Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.



Looking at data

The PoCverse

Scaling

11 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.



Much searching for straight lines on **log-log** or **double-logarithmic** plots.



Looking at data

The PoCverse

Scaling

11 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.

 Much searching for straight lines on **log-log** or **double-logarithmic** plots.

 Good practice: **Always, always, always use base 10.**



Looking at data

The PoCverse

Scaling

11 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.

 Much searching for straight lines on **log-log** or **double-logarithmic** plots.

 Good practice: **Always, always, always use base 10.**

 Yes, the Dozenalists  are right, 12 would be better.



Looking at data

The PoCverse

Scaling

11 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 Power-law relationships are linear in log-log space:

$$y = cx^\alpha$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.

 Much searching for straight lines on **log-log** or **double-logarithmic** plots.

 Good practice: **Always, always, always use base 10.**

 Yes, the Dozenalists  are right, 12 would be better.

 But: hands.¹ And social pressure.



Looking at data

The PoCverse

Scaling

11 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

 Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α , the scaling exponent.

 Much searching for straight lines on **log-log** or **double-logarithmic** plots.

 Good practice: **Always, always, always use base 10.**

 Yes, the Dozenalists  are right, 12 would be better.

 But: hands.¹ And social pressure.



¹Probably an accident of evolution—debated.

Looking at data

- Power-law relationships are linear in log-log space:

$$y = cx^\alpha$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

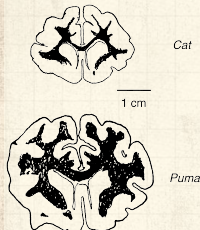
with slope equal to α , the scaling exponent.

- Much searching for straight lines on **log-log** or **double-logarithmic** plots.
- Good practice: **Always, always, always use base 10.**
- Yes, the Dozenalists  are right, 12 would be better.
- But: hands.¹ And social pressure.
- Talk only about orders of magnitude (powers of 10).



¹Probably an accident of evolution—debated.

A beautiful, heart-warming example:



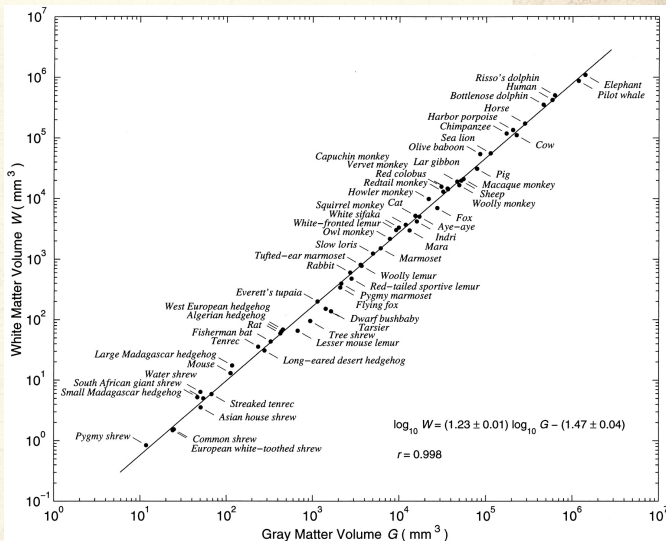
G = volume of
gray matter:
'computing
elements'



W = volume of
white matter:
'wiring'



$W \sim cG^{1.23}$



from Zhang & Sejnowski, PNAS (2000) [39]

Why is $\alpha \simeq 1.23$?

The PoCSverse

Scaling

13 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

-  G = Volume of gray matter (cortex/processors)
-  W = Volume of white matter (wiring)
-  T = Cortical thickness (wiring)
-  S = Cortical surface area
-  L = Average length of white matter fibers
-  p = density of axons on white matter/cortex interface



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

-  G = Volume of gray matter (cortex/processors)
-  W = Volume of white matter (wiring)
-  T = Cortical thickness (wiring)
-  S = Cortical surface area
-  L = Average length of white matter fibers
-  p = density of axons on white matter/cortex interface

A rough understanding:



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

-  G = Volume of gray matter (cortex/processors)
-  W = Volume of white matter (wiring)
-  T = Cortical thickness (wiring)
-  S = Cortical surface area
-  L = Average length of white matter fibers
-  p = density of axons on white matter/cortex interface

A rough understanding:

-  $G \sim ST$ (convolutions are okay)



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

-  G = Volume of gray matter (cortex/processors)
-  W = Volume of white matter (wiring)
-  T = Cortical thickness (wiring)
-  S = Cortical surface area
-  L = Average length of white matter fibers
-  p = density of axons on white matter/cortex interface

A rough understanding:

-  $G \sim ST$ (convolutions are okay)
-  $W \sim \frac{1}{2}pSL$



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

 G = Volume of gray matter (cortex/processors)

 W = Volume of white matter (wiring)

 T = Cortical thickness (wiring)

 S = Cortical surface area

 L = Average length of white matter fibers

 p = density of axons on white matter/cortex interface

A rough understanding:

 $G \sim ST$ (convolutions are okay)

 $W \sim \frac{1}{2}pSL$

 $G \sim L^3$



Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

-  G = Volume of gray matter (cortex/processors)
-  W = Volume of white matter (wiring)
-  T = Cortical thickness (wiring)
-  S = Cortical surface area
-  L = Average length of white matter fibers
-  p = density of axons on white matter/cortex interface

A rough understanding:

-  $G \sim ST$ (convolutions are okay)
-  $W \sim \frac{1}{2}pSL$
-  $G \sim L^3$
-  Eliminate S and L to find $W \propto G^{4/3}/T$



Why is $\alpha \simeq 1.23$?

The PoCSverse

Scaling

14 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A rough understanding:



We are here: $W \propto G^{4/3}/T$



Why is $\alpha \simeq 1.23$?

The PoCSverse

Scaling

14 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A rough understanding:



We are here: $W \propto G^{4/3}/T$



Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.



Why is $\alpha \simeq 1.23$?

The PoCverse

Scaling

14 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A rough understanding:



We are here: $W \propto G^{4/3}/T$



Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.



Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.



Why is $\alpha \simeq 1.23$?

The PoCverse

Scaling

14 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A rough understanding:



We are here: $W \propto G^{4/3}/T$



Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.



Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.



$\Rightarrow W \propto G^{4/3}/T \propto G^{1.23 \pm 0.02}$



Tricksiness:

The PoCverse

Scaling

15 of 114

Scaling-at-large

Allometry

Biology

Physics

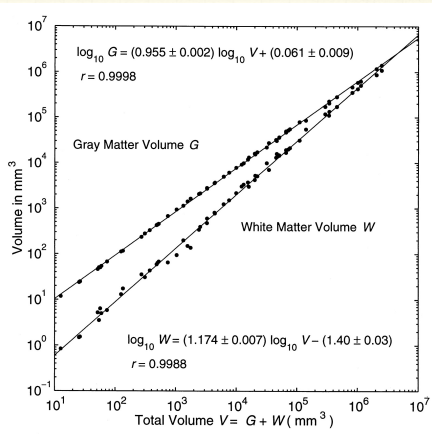
People

Money

Language

Technology

References



With $V = G + W$, some power laws must be approximations.



Tricksiness:

The PoCverse

Scaling

15 of 114

Scaling-at-large

Allometry

Biology

Physics

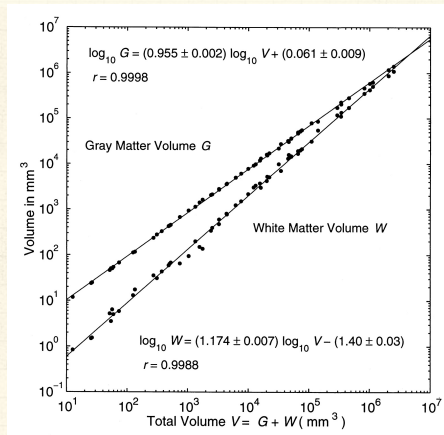
People

Money

Language

Technology

References



With $V = G + W$, some power laws must be approximations.

Measuring exponents is a hairy business...



Disappointing deviations from scaling:



Per George
Carlin 

Image from [here](#) 

The PoCVerse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:

The PoCVerse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

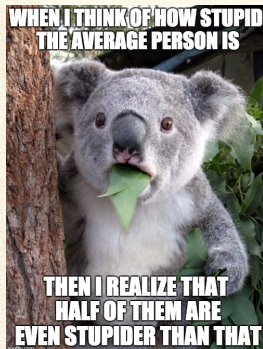
People

Money

Language

Technology

References



Per George
Carlin 



Yes, should be the
median.
#painful

Image from 



Disappointing deviations from scaling:



The koala ↗, a few roos short in the top paddock:

Per George Carlin ↗

Yes, should be the median.
#painful

Image from here ↗

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



The koala ↗, a few roos short in the top paddock:

Very small brains ↗ relative to body size.

Per George Carlin ↗

Yes, should be the median.
#painful

Image from here ↗

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



The koala ↗, a few roos short in the top paddock:



Very small brains ↗ relative to body size.



Wrinkle-free, smooth.



Per George Carlin ↗



Yes, should be the median.
#painful

Image from here ↗

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



The koala ↗, a few roos short in the top paddock:



Very small brains ↗ relative to body size.



Wrinkle-free, smooth.



Not many algorithms needed:



Per George Carlin ↗



Yes, should be the median.
#painful

Image from here ↗

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



The koala ↗, a few roos short in the top paddock:



Very small brains ↗ relative to body size.



Wrinkle-free, smooth.



Not many algorithms needed:



Only eat eucalyptus leaves (no water)



Per George Carlin ↗



Yes, should be the median.
#painful

Image from here ↗

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



 Per George Carlin 

 Yes, should be the median.
#painful

Image from [here](#) 

The koala , a few roos short in the top paddock:

 Very small brains  relative to body size.

 Wrinkle-free, smooth.

 Not many algorithms needed:

 Only eat eucalyptus leaves (no water)
(Will not eat leaves picked and presented to them)

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



Per George Carlin

Yes, should be the median.
#painful

Image from [here](#)

The koala, a few roos short in the top paddock:

Very small brains relative to body size.

Wrinkle-free, smooth.

Not many algorithms needed:

Only eat eucalyptus leaves (no water)

(Will not eat leaves picked and presented to them)

Move to the next tree.

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



Per George Carlin

Yes, should be the median.
#painful

Image from [here](#)

The koala, a few roos short in the top paddock:

Very small brains relative to body size.

Wrinkle-free, smooth.

Not many algorithms needed:

Only eat eucalyptus leaves (no water)

(Will not eat leaves picked and presented to them)

Move to the next tree.

Sleep.

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



Per George Carlin

Yes, should be the median.
#painful

Image from [here](#)

The koala, a few roos short in the top paddock:

Very small brains relative to body size.

Wrinkle-free, smooth.

Not many algorithms needed:

Only eat eucalyptus leaves (no water)

(Will not eat leaves picked and presented to them)

Move to the next tree.

Sleep.

Defend themselves if needed (tree-climbing crocodiles, humans).

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Disappointing deviations from scaling:



 Per George Carlin 

 Yes, should be the median.
#painful

Image from [here](#) 

The koala , a few roos short in the top paddock:

 Very small brains  relative to body size.

 Wrinkle-free, smooth.

 Not many algorithms needed:

 Only eat eucalyptus leaves (no water)

(Will not eat leaves picked and presented to them)

 Move to the next tree.

 Sleep.

 Defend themselves if needed (tree-climbing crocodiles, humans).

 Occasionally make more koalas.

The PoCverse

Scaling

16 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Good scaling:

General rules of thumb:

 *High quality:* scaling persists over three or more orders of magnitude for **each variable**.

The PoCSverse

Scaling

17 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Good scaling:

The PoCverse

Scaling

17 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

General rules of thumb:

 *High quality:* scaling persists over three or more orders of magnitude for **each variable**.

 *Medium quality:* scaling persists over three or more orders of magnitude for **only one variable** and at least one for **the other**.



Good scaling:

General rules of thumb:



High quality: scaling persists over three or more orders of magnitude for **each variable**.



Medium quality: scaling persists over three or more orders of magnitude for **only one variable** and at least one for **the other**.

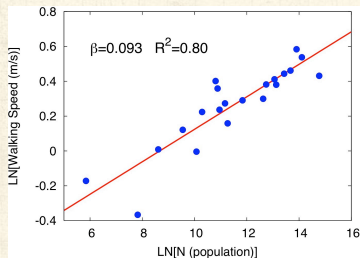


Very dubious: scaling 'persists' over less than an order of magnitude for **both variables**.



Unconvincing scaling:

Average walking speed as a function of city population:



Two problems:

1. use of natural log, and
2. minute variation in dependent variable.



from Bettencourt et al. (2007) ^[4]; otherwise totally great—more later.



Definitions

Power laws are the signature of **scale invariance**:

Scale invariant ‘objects’
look the ‘same’
when they are appropriately **rescaled**.



Definitions

The PoCVerse

Scaling

19 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Power laws are the signature of **scale invariance**:

Scale invariant ‘objects’
look the ‘same’
when they are appropriately **rescaled**.



Objects = geometric shapes, time series, functions, relationships, distributions,...



Definitions

The PoCSverse

Scaling

19 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Power laws are the signature of **scale invariance**:

Scale invariant ‘objects’
look the ‘same’
when they are appropriately **rescaled**.



Objects = geometric shapes, time series, functions, relationships, distributions,...



‘Same’ might be ‘statistically the same’



Definitions

Power laws are the signature of **scale invariance**:

Scale invariant ‘objects’
look the ‘same’
when they are appropriately **rescaled**.



Objects = geometric shapes, time series, functions, relationships, distributions,...



‘Same’ might be ‘statistically the same’



To **rescale** means to change the units of measurement for the relevant variables



Scale invariance

The PoCSverse

Scaling

20 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Our friend $y = cx^\alpha$:



If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,



Scale invariance

The PoCVerse

Scaling

20 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Our friend $y = cx^\alpha$:



If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,



then

$$r^\alpha y' = c(rx')^\alpha$$



Scale invariance

The PoCverse

Scaling

20 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Our friend $y = cx^\alpha$:

 If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,

 then

$$r^\alpha y' = c(rx')^\alpha$$



$$\Rightarrow y' = cr^\alpha x'^\alpha r^{-\alpha}$$



Scale invariance

Our friend $y = cx^\alpha$:

 If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,

 then

$$r^\alpha y' = c(rx')^\alpha$$



$$\Rightarrow y' = cr^\alpha x'^\alpha r^{-\alpha}$$



$$\Rightarrow y' = cx'^\alpha$$



Scale invariance

The PoCSverse

Scaling

21 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$



Scale invariance

The PoCSverse

Scaling

21 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$

 Original form cannot be recovered.



Scale invariance

The PoCverse

Scaling

21 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$

 Original form cannot be recovered.

 **Scale matters** for the exponential.



Scale invariance

The PoCverse

Scaling

21 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$

 Original form cannot be recovered.

 **Scale matters** for the exponential.

More on $y = ce^{-\lambda x}$:



Scale invariance

The PoCverse

Scaling

21 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$

 Original form cannot be recovered.

 **Scale matters** for the exponential.

More on $y = ce^{-\lambda x}$:

 Say $x_0 = 1/\lambda$ is the **characteristic scale**.



Scale invariance

Compare with $y = ce^{-\lambda x}$:

 If we rescale x as $x = rx'$, then

$$y = ce^{-\lambda rx'}$$

 Original form cannot be recovered.

 **Scale matters** for the exponential.

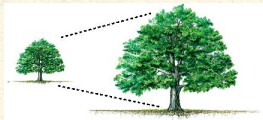
More on $y = ce^{-\lambda x}$:

 Say $x_0 = 1/\lambda$ is the **characteristic scale**.

 For $x \gg x_0$, y is small,
while for $x \ll x_0$, y is large.

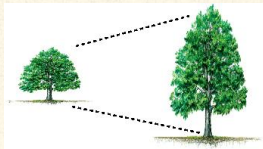


Isometry:



Dimensions scale linearly with each other.

Allometry:

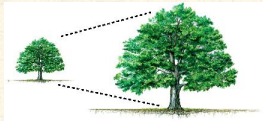


Dimensions scale nonlinearly.

Allometry:

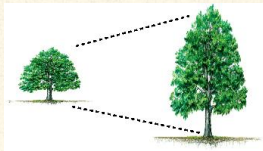


Isometry:



Dimensions scale linearly with each other.

Allometry:



Dimensions scale nonlinearly.

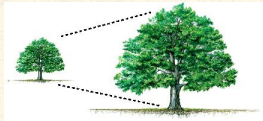
Allometry:



Refers to differential growth rates of the parts of a living organism's body part or process.

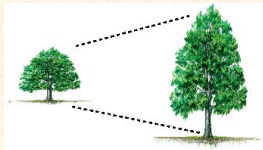


Isometry:



Dimensions scale linearly with each other.

Allometry:



Dimensions scale nonlinearly.

Allometry:

 Refers to differential growth rates of the parts of a living organism's body part or process.

 First proposed by Huxley and Teissier, Nature, 1936
“Terminology of relative growth” [15, 34]



Definitions

The PoCSverse

Scaling

23 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Isometry versus Allometry:



Iso-metry = ‘same measure’



Allo-metry = ‘other measure’



Definitions

The PoCverse

Scaling

23 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Isometry versus Allometry:

 Iso-metry = 'same measure'

 Allo-metry = 'other measure'

We use allometric scaling to refer to both:



Definitions

The PoCverse

Scaling

23 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Isometry versus Allometry:

 Iso-metry = 'same measure'

 Allo-metry = 'other measure'

We use allometric scaling to refer to both:

1. Nonlinear scaling of a dependent variable on an independent one (e.g., $y \propto x^{1/3}$)



Isometry versus Allometry:



Iso-metry = ‘same measure’



Allo-metry = ‘other measure’

We use allometric scaling to refer to both:

1. Nonlinear scaling of a dependent variable on an independent one (e.g., $y \propto x^{1/3}$)
2. The relative scaling of correlated measures (e.g., white and gray matter).



An interesting, earlier treatise on scaling:

McMahon and
Bonner, 1983 [26]

ON SIZE AND LIFE

THOMAS A. McMAHON AND JOHN TYLER BONNER



The PoCVerse

Scaling

24 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

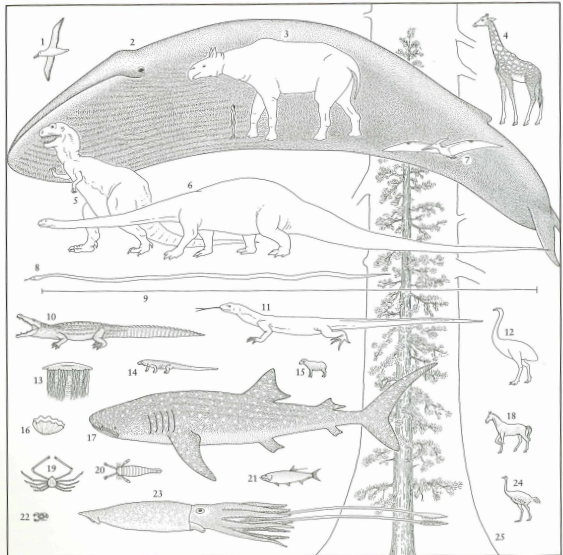
Technology

References



The many scales of life:

The biggest living things (left). All the organisms are drawn to the same scale. 1, The largest flying bird (albatross); 2, the largest known animal (the blue whale); 3, the largest extinct land mammal (*Baluchitherium*) with a human figure shown for scale; 4, the tallest living land animal (giraffe); 5, *Tyrannosaurus*; 6, *Diplodocus*; 7, one of the largest flying reptiles (*Pteranodon*); 8, the largest extinct snake; 9, the length of the largest tapeworm found in man; 10, the largest living reptile (West African crocodile); 11, the largest extinct lizard; 12, the largest extinct bird (*Aepyornis*); 13, the largest jellyfish (*Cyanea*); 14, the largest living lizard (Komodo dragon); 15, sheep; 16, the largest bivalve mollusc (*Tridacna*); 17, the largest fish (whale shark); 18, horse; 19, the largest crustacean (Japanese spider crab); 20, the largest sea scorpion (Euryp-terid); 21, large tarpon; 22, the largest lobster; 23, the largest mollusc (deep-water squid, *Architeuthis*); 24, ostrich; 25, the lower 105 feet of the largest organism (giant sequoia), with a 100-foot larch superposed.

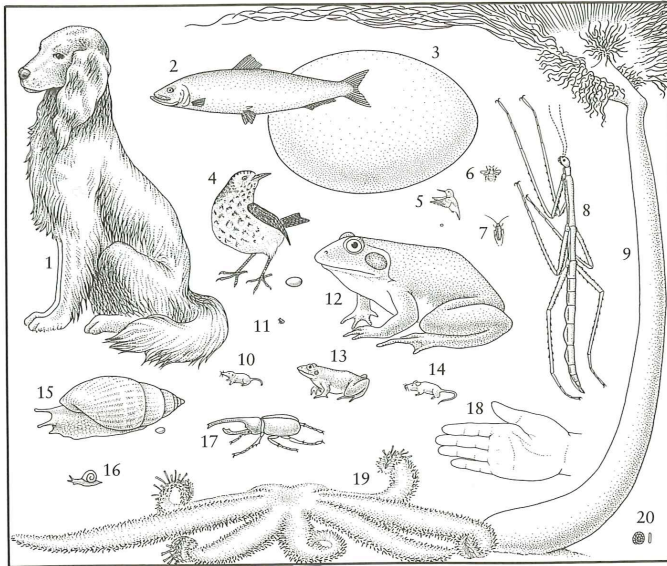


p. 2, McMahon and
Bonner^[26]

The many scales of life:

Medium-sized creatures (above). 1, Dog; 2, common herring; 3, the largest egg (*Aepyornis*); 4, song thrush with egg; 5, the smallest bird (hummingbird) with egg; 6, queen bee; 7, common cockroach; 8, the largest stick insect; 9, the largest polyp (*Branchiocerianthus*); 10, the smallest mammal (flying shrew); 11, the smallest vertebrate (a tropical frog); 12, the largest frog (goliath frog); 13, common grass frog; 14, house mouse; 15, the largest land snail (*Achatina*) with egg; 16, common snail; 17, the largest beetle (goliath beetle); 18, human hand; 19, the largest starfish (*Luidia*); 20, the largest free-moving protozoan (an extinct nummulite).

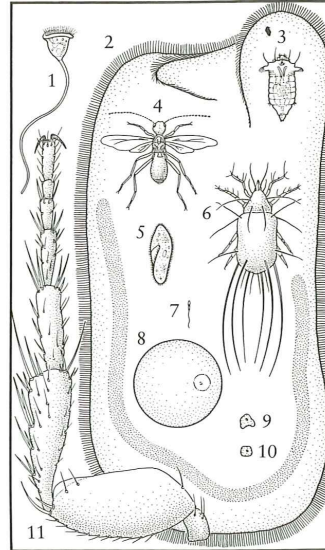
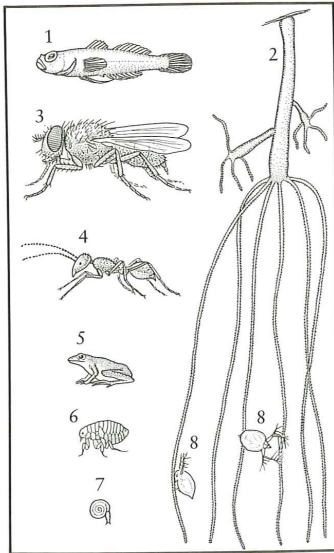
p. 3, McMahon and Bonner^[26]
More on the Elephant Bird [here](#) ↗.



The many scales of life:

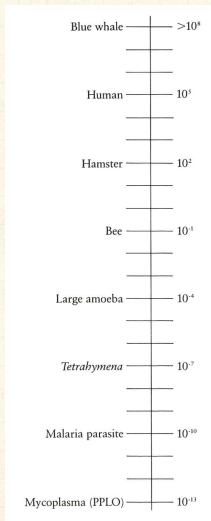
Small, "naked-eye" creatures (lower left). 1, One of the smallest fishes (*Trimmatom nanus*); 2, common brown hydra, expanded; 3, housefly; 4, medium-sized ant; 5, the smallest vertebrate (a tropical frog, the same as the one numbered 11 in the figure above); 6, flea (*Xenopsylla cheopis*); 7, the smallest land snail; 8, common water flea (*Daphnia*).

The smallest "naked-eye" creatures and some large microscopic animals and cells (below right). 1, *Vorticella*, a ciliate; 2, the largest ciliate protozoan (*Bursaria*); 3, the smallest many-celled animal (a rotifer); 4, smallest flying insect (*Elaphis*); 5, another ciliate (*Paramecium*); 6, cheese mite; 7, human sperm; 8, human ovum; 9, dysentery amoeba; 10, human liver cell; 11, the foreleg of the flea (numbered 6 in the figure to the left).



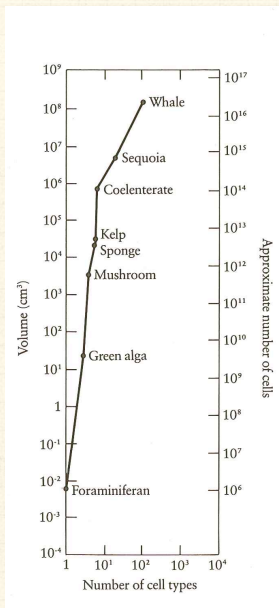
p. 3, McMahon and
Bonner^[26]

Size range (in grams) and cell differentiation:



10^{-13} to 10^8 g, p. 3,

McMahon and Bonner [26]



The PoCSeve

Scaling

28 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

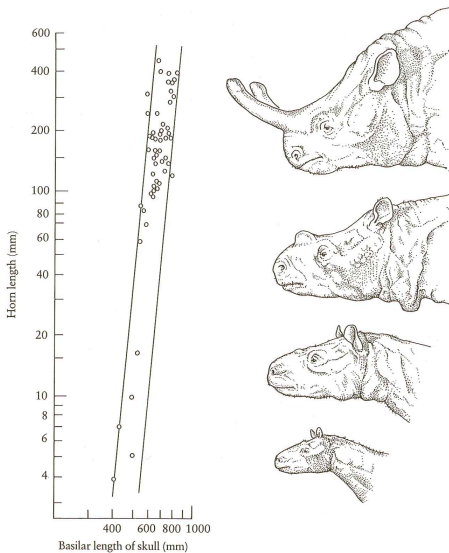
Language

Technology

References



Titanotheres horns: $L_{\text{horn}} \sim L_{\text{skull}}^4$



The PoCVerse

Scaling

29 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



p. 36, McMahon and Bonner^[26]; a bit dubious.

Non-uniform growth:

The PoCServe

Scaling

30 of 114

Scaling-at-large

Allometry

Biology

Physics

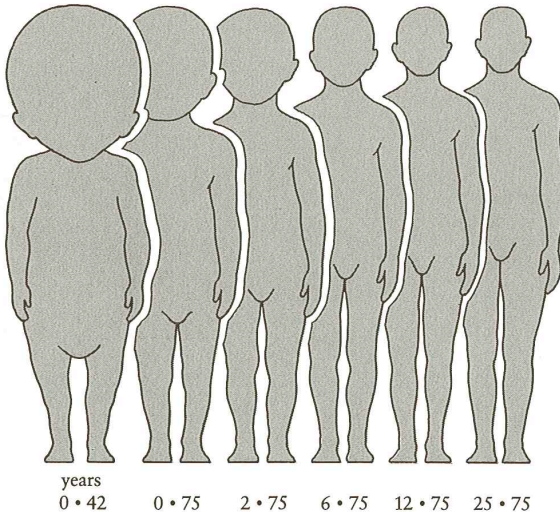
People

Money

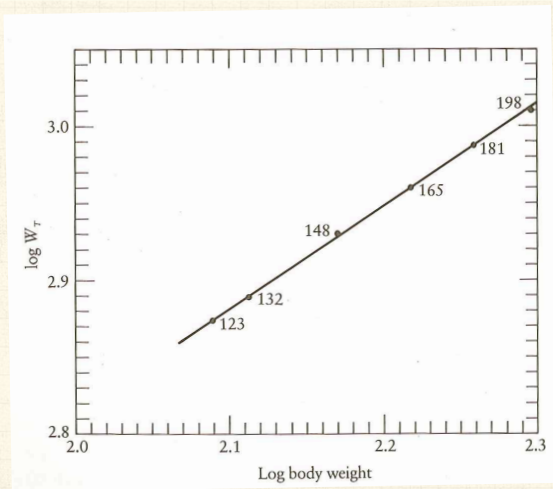
Language

Technology

References



Weightlifting: $M_{\text{world record}} \propto M_{\text{lifter}}^{2/3}$



Idea: Power $\sim$ cross-sectional area of isometric lifters.

p. 53, McMahon and Bonner [26]

The PoCverse

Scaling

32 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

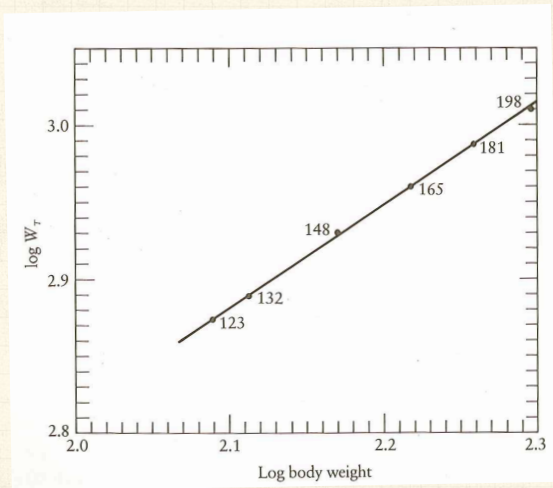
Language

Technology

References



Weightlifting: $M_{\text{world record}} \propto M_{\text{lifter}}^{2/3}$



Idea: Power $\sim$ cross-sectional area of isometric lifters.
But modern data suggests an exponent of $1/2$.

p. 53, McMahon and Bonner [26]

The PoCverse

Scaling

32 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

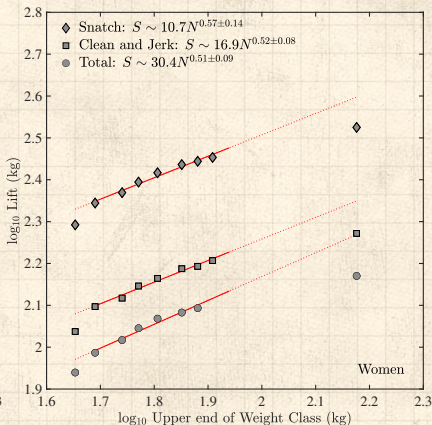
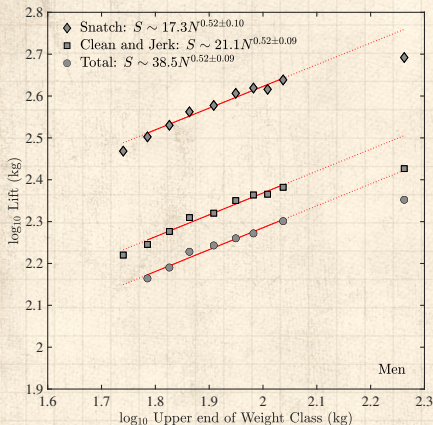
Language

Technology

References

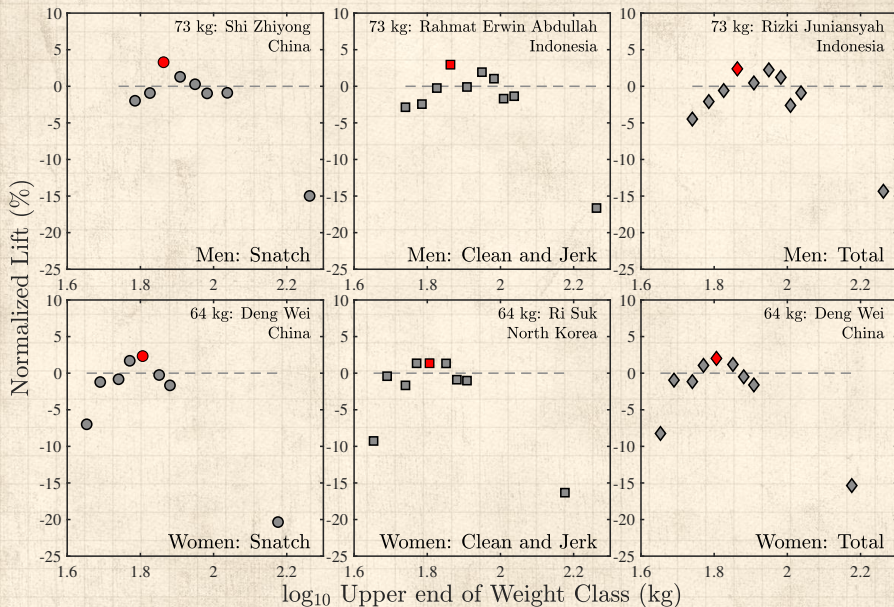


Evidence for a 1/2 scaling exponent for weightlifting:



Li Wenwen's gold medal joy in Paris: [Enjoy](#) (at 2:25 with bonus Australian commentary).

The “best” overall lifters:



Stories—The Fraction Assassin:²

The PoCVerse

Scaling

35 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Animal power

The PoCSverse

Scaling

36 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental biological and ecological constraint:

$$P = c M^\alpha$$

P = basal metabolic rate

M = organismal body mass



Animal power

The PoCSverse

Scaling

36 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental biological and ecological constraint:

$$P = c M^{\alpha}$$

P = basal metabolic rate

M = organismal body mass



$$P = c M^\alpha$$

Prefactor c depends on **body plan** and **body temperature**:

The PoCSverse

Scaling

37 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



$$P = c M^\alpha$$

Prefactor c depends on **body plan** and **body temperature**:

Birds	39–41 °C
Eutherian Mammals	36–38 °C
Marsupials	34–36 °C
Monotremes	30–31 °C



What one might expect:

$$\alpha = 2/3$$

The PoCSverse

Scaling

38 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



What one might expect:

$\alpha = 2/3$ because ...



Dimensional analysis suggests
an energy balance surface law:

$$P \propto S \propto V^{2/3} \propto M^{2/3}$$

The PoCVerse

Scaling

38 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



What one might expect:

$\alpha = 2/3$ because ...

 Dimensional analysis suggests
an energy balance surface law:

$$P \propto S \propto V^{2/3} \propto M^{2/3}$$

 Assumes isometric scaling (not quite the spherical cow).

The PoCSverse

Scaling

38 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



What one might expect:

$\alpha = 2/3$ because ...

 Dimensional analysis suggests
an energy balance surface law:

$$P \propto S \propto V^{2/3} \propto M^{2/3}$$

 Assumes isometric scaling (not quite the spherical cow).

 **Lognormal fluctuations:**
Gaussian fluctuations in $\log P$ around $\log cM^\alpha$.

The PoCVerse

Scaling

38 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The PoCSverse

Scaling

38 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

$\alpha = 2/3$ because ...



Dimensional analysis suggests
an energy balance surface law:

$$P \propto S \propto V^{2/3} \propto M^{2/3}$$



Assumes isometric scaling (not quite the spherical cow).



Lognormal fluctuations:

Gaussian fluctuations in $\log P$ around $\log cM^\alpha$.



Stefan-Boltzmann law  for radiated energy:

$$\frac{dE}{dt} = \sigma \varepsilon S T^4 \propto S$$



Sharks are super-duper isometric:



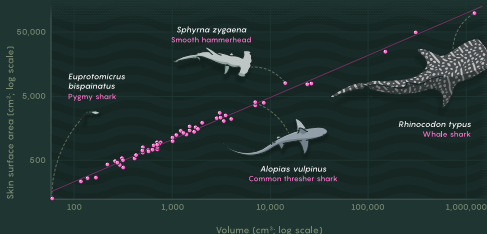
“The geometry of life: Testing the scaling of whole-organism surface area and volume using sharks”

Gayford et al.,

Royal Society Open Science, **12**, 242205, 2025. ^[11]

The Geometric Scaling of Sharks

The biggest study of geometric scaling in large animals found that life follows math's rules. As a 3D object grows while maintaining its shape, its surface area scales as the two-thirds power of volume. This is also true for a diverse set of shark species.



Quanta Magazine article: “Shark Data Suggests Animals Scale Like Geometric Objects”

The PoCVerse

Scaling

39 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The prevailing belief of the Church of Quarterology:

The PoCVerse

Scaling

40 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

$$\alpha = 3/4$$

$$P \propto M^{3/4}$$



The prevailing belief of the Church of Quarterology:

The PoCVerse

Scaling

40 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

$$\alpha = 3/4$$

$$P \propto M^{3/4}$$

Huh?



The prevailing belief of the Church of Quarterology:

The PoCVerse

Scaling

41 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Most obvious concern:

$$3/4 - 2/3 = 1/12$$



An exponent higher than $2/3$ points suggests a fundamental inefficiency in biology.



The prevailing belief of the Church of Quarterology:

The PoCVerse

Scaling

41 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Most obvious concern:

$$3/4 - 2/3 = 1/12$$

-  An exponent higher than $2/3$ points suggests a fundamental inefficiency in biology.
-  Organisms must somehow be running 'hotter' than they need to balance heat loss.



Related putative scalings:

The PoCSverse

Scaling

42 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Wait! There's more!:

-  number of capillaries $\propto M^{3/4}$
-  time to reproductive maturity $\propto M^{1/4}$
-  heart rate $\propto M^{-1/4}$
-  cross-sectional area of aorta $\propto M^{3/4}$
-  population density $\propto M^{-3/4}$



The great 'law' of heartbeats:

Assuming:

-  Average lifespan $\propto M^{\beta}$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

The PoCSverse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The great 'law' of heartbeats:

Assuming:

-  Average lifespan $\propto M^\beta$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

The PoCVerse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The great 'law' of heartbeats:

Assuming:

-  Average lifespan $\propto M^{\beta}$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

-  Average number of heart beats in a lifespan

The PoCVerse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The great 'law' of heartbeats:

The PoCSverse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Assuming:

-  Average lifespan $\propto M^\beta$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

-  Average number of heart beats in a lifespan
 $\simeq (\text{Average lifespan}) \times (\text{Average heart rate})$



The great 'law' of heartbeats:

The PoCSverse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Assuming:

-  Average lifespan $\propto M^{\beta}$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

$$\begin{aligned} \text{Average number of heart beats in a lifespan} \\ \simeq (\text{Average lifespan}) \times (\text{Average heart rate}) \\ \propto M^{\beta-\beta} \end{aligned}$$



The great 'law' of heartbeats:

Assuming:

-  Average lifespan $\propto M^\beta$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

$$\begin{aligned} \text{Average number of heart beats in a lifespan} \\ \simeq (\text{Average lifespan}) \times (\text{Average heart rate}) \\ \propto M^{\beta-\beta} \\ \propto M^0 \end{aligned}$$

The PoCVerse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The great 'law' of heartbeats:

The PoCSverse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Assuming:

-  Average lifespan $\propto M^\beta$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

$$\begin{aligned} \text{Average number of heart beats in a lifespan} \\ \simeq (\text{Average lifespan}) \times (\text{Average heart rate}) \\ \propto M^{\beta-\beta} \\ \propto M^0 \end{aligned}$$

-  Number of heartbeats per life time is independent of organism size!



The great 'law' of heartbeats:

The PoCVerse

Scaling

43 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Assuming:

-  Average lifespan $\propto M^{\beta}$
-  Average heart rate $\propto M^{-\beta}$
-  Irrelevant but perhaps $\beta = 1/4$.

Then:

$$\begin{aligned} \text{Average number of heart beats in a lifespan} \\ \simeq (\text{Average lifespan}) \times (\text{Average heart rate}) \\ \propto M^{\beta-\beta} \\ \propto M^0 \end{aligned}$$

-  Number of heartbeats per life time is independent of organism size!
-  ≈ 1.5 billion....





Allegedly (data is messy): [21, 19]



“An equilibrium theory of insular zoogeography” ↗
MacArthur and Wilson,
Evolution, **17**, 373–387, 1963. [21]



$$N_{\text{species}} \propto A^{\beta}$$



According to physicists—on islands: $\beta \approx 1/4$.



Also—on continuous land: $\beta \approx 1/8$.



Cancer:



“Variation in cancer risk among tissues can be explained by the number of stem cell divisions” ↗

Tomasetti and Vogelstein,
Science, **347**, 78–81, 2015. [36]

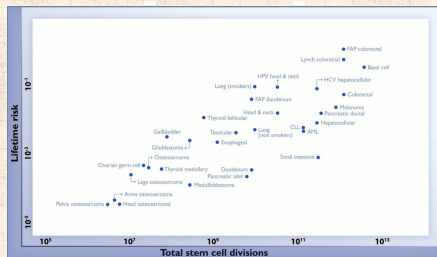


Fig. 1. The relationship between the number of stem cell divisions in the lifetime of a given tissue and the lifetime risk of cancer in that tissue. Values are from Table S1, the derivation of which is discussed in the supplementary materials.

Roughly: $p \sim r^{2/3}$ where p = life time probability and r = rate of stem cell replication.

The PoCSeve

Scaling

46 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



“How fast do living organisms move: Maximum speeds from bacteria to elephants and whales” ↗

Meyer-Vernet and Rospars,
American Journal of Physics, **83**, 719–722, 2015. [28]

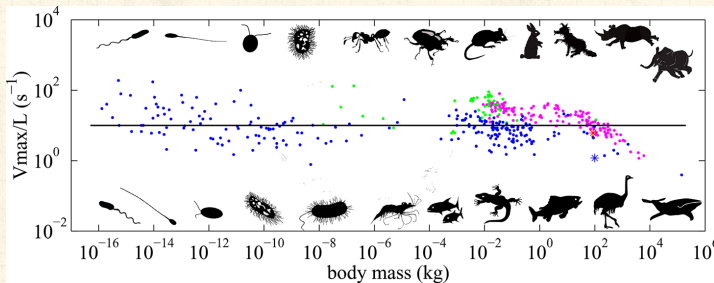


Fig. 1. Maximum relative speed versus body mass for 202 running species (157 mammals plotted in magenta and 45 non-mammals plotted in green), 127 swimming species and 91 micro-organisms (plotted in blue). The sources of the data are given in Ref. 16. The solid line is the maximum relative speed [Eq. (13)] estimated in Sec. III. The human world records are plotted as asterisks (upper for running and lower for swimming). Some examples of organisms of various masses are sketched in black (drawings by François Meyer).

Insert assignment question ↗



“A general scaling law reveals why the largest animals are not the fastest” 

Hirt et al.,

Nature Ecology & Evolution, **1**, 1116, 2017. ^[12]

The PoCSverse

Scaling

48 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

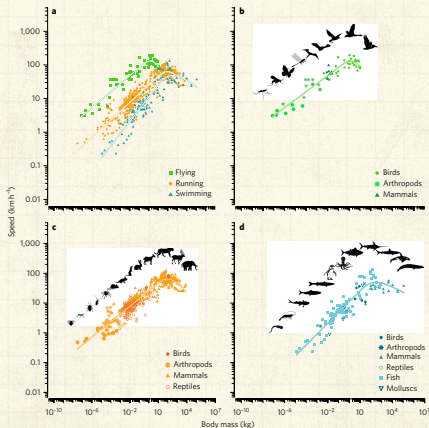


Figure 2 | Empirical data and time-dependent model fit for the allometric scaling of maximum speed. **a**, Comparison of scaling for the different locomotion modes (flying, running, swimming). **b–d**, Taxonomic differences are illustrated separately for flying (**b**; $n=55$), running (**c**; $n=458$) and swimming (**d**; $n=109$) animals. Overall model fit: $R^2=0.893$. The residual variance does not exhibit a signature of taxonomy (only a weak effect of thermoregulation; see Methods).





“A general scaling law reveals why the largest animals are not the fastest” 

Hirt et al.,

Nature Ecology & Evolution, **1**, 1116, 2017. ^[12]

The PoCSevere

Scaling

49 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

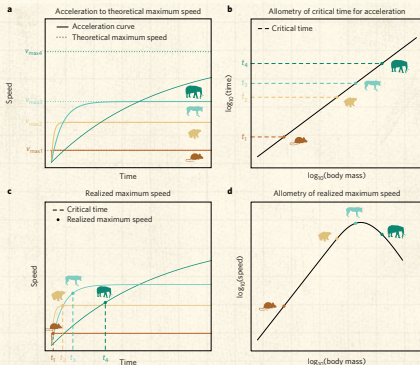
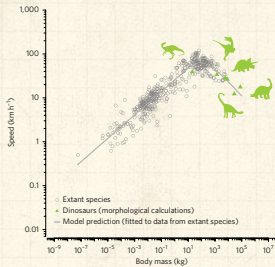


Figure 1 | Concept of time-dependent and mass-dependent realized maximum speed of animals. **a**, Acceleration of animals follows a saturation curve (solid lines) approaching the theoretical maximum speed (dotted lines) depending on body mass (colour code). **b**, The time available for acceleration increases with body mass following a power law. **c**, **d**, This critical time determines the realized maximum speed (**c**), yielding a hump-shaped increase of maximum speed with body mass (**d**).



Theoretical story:



Maximum speed increases with size:

$$v_{\max} = aM^b$$

Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

Note: [28] not cited.



Theoretical story:

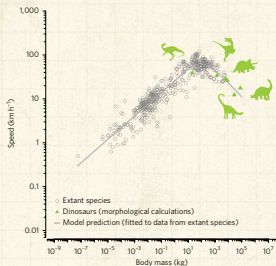


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.



Maximum speed increases with size:

$$v_{\max} = aM^b$$



Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$

The PoCServe

Scaling

50 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Note: [28] not cited.



Theoretical story:

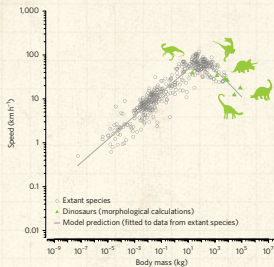


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.



Maximum speed increases with size:

$$v_{\max} = aM^b$$



Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$



$$k \sim F_{\max}/M \sim cM^{d-1}$$

Literature: $0.75 \lesssim d \lesssim 0.94$

Note: [28] not cited.



Theoretical story:

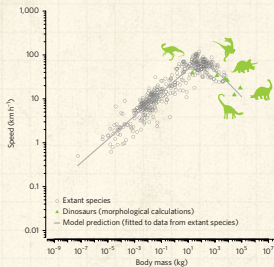


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

- Maximum speed increases with size:

$$v_{\max} = aM^b$$

- Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$

- $k \sim F_{\max}/M \sim cM^{d-1}$

Literature: $0.75 \lesssim d \lesssim 0.94$

- Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$

Literature: $0.76 \lesssim g \lesssim 1.27$

Note: [28] not cited.



Theoretical story:

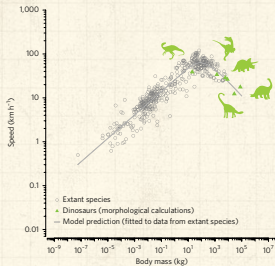


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

Maximum speed increases with size:

$$v_{\max} = aM^b$$

Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$

$$k \sim F_{\max}/M \sim cM^{d-1}$$

Literature: $0.75 \lesssim d \lesssim 0.94$

Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$

Literature: $0.76 \lesssim g \lesssim 1.27$

$$v_{\max} = aM^b \left(1 - e^{-hM^i}\right)$$

Note: [28] not cited.



Theoretical story:

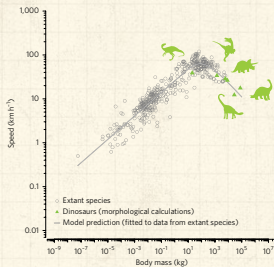


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

Maximum speed increases with size:

$$v_{\max} = aM^b$$

Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$

$$k \sim F_{\max}/M \sim cM^{d-1}$$

Literature: $0.75 \lesssim d \lesssim 0.94$

Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$

Literature: $0.76 \lesssim g \lesssim 1.27$

$$v_{\max} = aM^b \left(1 - e^{-hM^i}\right)$$

$$i = d - 1 + g \text{ and } h = cf$$

Note: [28] not cited.



Theoretical story:

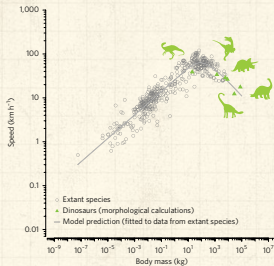


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

Maximum speed increases with size:

$$v_{\max} = aM^b$$

Takes a while to get going:

$$v(t) = v_{\max}(1 - e^{-kt})$$

$$k \sim F_{\max}/M \sim cM^{d-1}$$

$$\text{Literature: } 0.75 \lesssim d \lesssim 0.94$$

Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$

$$\text{Literature: } 0.76 \lesssim g \lesssim 1.27$$

$$v_{\max} = aM^b \left(1 - e^{-hM^i}\right)$$

$$i = d - 1 + g \text{ and } h = cf$$

Literature search for for maximum speeds of running, flying and swimming animals.

Search terms: “maximum speed”, “escape speed”, and “sprint speed”.

Note: [28] not cited.



“Scaling in athletic world records”

Savaglio and Carbone,

Nature, **404**, 244, 2000. [33]



Mean speed $\langle s \rangle$ decays with race time τ :

$$\langle s \rangle \sim \tau^{-\beta}$$

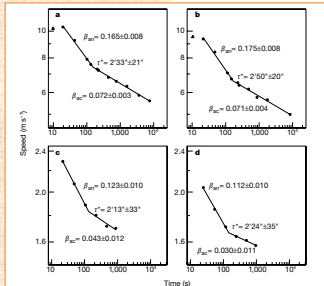


Figure 1 Plots of world record mean speeds against the record time (at November 1999). **a,b**, Running, and **c,d**, swimming records: for men (**a,c**), we consider 11 races (200 m, 400 m, 800 m, 1,600 m, 1,500 m, the mile, 3,000 m, 5,000 m, 10,000 m, 1 hour, and marathon); the same races are considered for women (**b,d**), apart from the 1 hour race. Lines represent the best fits. The scaling exponents β and characteristic times τ^* of the breakpoints are shown; characteristic times have been determined by using a χ^2 minimization on a broken power law. Triangles in **a,b** represent the 100 m race, which is excluded from the analysis because the mean speed is strongly affected by the standing start of athletes.



EEK: Small scaling regimes



“Scaling in athletic world records”

Savaglio and Carbone,

Nature, **404**, 244, 2000. [33]



Mean speed $\langle s \rangle$ decays with race time τ :

$$\langle s \rangle \sim \tau^{-\beta}$$



Break in scaling at around $\tau \simeq 150\text{--}170$ seconds

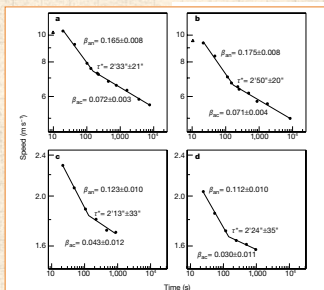


Figure 1 Plots of world record mean speeds against the record time (at November 1999). **a,b**, Running, and **c,d**, swimming records: for men (**a,c**), we consider 11 races (200 m, 400 m, 800 m, 1,600 m, 1,500 m, the mile, 3,000 m, 5,000 m, 10,000 m, 1 hour, and marathon); the same races are considered for women (**b,d**), apart from the 1 hour race. Lines represent the best fits. The scaling exponents β and characteristic times τ^* of the breakpoints are shown; characteristic times have been determined by using a χ^2 minimization on a broken power law. Triangles in **a,b** represent the 100 m race, which is excluded from the analysis because the mean speed is strongly affected by the standing start of athletes.



EEK: Small scaling regimes



“Scaling in athletic world records”

Savaglio and Carbone,

Nature, **404**, 244, 2000. [33]



Mean speed $\langle s \rangle$ decays with race time τ :

$$\langle s \rangle \sim \tau^{-\beta}$$



Break in scaling at around $\tau \simeq 150\text{--}170$ seconds



Anaerobic–aerobic transition

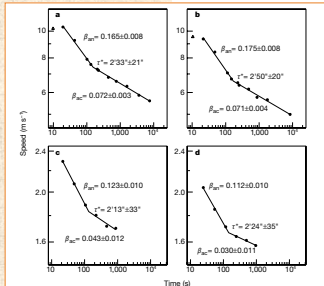


Figure 1 Plots of world record mean speeds against the record time (at November 1999). **a,b**, Running, and **c,d**, swimming records; for men (**a,c**), we consider 11 races (200 m, 400 m, 800 m, 1,600 m, 1,500 m, the mile, 3,000 m, 5,000 m, 10,000 m, 1 hour, and marathon); the same races are considered for women (**b,d**), apart from the 1 hour race. Lines represent the best fits. The scaling exponents β and characteristic times τ^* of the breakpoints are shown; characteristic times have been determined by using a χ^2 minimization on a broken power law. Triangles in **a,b** represent the 100 m race, which is excluded from the analysis because the mean speed is strongly affected by the standing start of athletes.



EEK: Small scaling regimes



“Scaling in athletic world records”

Savaglio and Carbone,

Nature, **404**, 244, 2000. [33]



Mean speed $\langle s \rangle$ decays with race time τ :

$$\langle s \rangle \sim \tau^{-\beta}$$



Break in scaling at around $\tau \simeq 150\text{--}170$ seconds



Anaerobic–aerobic transition



Roughly 1 km running race

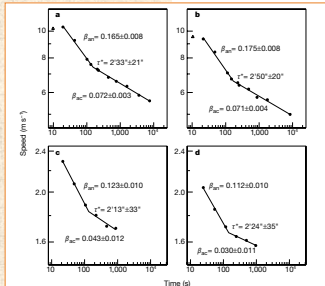


Figure 1 Plots of world record mean speeds against the record time (at November 1999). **a,b**, Running, and **c,d**, swimming records; for men **a,c**, we consider 11 races (200 m, 400 m, 800 m, 1,000 m, 1,500 m, the mile, 3,000 m, 5,000 m, 10,000 m, 1 hour, and marathon); the same races are considered for women **b,d**, apart from the 1 hour race. Lines represent the best fits. The scaling exponents β and characteristic times τ^* of the breakpoints are shown; characteristic times have been determined by using a χ^2 minimization on a broken power law. Triangles in **a,b** represent the 100 m race, which is excluded from the analysis because the mean speed is strongly affected by the standing start of athletes.



EEK: Small scaling regimes

“Scaling in athletic world records”

Savaglio and Carbone,

Nature, **404**, 244, 2000. [33]



Mean speed $\langle s \rangle$ decays with race time τ :

$$\langle s \rangle \sim \tau^{-\beta}$$



Break in scaling at around $\tau \simeq 150\text{--}170$ seconds



Anaerobic–aerobic transition



Roughly 1 km running race



Running decays faster than swimming

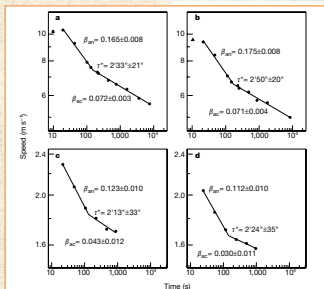
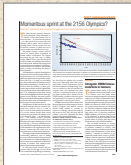


Figure 1 Plots of world record mean speeds against the record time (at November 1999). **a,b**, Running, and **c,d**, swimming records: for men (**a,b**), we consider 11 races (200 m, 400 m, 800 m, 1,000 m, 1,500 m, the mile, 3,000 m, 5,000 m, 10,000 m, 1 hour, and marathon); the same races are considered for women (**c,d**), apart from the 1 hour race. Lines represent the best fits. The scaling exponents β and characteristic times τ^* of the breakpoints are shown; characteristic times have been determined by using a χ^2 minimization on a broken power law. Triangles in **a,b** represent the 100 m race, which is excluded from the analysis because the mean speed is strongly affected by the standing start of athletes.



EEK: Small scaling regimes





“Athletics: Momentous sprint at the 2156 Olympics?” [↗](#)
 Tatem et al.,
 Nature, **431**, 525–525, 2004. ^[35]

Linear extrapolation for the 100 metres:

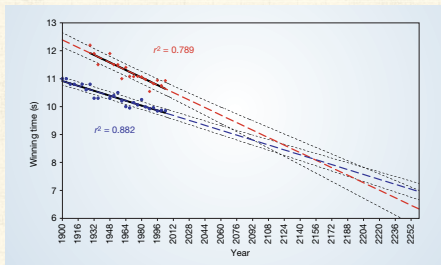


Figure 1 The winning Olympic 100-metre sprint times for men (blue points) and women (red points), with superimposed best-fit linear regression lines (solid black lines) and coefficients of determination. The regression lines are extrapolated (broken blue and red lines for men and women, respectively) and 95% confidence intervals (dotted black lines) based on the available points are superimposed. The projections intersect just before the 2156 Olympics, when the winning women's 100-metre sprint time of 8.079 s will be faster than the men's at 8.098 s.

Tatem: [↗](#) “If I’m wrong anyone is welcome to come and question me about the result after the 2156 Olympics.”

The PoCSeve

Scaling

52 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



“Duration of urination does not change with body size” ,

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

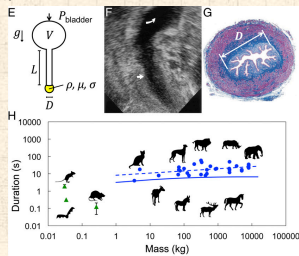


Table 1. Measured allometric relationships for the urinary system of animals

Variable	Unit	Best fit	R^2	N
Duration of urination	T s	$8.2 M^{0.13}$	0.2	32
Urethral length	L mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α —	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder} kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” 
 Yang et al., Proceedings of the National Academy of
 Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

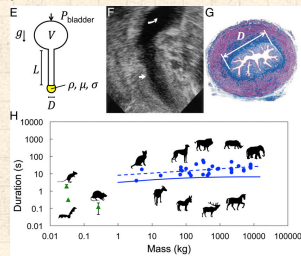


Table 1. Measured allometric relationships for the urinary system of animals

Variable	Unit	Best fit	R^2	N
Duration of urination	T s	$8.2 M^{0.13}$	0.2	32
Urethral length	L mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α —	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder} kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” ,

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

 $M = 3 \times 10^1 \text{ g to } 8 \times 10^6 \text{ g}$

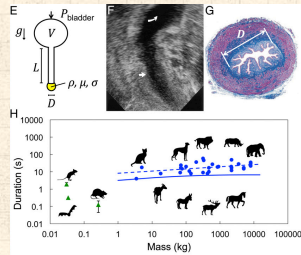


Table 1. Measured allometric relationships for the urinary system of animals

Variable	Unit	Best fit	R^2	N
Duration of urination	T s	$8.2 M^{0.13}$	0.2	32
Urethral length	L mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α —	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder} kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” 

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

 $M = 3 \times 10^1 \text{ g to } 8 \times 10^6 \text{ g}$

 For $\geq 3 \times 10^3 \text{ g}$, $T \sim M^{1/6}$

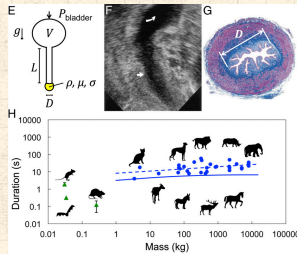


Table 1. Measured allometric relationships for the urinary system of animals

	Variable	Unit	Best fit	R^2	N
Duration of urination	T	s	$8.2 M^{0.13}$	0.2	32
Urethral length	L	mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D	mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α	—	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V	mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder}	kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f	mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m	mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” ,

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

 $M = 3 \times 10^1 \text{ g to } 8 \times 10^6 \text{ g}$

 For $\geq 3 \times 10^3 \text{ g}$, $T \sim M^{1/6}$

 Duration $\sim 21 \pm 13$ seconds

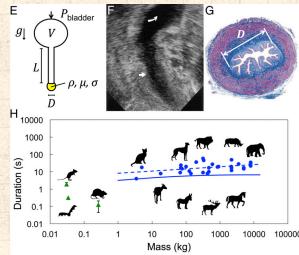


Table 1. Measured allometric relationships for the urinary system of animals

	Variable	Unit	Best fit	R^2	N
Duration of urination	T	s	$8.2 M^{0.13}$	0.2	32
Urethral length	L	mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D	mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α	—	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V	mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder}	kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f	mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m	mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” 

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

 32 mammals at Zoo Atlanta

 Figs. 1 and 2 are NSFTCR³

 $M = 3 \times 10^1 \text{ g to } 8 \times 10^6 \text{ g}$

 For $\geq 3 \times 10^3 \text{ g}$, $T \sim M^{1/6}$

 Duration $\sim 21 \pm 13$ seconds

 Smaller mammals: $T \sim M^0$

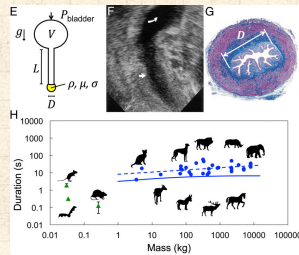


Table 1. Measured allometric relationships for the urinary system of animals

	Variable	Unit	Best fit	R^2	N
Duration of urination	T	s	$8.2 M^{0.13}$	0.2	32
Urethral length	L	mm	$35 M^{0.43}$	0.9	47
Urethral diameter	D	mm	$2.0 M^{0.39}$	0.9	22
Shape factor	α	—	$0.2 M^{-0.05}$	0.5	5
Bladder capacity	V	mL	$4.6 M^{0.97}$	0.9	9
Bladder pressure	P_{bladder}	kPa	$5.2 M^{-0.01}$	0.02	8
Flow rate for females	Q_f	mL/s	$1.8 M^{0.66}$	0.9	16
Flow rate for males	Q_m	mL/s	$0.3 M^{0.92}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



“Duration of urination does not change with body size” ,

Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. ^[38]

-  32 mammals at Zoo Atlanta
-  Figs. 1 and 2 are NSFTCR³
-  $M = 3 \times 10^1 \text{ g to } 8 \times 10^6 \text{ g}$
-  For $\geq 3 \times 10^3 \text{ g}$, $T \sim M^{1/6}$
-  Duration $\sim 21 \pm 13$ seconds
-  Smaller mammals: $T \sim M^0$
-  Duration ~ 0.02 to 2 seconds

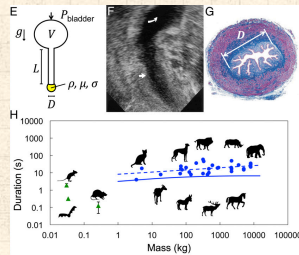


Table 1. Measured allometric relationships for the urinary system of animals

Variable	Unit	Best fit	R^2	N
Duration of urination	T	s	$8.2 M^{0.13}$	0.2 32
Urethral length	L	mm	$35 M^{0.43}$	0.9 47
Urethral diameter	D	mm	$2.0 M^{0.39}$	0.9 22
Shape factor	α	—	$0.2 M^{-0.05}$	0.5 5
Bladder capacity	V	mL	$4.6 M^{0.97}$	0.9 9
Bladder pressure	P_{bladder}	kPa	$5.2 M^{-0.01}$	0.02 8
Flow rate for females	Q_f	mL/s	$1.8 M^{0.66}$	0.9 16
Flow rate for males	Q_m	mL/s	$0.3 M^{0.92}$	0.9 15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.



Where this was always going:⁴

The PoCSverse

Scaling

54 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



⁴David Hu's papers on the fluid mechanics of interesting things [!\[\]\(bcb1dee6692db79eb60fa53f10033711_img.jpg\)](#)

Where this was always going:⁴

 Ig Nobel in Physics in 2015 

The PoCSverse

Scaling

54 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



⁴David Hu's papers on the fluid mechanics of interesting things 

Where this was always going:⁴

 Ig Nobel in Physics in 2015 

 And again in 2019 for a paper on a peculiarity of wombats [?] 

The PoCSverse

Scaling

54 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



⁴David Hu's papers on the fluid mechanics of interesting things 

Where this was always going:⁴

 Ig Nobel in Physics in 2015 

 And again in 2019 for a paper on a peculiarity of wombats^[?] 



⁴David Hu's papers on the fluid mechanics of interesting things 

From How do wombats poop cubes? Scientists get to the bottom of the mystery ↗, Science, 2021/01/27:

‘That just leaves one mystery: why wombats evolved cubic poop in the first place.

The PoCVerse

Scaling

55 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



From How do wombats poop cubes? Scientists get to the bottom of the mystery ↗, Science, 2021/01/27:

‘That just leaves one mystery: why wombats evolved cubic poop in the first place.

Hu speculates that because the animals climb up on rocks and logs to mark their territory, the flat-sided [poops] aren’t as likely to roll off from these high perches.

...

The PoCVerse

Scaling

55 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



From How do wombats poop cubes? Scientists get to the bottom of the mystery ↗, Science, 2021/01/27:

‘That just leaves one mystery: why wombats evolved cubic poop in the first place.

Hu speculates that because the animals climb up on rocks and logs to mark their territory, the flat-sided [poops] aren’t as likely to roll off from these high perches.

...

In the meantime, Hu also thinks this knowledge could help researchers raising wombats in captivity.



From How do wombats poop cubes? Scientists get to the bottom of the mystery ↗, Science, 2021/01/27:

‘That just leaves one mystery: why wombats evolved cubic poop in the first place.

Hu speculates that because the animals climb up on rocks and logs to mark their territory, the flat-sided [poops] aren’t as likely to roll off from these high perches.

...

In the meantime, Hu also thinks this knowledge could help researchers raising wombats in captivity.

“Sometimes their [poops] aren’t as cubic as the [wild] ones,” he says.



From How do wombats poop cubes? Scientists get to the bottom of the mystery ↗, Science, 2021/01/27:

‘That just leaves one mystery: why wombats evolved cubic poop in the first place.

Hu speculates that because the animals climb up on rocks and logs to mark their territory, the flat-sided [poops] aren’t as likely to roll off from these high perches.

...

In the meantime, Hu also thinks this knowledge could help researchers raising wombats in captivity.

“Sometimes their [poops] aren’t as cubic as the [wild] ones,” he says.

The squarer the poop, the healthier the wombat.’

The PoCverse

Scaling

55 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Engines:

The PoCVerse

Scaling

56 of 114

Scaling-at-large

Allometry

Biology

Physics

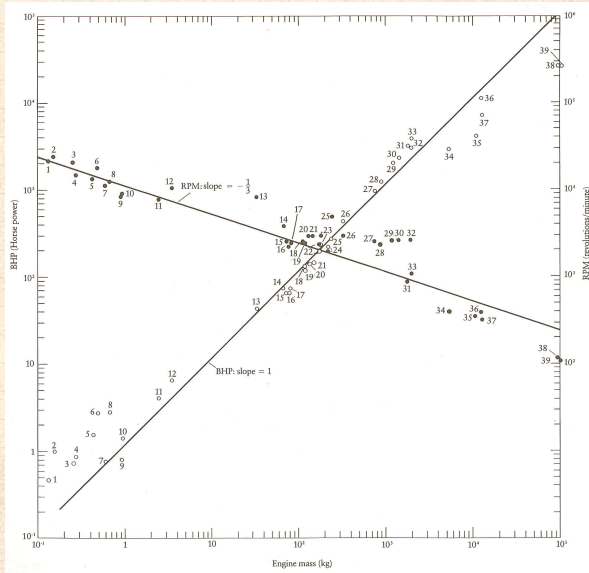
People

Money

Language

Technology

References

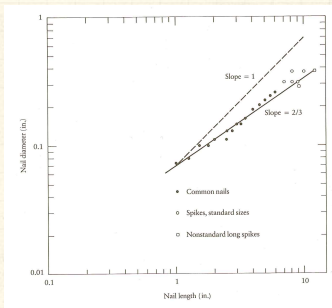


BHP = brake horse power



The allometry of nails:

Observed: Diameter $\propto$ Length^{2/3} or $d \propto \ell^{2/3}$.



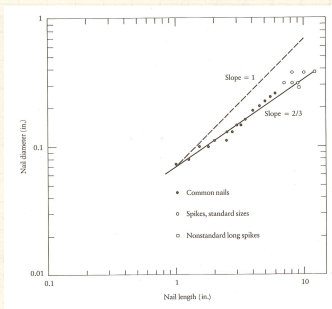
Since $\ell d^2 \propto$ Volume v :

p. 58–59, McMahon and Bonner [26]



The allometry of nails:

Observed: Diameter $\propto$ Length $^{2/3}$ or $d \propto \ell^{2/3}$.



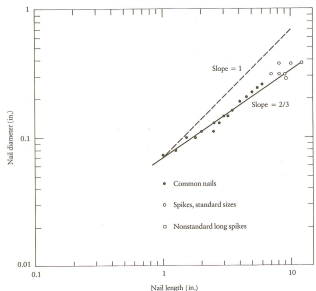
Since $\ell d^2 \propto$ Volume v :

 Diameter $\propto$



The allometry of nails:

Observed: Diameter $\propto$ Length $^{2/3}$ or $d \propto \ell^{2/3}$.



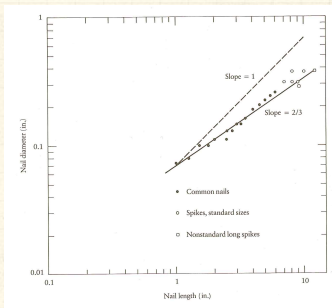
Since $\ell d^2 \propto$ Volume v :

 Diameter $\propto$ Mass $^{2/7}$ or $d \propto v^{2/7}$.



The allometry of nails:

Observed: Diameter $\propto$ Length $^{2/3}$ or $d \propto \ell^{2/3}$.



Since $\ell d^2 \propto$ Volume v :

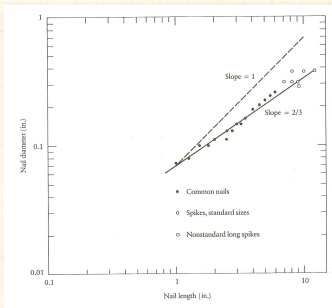
 Diameter $\propto$ Mass $^{2/7}$ or $d \propto v^{2/7}$.

 Length $\propto$



The allometry of nails:

Observed: Diameter $\propto$ Length $^{2/3}$ or $d \propto \ell^{2/3}$.



Since $\ell d^2 \propto$ Volume v :

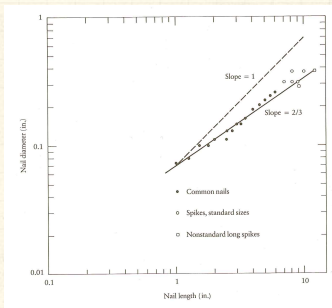
 Diameter $\propto$ Mass $^{2/7}$ or $d \propto v^{2/7}$.

 Length $\propto$ Mass $^{3/7}$ or $\ell \propto v^{3/7}$.



The allometry of nails:

Observed: Diameter $\propto$ Length $^{2/3}$ or $d \propto \ell^{2/3}$.



Since $\ell d^2 \propto$ Volume v :

 Diameter $\propto$ Mass $^{2/7}$ or $d \propto v^{2/7}$.

 Length $\propto$ Mass $^{3/7}$ or $\ell \propto v^{3/7}$.

 Nails lengthen faster than they broaden (c.f. trees).

p. 58–59, McMahon and Bonner ^[26]



The allometry of nails:

A buckling instability?:

The PoCSverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The allometry of nails:

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

The PoCSverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



The allometry of nails:

The PoCVerse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .



The allometry of nails:

The PoCverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .

 Match forces independent of nail size: $d^4/\ell^2 \propto d$.



The allometry of nails:

The PoCverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .

 Match forces independent of nail size: $d^4/\ell^2 \propto d$.

 Leads to $d \propto \ell^{2/3}$.



The allometry of nails:

The PoCverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .

 Match forces independent of nail size: $d^4/\ell^2 \propto d$.

 Leads to $d \propto \ell^{2/3}$.

 Argument made by Galileo ^[10] in 1638 in “Discourses on Two New Sciences.”  Also, see here. 



The allometry of nails:

The PoCverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .

 Match forces independent of nail size: $d^4/\ell^2 \propto d$.

 Leads to $d \propto \ell^{2/3}$.

 Argument made by Galileo ^[10] in 1638 in “Discourses on Two New Sciences.”  Also, see here. 

 Another smart person’s contribution: Euler, 1757 



The allometry of nails:

The PoCverse

Scaling

58 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

A buckling instability?:

 Physics/Engineering result : Columns buckle under a load which depends on d^4/ℓ^2 .

 To drive nails in, posit resistive force $\propto$ nail circumference = πd .

 Match forces independent of nail size: $d^4/\ell^2 \propto d$.

 Leads to $d \propto \ell^{2/3}$.

 Argument made by Galileo ^[10] in 1638 in “Discourses on Two New Sciences.”  Also, see here. 

 Another smart person’s contribution: Euler, 1757 

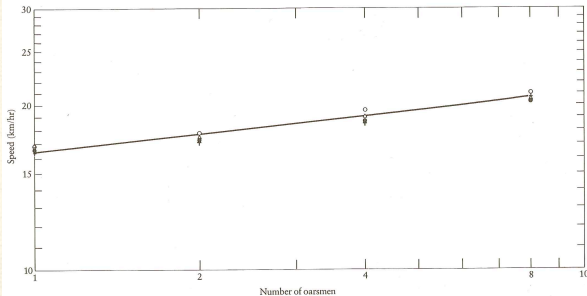
 Also see McMahon, “Size and Shape in Biology,” Science, 1973. ^[25]



Rowing: Speed $\propto$ (number of rowers)^{1/9}

Shell dimensions and performances.

No. of oarsmen	Modifying description	Length, l (m)	Beam, b (m)	l/b	Boat mass per oarsman (kg)	Time for 2000 m (min)			
						I	II	III	IV
8	Heavyweight	18.28	0.610	30.0	14.7	5.87	5.92	5.82	5.73
8	Lightweight	18.28	0.598	30.6	14.7				
4	With coxswain	12.80	0.574	22.3	18.1				
4	Without coxswain	11.75	0.574	21.0	18.1	6.33	6.42	6.48	6.13
2	Double scull	9.76	0.381	25.6	13.6				
2	Pair-oared shell	9.76	0.356	27.4	13.6	6.87	6.92	6.95	6.77
1	Single scull	7.93	0.293	27.0	16.3	7.16	7.25	7.28	7.17



Very weak scaling and size variation but it's theoretically explainable ...

The PoCverse

Scaling

59 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Scaling in elementary laws of physics:



Inverse-square law of gravity and Coulomb's law:

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$



Force is diminished by expansion of space away from source.



Scaling in elementary laws of physics:



Inverse-square law of gravity and Coulomb's law:

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$



Force is diminished by expansion of space away from source.



The square is $d - 1 = 3 - 1 = 2$, the dimension of a sphere's surface.



Scaling in elementary laws of physics:

 Inverse-square law of gravity and Coulomb's law:

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$

 Force is diminished by expansion of space away from source.

 The square is $d - 1 = 3 - 1 = 2$, the dimension of a sphere's surface.

 We'll see a gravity law applies for a range of human phenomena.



Dimensional Analysis:

The PoCVerse

Scaling

61 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

The Buckingham π theorem ⁵

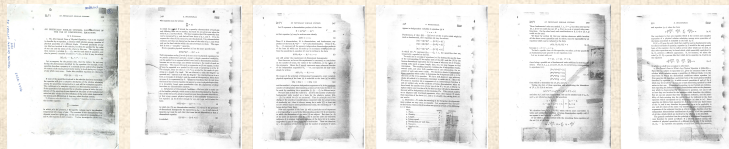


“On Physically Similar Systems: Illustrations of the Use of Dimensional Equations” 

E. Buckingham,

Phys. Rev., **4**, 345–376, 1914. ^[7]

As captured in the 1990s in the MIT physics library:



⁵Stigler’s Law of Eponymy  applies yet again. See [here](#) . More later.

Dimensional Analysis:⁶

Fundamental equations cannot depend on units:

The PoCSverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCSverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

 System involves n related quantities with some unknown equation
 $f(q_1, q_2, \dots, q_n) = 0$.



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

 System involves n related quantities with some unknown equation
 $f(q_1, q_2, \dots, q_n) = 0$.

 Geometric ex.: area of a square, side length ℓ :
 $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

 System involves n related quantities with some unknown equation $f(q_1, q_2, \dots, q_n) = 0$.

 Geometric ex.: area of a square, side length ℓ :
 $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.

 Rewrite as a relation of $p \leq n$ independent dimensionless parameters  where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

System involves n related quantities with some unknown equation
 $f(q_1, q_2, \dots, q_n) = 0$.

Geometric ex.: area of a square, side length ℓ :
 $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.

Rewrite as a relation of $p \leq n$ independent dimensionless
parameters  where p is the number of independent dimensions
(mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$

e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

 System involves n related quantities with some unknown equation $f(q_1, q_2, \dots, q_n) = 0$.

 Geometric ex.: area of a square, side length ℓ :
 $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.

 Rewrite as a relation of $p \leq n$ independent dimensionless parameters  where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$

 e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.

 Another example: $F = ma \Rightarrow F/ma - 1 = 0$.



⁶Length is a dimension, furlongs and smoots  are units

Dimensional Analysis:⁶

The PoCverse

Scaling

62 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Fundamental equations cannot depend on units:

 System involves n related quantities with some unknown equation $f(q_1, q_2, \dots, q_n) = 0$.

 Geometric ex.: area of a square, side length ℓ :
 $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.

 Rewrite as a relation of $p \leq n$ independent dimensionless parameters  where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$

 e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.

 Another example: $F = ma \Rightarrow F/ma - 1 = 0$.

 Plan: solve problems using only backs of envelopes.



⁶Length is a dimension, furlongs and smoots  are units

Example:

Simple pendulum:



Idealized mass/platypus
swinging forever.

The PoCverse

Scaling

63 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

The PoCVerse

Scaling

63 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,

The PoCverse

Scaling

63 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,

The PoCverse

Scaling

63 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,
3. gravitational acceleration g , and



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,
3. gravitational acceleration g , and
4. pendulum's period τ .



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,
3. gravitational acceleration g , and
4. pendulum's period τ .



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,
3. gravitational acceleration g , and
4. pendulum's period τ .



Variable dimensions: $[\ell] = L$, $[m] = M$, $[g] = LT^{-2}$, and $[\tau] = T$.



Example:

Simple pendulum:



Idealized mass/platypus swinging forever.



Four quantities:

1. Length ℓ ,
2. mass m ,
3. gravitational acceleration g , and
4. pendulum's period τ .



Variable dimensions: $[\ell] = L$, $[m] = M$, $[g] = LT^{-2}$, and $[\tau] = T$.



Turn over your envelopes and find some π 's.



A little formalism:

The PoCSverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:



Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

The PoCSverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:



Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).



Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

The PoCverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

The PoCverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:

Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

The PoCverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:

Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,

The PoCverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

 Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

 For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

The PoCverse

Scaling

64 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

 Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

 For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

 So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.



A little formalism:

Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.

We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

 Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

 For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

 So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.

 We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.

 We now need: $x_1 + x_3 = 0$, $x_2 = 0$, and $-2x_3 + x_4 = 0$.



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

 Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

 For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

 So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.

 We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.

 We now need: $x_1 + x_3 = 0$, $x_2 = 0$, and $-2x_3 + x_4 = 0$.

 Time for



A little formalism:

 Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).

 Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.

 We (desperately) want to find all sets of powers x_j that create dimensionless quantities.

 Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.

 For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$,
with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.

 So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.

 We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.

 We now need: $x_1 + x_3 = 0$, $x_2 = 0$, and $-2x_3 + x_4 = 0$.

 Time for **matrixology** ...



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The PoCVerse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3$

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0$

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$



In general: Create a matrix $\mathbf{A}$ where ij th entry is the power of dimension i in the j th variable, and solve by row reduction to find basis null vectors.

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$



In general: Create a matrix $\mathbf{A}$ where ij th entry is the power of dimension i in the j th variable, and solve by row reduction to find basis null vectors.



We (you) find: $\pi_1 = \ell/g\tau^2 = \text{const.}$

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$



In general: Create a matrix $\mathbf{A}$ where ij th entry is the power of dimension i in the j th variable, and solve by row reduction to find basis null vectors.



We (you) find: $\pi_1 = \ell/g\tau^2 = \text{const.}$ Upshot: $\tau \propto \sqrt{\ell}$.

The PoCverse

Scaling

65 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Well, of course there are matrices:



Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.



Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.



Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$



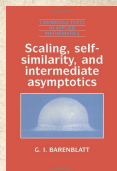
In general: Create a matrix $\mathbf{A}$ where ij th entry is the power of dimension i in the j th variable, and solve by row reduction to find basis null vectors.



We (you) find: $\pi_1 = \ell/g\tau^2 = \text{const.}$ Upshot: $\tau \propto \sqrt{\ell}$.

Insert assignment question





“Scaling, self-similarity, and intermediate asymptotics”

by G. I. Barenblatt (1996). ^[2]

The PoCVerse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

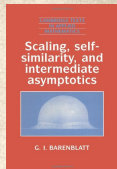
Money

Language

Technology

References





“Scaling, self-similarity, and intermediate asymptotics” [a](#) [↗](#)

by G. I. Barenblatt (1996). [2]

G. I. Taylor, magazines, and classified secrets:

The PoCVerse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

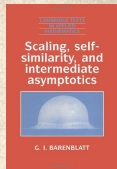
Money

Language

Technology

References





“Scaling, self-similarity, and intermediate asymptotics”

by G. I. Barenblatt (1996). ^[2]

The PoCVerse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

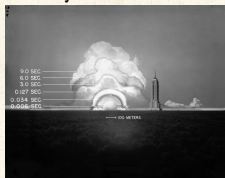
References

G. I. Taylor, magazines, and classified secrets:

1945

New Mexico

Trinity test:



Self-similar blast wave:



Radius: $[R] = L$,

Time: $[t] = T$,

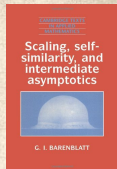
Density of air: $[\rho] = M/L^3$,

Energy: $[E] = ML^2/T^2$.



Four variables, three dimensions.





“Scaling, self-similarity, and intermediate asymptotics”

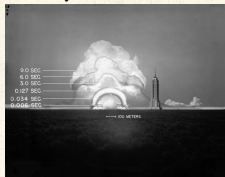
by G. I. Barenblatt (1996). [2]

G. I. Taylor, magazines, and classified secrets:

1945

New Mexico

Trinity test:



Self-similar blast wave:



Radius: $[R] = L$,

Time: $[t] = T$,

Density of air: $[\rho] = M/L^3$,

Energy: $[E] = ML^2/T^2$.



Four variables, three dimensions.



One dimensionless variable:

$E = \text{constant} \times \rho R^5/t^2$.

The PoCverse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

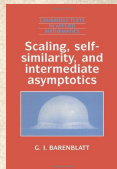
Money

Language

Technology

References





“Scaling, self-similarity, and intermediate asymptotics”

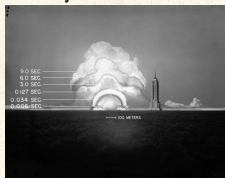
by G. I. Barenblatt (1996). ^[2]

G. I. Taylor, magazines, and classified secrets:

1945

New Mexico

Trinity test:



Self-similar blast wave:



Radius: $[R] = L$,

Time: $[t] = T$,

Density of air: $[\rho] = M/L^3$,

Energy: $[E] = ML^2/T^2$.



Four variables, three dimensions.



One dimensionless variable:

$E = \text{constant} \times \rho R^5/t^2$.



Scaling: Speed decays as $1/R^{3/2}$.

The PoCverse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

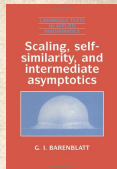
Money

Language

Technology

References





“Scaling, self-similarity, and intermediate asymptotics” [a](#) [↗](#)

by G. I. Barenblatt (1996). ^[2]

The PoCverse

Scaling

66 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

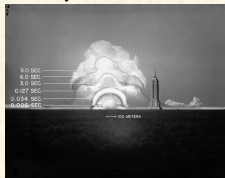
References

G. I. Taylor, magazines, and classified secrets:

1945

New Mexico

Trinity test:



Self-similar blast wave:



Radius: $[R] = L$,

Time: $[t] = T$,

Density of air: $[\rho] = M/L^3$,

Energy: $[E] = ML^2/T^2$.



Four variables, three dimensions.



One dimensionless variable:

$E = \text{constant} \times \rho R^5 / t^2$.



Scaling: Speed decays as $1/R^{3/2}$.

Related: Radiolab's [Elements](#) [↗](#) on the Cold War, the Bomb Pulse, and the dating of cell age (33:30).



Sorting out base units of fundamental measurement:

The PoCverse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

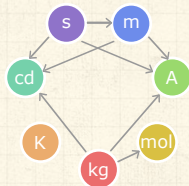
Money

Language

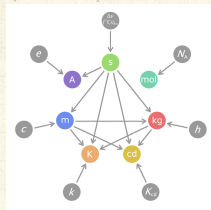
Technology

References

SI base units were redefined in 2019: 



by Dono/Wikipedia



by Wikipetzi/Wikipedia



Sorting out base units of fundamental measurement:

The PoCVerse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

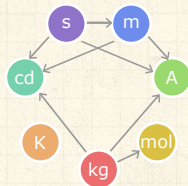
Money

Language

Technology

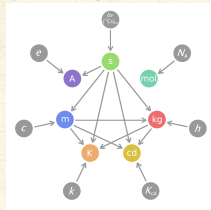
References

SI base units were redefined in 2019: 



Now: kilogram is an artifact  in
Sèvres, France.

by Dono/Wikipedia



by Wikipetzi/Wikipedia



Sorting out base units of fundamental measurement:

The PoCverse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

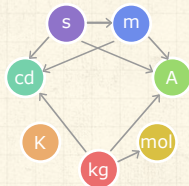
Money

Language

Technology

References

SI base units were redefined in 2019: 

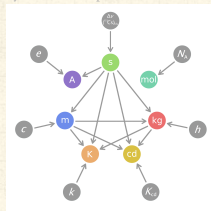


Now: kilogram is an artifact  in
Sèvres, France.



Defined by fixing Planck's constant as
 $6.62607015 \times 10^{-34} \text{ s}^{-1} \cdot \text{m}^2 \cdot \text{kg}^{-1}$

by Dono/Wikipedia



by Wikipetzi/Wikipedia



Sorting out base units of fundamental measurement:

The PoCverse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

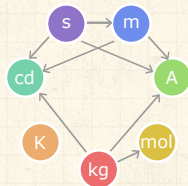
Money

Language

Technology

References

SI base units were redefined in 2019: 

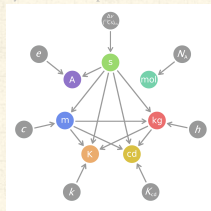


Now: kilogram is an artifact  in
Sèvres, France.



Defined by fixing Planck's constant as
 $6.62607015 \times 10^{-34} \text{ s}^{-1} \cdot \text{m}^2 \cdot \text{kg}^7$

by Dono/Wikipedia



by Wikipetzi/Wikipedia



⁷Not without some arguing ...

Sorting out base units of fundamental measurement:

The PoCverse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

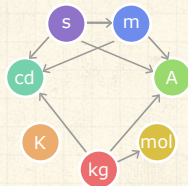
Money

Language

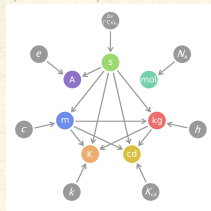
Technology

References

SI base units were redefined in 2019: 



by Dono/Wikipedia



by Wikipetzi/Wikipedia



Now: kilogram is an artifact  in
Sèvres, France.



Defined by fixing Planck's constant as
 $6.62607015 \times 10^{-34} \text{ s}^{-1} \cdot \text{m}^2 \cdot \text{kg}^7$



Metre chosen to fix speed of light at
 $299,792,458 \text{ m} \cdot \text{s}^{-1}$.



⁷Not without some arguing ...

Sorting out base units of fundamental measurement:

The PoCVerse

Scaling

67 of 114

Scaling-at-large

Allometry

Biology

Physics

People

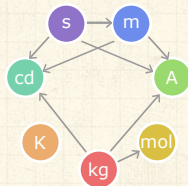
Money

Language

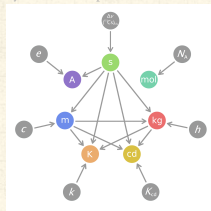
Technology

References

SI base units were redefined in 2019: 



by Dono/Wikipedia



by Wikipetzi/Wikipedia



Now: kilogram is an artifact  in
Sèvres, France.



Defined by fixing Planck's constant as
 $6.62607015 \times 10^{-34} \text{ s}^{-1} \cdot \text{m}^2 \cdot \text{kg}^{-1}$.



Metre chosen to fix speed of light at
 $299,792,458 \text{ m} \cdot \text{s}^{-1}$.



Radiolab piece: $\leq \text{kg}$ 



⁷Not without some arguing ...

Turbulence:

The PoCVerse

Scaling

68 of 114

Scaling-at-large

Allometry

Biology

Physics

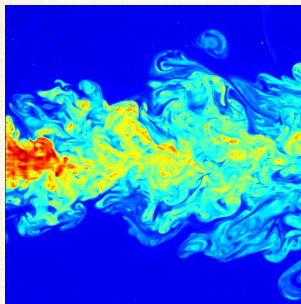
People

Money

Language

Technology

References



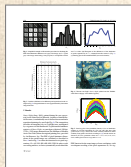
Big whirls have little whirls
That heed on their velocity,
And little whirls have littler whirls
And so on to viscosity.

— Lewis Fry Richardson 

 Image from here .

 Jonathan Swift (1733): “Big fleas have little fleas upon their backs to bite ’em, And little fleas have lesser fleas, and so, ad infinitum.” The Siphonaptera. 





“Turbulent luminance in impassioned van Gogh paintings”

Aragón et al.,

J. Math. Imaging Vis., **30**, 275–283, 2008. ^[1]



Examined the probability pixels a distance R apart share the same luminance.



“Van Gogh painted perfect turbulence”  by Phillip Ball, July 2006.



Apparently not observed in other famous painter’s works or when van Gogh was stable.



Oops: Small ranges and natural log used.



Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

 $E(k)$ = energy spectrum function.

 ϵ = rate of energy dissipation.

 $k = 2\pi/\lambda$ = wavenumber.

The PoCverse

Scaling

70 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

 $E(k)$ = energy spectrum function.

 ϵ = rate of energy dissipation.

 $k = 2\pi/\lambda$ = wavenumber.

The PoCverse

Scaling

70 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

 $E(k)$ = energy spectrum function.

 ϵ = rate of energy dissipation.

 $k = 2\pi/\lambda$ = wavenumber.

 Energy is distributed across all modes, decaying with wave number.

The PoCverse

Scaling

70 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

 $E(k)$ = energy spectrum function.

 ϵ = rate of energy dissipation.

 $k = 2\pi/\lambda$ = wavenumber.

 Energy is distributed across all modes, decaying with wave number.

 No internal characteristic scale to turbulence.

The PoCverse

Scaling

70 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

 $E(k)$ = energy spectrum function.

 ϵ = rate of energy dissipation.

 $k = 2\pi/\lambda$ = wavenumber.

 Energy is distributed across all modes, decaying with wave number.

 No internal characteristic scale to turbulence.

 Stands up well experimentally and there has been no other advance of similar magnitude.

The PoCverse

Scaling

70 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



“The Geometry of Nature”: Fractals



8



“Anomalous” scaling of lengths, areas, volumes relative to each other.



“The Geometry of Nature”: Fractals



⁸



“Anomalous” scaling of lengths, areas, volumes relative to each other.



⁸Note to self: Make millions with the “Fractal Diet”

“The Geometry of Nature”: Fractals



“Anomalous” scaling of lengths, areas, volumes relative to each other.



The enduring question: how do self-similar geometries form?

⁸



⁸Note to self: Make millions with the “Fractal Diet”

“The Geometry of Nature”: Fractals



“Anomalous” scaling of lengths, areas, volumes relative to each other.



The enduring question: how do self-similar geometries form?



Robert E. Horton : Self-similarity of river (branching) networks (1945). ^[13]



⁸Note to self: Make millions with the “Fractal Diet”

“The Geometry of Nature”: Fractals



“Anomalous” scaling of lengths, areas, volumes relative to each other.



The enduring question: how do self-similar geometries form?



Robert E. Horton : Self-similarity of river (branching) networks (1945). ^[13]



Harold Hurst —Roughness of time series (1951). ^[14]



⁸Note to self: Make millions with the “Fractal Diet”

“The Geometry of Nature”: Fractals



“Anomalous” scaling of lengths, areas, volumes relative to each other.



The enduring question: how do self-similar geometries form?



Robert E. Horton : Self-similarity of river (branching) networks (1945). ^[13]



Harold Hurst —Roughness of time series (1951). ^[14]



Lewis Fry Richardson —Coastlines (1961).



⁸Note to self: Make millions with the “Fractal Diet”

“The Geometry of Nature”: Fractals



“Anomalous” scaling of lengths, areas, volumes relative to each other.



The enduring question: how do self-similar geometries form?



Robert E. Horton : Self-similarity of river (branching) networks (1945). ^[13]



Harold Hurst —Roughness of time series (1951). ^[14]



Lewis Fry Richardson —Coastlines (1961).



Benoît B. Mandelbrot —Introduced the term “Fractals” and explored them everywhere, 1960s on. ^[22, 23, 24]



⁸Note to self: Make millions with the “Fractal Diet”

Scaling in Cities:

The PoCverse

Scaling

72 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



“Growth, innovation, scaling, and the pace of life in cities” 

Bettencourt et al.,
Proc. Natl. Acad. Sci., **104**, 7301–7306, 2007. [4]



Quantified levels of

-  Infrastructure
-  Wealth
-  Crime levels
-  Disease
-  Energy consumption

as a function of city size N (population).



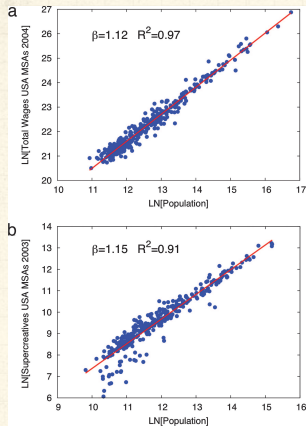


Fig. 1. Examples of scaling relationships. (a) Total wages per MSA in 2004 for the U.S. (blue points) vs. metropolitan population. (b) Supercreative employment per MSA in 2003, for the U.S. (blue points) vs. metropolitan population. Best-fit scaling relations are shown as solid lines.

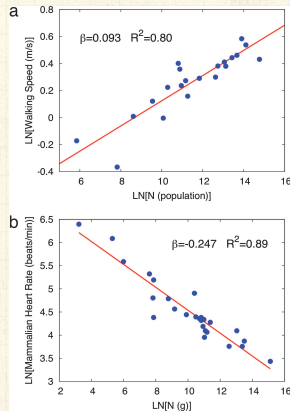


Fig. 2. The pace of urban life increases with city size in contrast to the pace of biological life, which decreases with organism size. (a) Scaling of walking speed vs. population for cities around the world. (b) Heart rate vs. the size (mass) of organisms.



Table 1. Scaling exponents for urban indicators vs. city size

Y	β	95% CI	Adj- R^2	Observations	Country-year
New patents	1.27	[1.25,1.29]	0.72	331	U.S. 2001
Inventors	1.25	[1.22,1.27]	0.76	331	U.S. 2001
Private R&D employment	1.34	[1.29,1.39]	0.92	266	U.S. 2002
"Supercreative" employment	1.15	[1.11,1.18]	0.89	287	U.S. 2003
R&D establishments	1.19	[1.14,1.22]	0.77	287	U.S. 1997
R&D employment	1.26	[1.18,1.43]	0.93	295	China 2002
Total wages	1.12	[1.09,1.13]	0.96	361	U.S. 2002
Total bank deposits	1.08	[1.03,1.11]	0.91	267	U.S. 1996
GDP	1.15	[1.06,1.23]	0.96	295	China 2002
GDP	1.26	[1.09,1.46]	0.64	196	EU 1999–2003
GDP	1.13	[1.03,1.23]	0.94	37	Germany 2003
Total electrical consumption	1.07	[1.03,1.11]	0.88	392	Germany 2002
New AIDS cases	1.23	[1.18,1.29]	0.76	93	U.S. 2002–2003
Serious crimes	1.16	[1.11, 1.18]	0.89	287	U.S. 2003
Total housing	1.00	[0.99,1.01]	0.99	316	U.S. 1990
Total employment	1.01	[0.99,1.02]	0.98	331	U.S. 2001
Household electrical consumption	1.00	[0.94,1.06]	0.88	377	Germany 2002
Household electrical consumption	1.05	[0.89,1.22]	0.91	295	China 2002
Household water consumption	1.01	[0.89,1.11]	0.96	295	China 2002
Gasoline stations	0.77	[0.74,0.81]	0.93	318	U.S. 2001
Gasoline sales	0.79	[0.73,0.80]	0.94	318	U.S. 2001
Length of electrical cables	0.87	[0.82,0.92]	0.75	380	Germany 2002
Road surface	0.83	[0.74,0.92]	0.87	29	Germany 2002

Data sources are shown in [SI Text](#). CI, confidence interval; Adj- R^2 , adjusted R^2 ; GDP, gross domestic product.



Scaling in Cities:

Intriguing findings:

 Global supply costs scale **sublinearly** with N ($\beta < 1$).

 Returns to scale for infrastructure.

The PoCSverse

Scaling

75 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Scaling in Cities:

The PoCVerse

Scaling

75 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Intriguing findings:

-  Global supply costs scale **sublinearly** with $N(\beta < 1)$.
 -  Returns to scale for infrastructure.
-  Total individual costs scale **linearly** with $N(\beta = 1)$
 -  Individuals consume similar amounts independent of city size.



Scaling in Cities:

The PoCVerse

Scaling

75 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Intriguing findings:

-  Global supply costs scale **sublinearly** with $N(\beta < 1)$.
 -  Returns to scale for infrastructure.
-  Total individual costs scale **linearly** with $N(\beta = 1)$
 -  Individuals consume similar amounts independent of city size.
-  Social quantities scale **superlinearly** with $N(\beta > 1)$
 -  Creativity (# patents), wealth, disease, crime, ...



Scaling in Cities:

The PoCVerse

Scaling

75 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Intriguing findings:

-  Global supply costs scale **sublinearly** with $N (\beta < 1)$.
 -  Returns to scale for infrastructure.
-  Total individual costs scale **linearly** with $N (\beta = 1)$
 -  Individuals consume similar amounts independent of city size.
-  Social quantities scale **superlinearly** with $N (\beta > 1)$
 -  Creativity (# patents), wealth, disease, crime, ...

Density doesn't seem to matter...

-  Surprising given that across the world, we observe two orders of magnitude variation in area covered by agglomerations  of fixed populations.





“Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities” 

Bettencourt et al.,
PLoS ONE, 5, e13541, 2010. ^[5]

Comparing city features across populations:



Cities = Metropolitan Statistical Areas (MSAs)





“Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities”

Bettencourt et al.,
PLoS ONE, 5, e13541, 2010. ^[5]

Comparing city features across populations:



Cities = Metropolitan Statistical Areas (MSAs)



Story: Fit scaling law and examine residuals





“Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities”

Bettencourt et al.,
PLoS ONE, 5, e13541, 2010. [5]

Comparing city features across populations:

-  Cities = Metropolitan Statistical Areas (MSAs)
-  Story: Fit scaling law and examine residuals
-  Does a city have more or less crime than expected when normalized for population?





“Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities”

Bettencourt et al.,
PLoS ONE, 5, e13541, 2010. ^[5]

Comparing city features across populations:

-  Cities = Metropolitan Statistical Areas (MSAs)
-  Story: Fit scaling law and examine residuals
-  Does a city have more or less crime than expected when normalized for population?
-  Same idea as Encephalization Quotient (EQ).



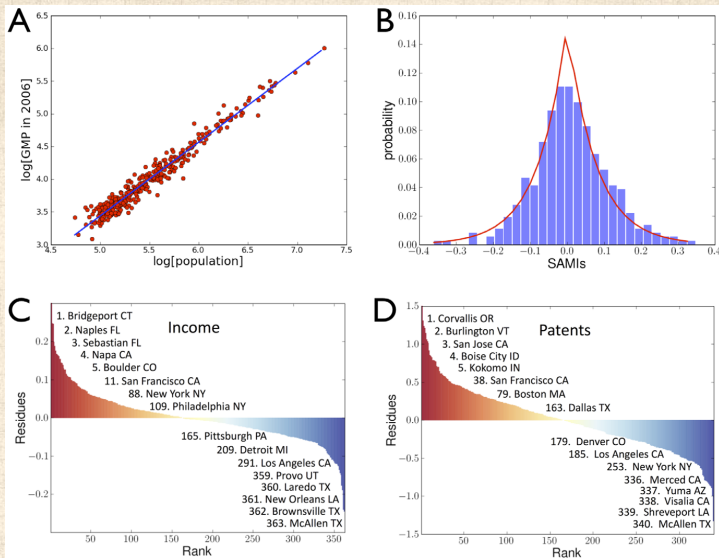


Figure 1. Urban Agglomeration effects result in per capita nonlinear scaling of urban metrics. Subtracting these effects produces a truly local measure of urban dynamics and a reference scale for ranking cities. a) A typical superlinear scaling law (solid line): Gross Metropolitan Product of US MSAs in 2006 (red dots) vs. population; the slope of the solid line has exponent, $\beta = 1.126$ (95% CI [1.101, 1.149]). b) Histogram showing frequency of residuals, (SAMIs, see Eq. (2)); the statistics of residuals is well described by a Laplace distribution (red line). Scale independent ranking (SAMIs) for US MSAs by c) personal income and d) patenting (red denotes above average performance, blue below). For more details see Text S1, Table S1 and Figure S1.



A possible theoretical explanation?



“The origins of scaling in cities” 

Luís M. A. Bettencourt,
Science, **340**, 1438–1441, 2013. ^[3]

#sixthology



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, **4**, 6239, 2014. ^[27]



Bettencourt *et al.*’s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)

The PoCverse

Scaling

79 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, **4**, 6239, 2014. ^[27]



Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)



Homicide, traffic, and suicide ^[9] all tied to social context in complex, different ways.



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, **4**, 6239, 2014. ^[27]



Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)



Homicide, traffic, and suicide ^[9] all tied to social context in complex, different ways.



For cities in Brazil, Melo *et al.* show:



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, **4**, 6239, 2014. ^[27]



Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)



Homicide, traffic, and suicide ^[9] all tied to social context in complex, different ways.



For cities in Brazil, Melo *et al.* show:



Homicide appears to follow superlinear scaling
($\beta = 1.24 \pm 0.01$)



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, 4, 6239, 2014. ^[27]



Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)



Homicide, traffic, and suicide ^[9] all tied to social context in complex, different ways.



For cities in Brazil, Melo *et al.* show:



Homicide appears to follow superlinear scaling
($\beta = 1.24 \pm 0.01$)



Traffic accident deaths appear to follow linear scaling
($\beta = 0.99 \pm 0.02$)



Non-simple scaling for death:



“Statistical signs of social influence on suicides” 

Melo et al.,

Scientific Reports, 4, 6239, 2014. ^[27]

-  Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)
-  Homicide, traffic, and suicide ^[9] all tied to social context in complex, different ways.
-  For cities in Brazil, Melo *et al.* show:
 -  Homicide appears to follow superlinear scaling ($\beta = 1.24 \pm 0.01$)
 -  Traffic accident deaths appear to follow linear scaling ($\beta = 0.99 \pm 0.02$)
 -  Suicide appears to follow sublinear scaling. ($\beta = 0.84 \pm 0.02$)



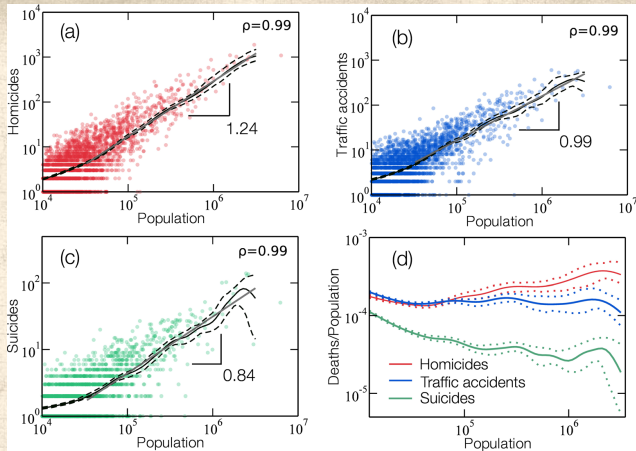


Figure 1 | Scaling relations for homicides, traffic accidents, and suicides for the year of 2009 in Brazil. The small circles show the total number of deaths by (a) homicides (red), (b) traffic accidents (blue), and (c) suicides (green) vs the population of each city. Each graph represents only one urban indicator, and the solid gray line indicate the best fit for a power-law relation, using OLS regression, between the average total number of deaths and the city size (population). To reduce the fluctuations we also performed a Nadaraya-Watson kernel regression^{17,18}. The dashed lines show the 95% confidence band for the Nadaraya-Watson kernel regression. The ordinary least-squares (OLS)¹⁹ fit to the Nadaraya-Watson kernel regression applied to the data on homicides in (a) reveals an allometric exponent $\beta = 1.24 \pm 0.01$, with a 95% confidence interval estimated by bootstrap. This is compatible with previous results obtained for U.S.² that also indicate a super-linear scaling relation with population and an exponent $\beta = 1.16$. Using the same procedure, we find $\beta = 0.99 \pm 0.02$ and 0.84 ± 0.02 for the numbers of deaths in traffic accidents (b) and suicides (c), respectively. The values of the Pearson correlation coefficients ρ associated with these scaling relations are shown in each plot. This non-linear behavior observed for homicides and suicides certainly reflects the complexity of human social relations and strongly suggests that the topology of the social network plays an important role on the rate of these events. (d) The solid lines show the Nadaraya-Watson kernel regression rate of deaths (total number of deaths divided by the population of a city) for each urban indicator, namely, homicides (red), traffic accidents (blue), and suicides (green). The dashed lines represent the 95% confidence bands. While the rate of fatal traffic accidents remains approximately invariant, the rate of homicides systematically increases, and the rate of suicides decreases with population.

Dynamics (Brazil):

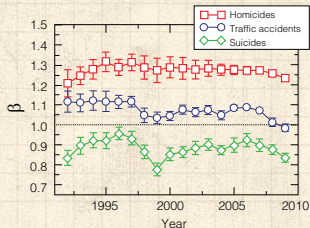
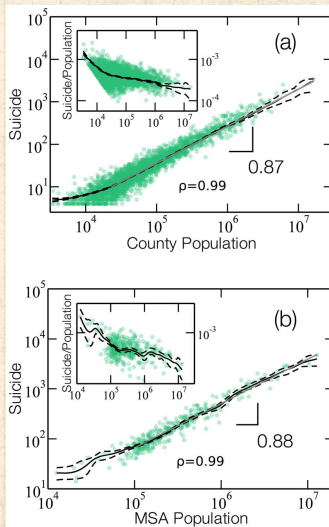
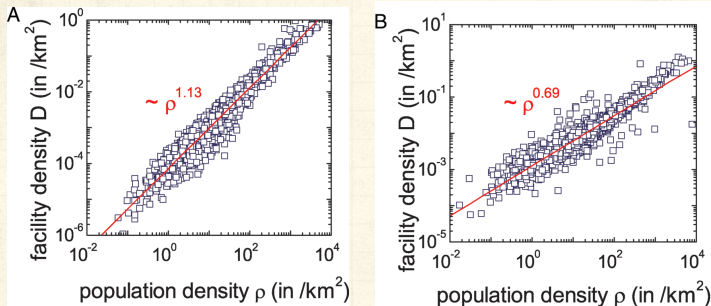


Figure 2 | Temporal evolution of allometric exponent β for homicides (red squares), deaths in traffic accidents (blue circles), and suicides (green diamonds). Time evolution of the power-law exponent β for each behavioral urban indicator in Brazil from 1992 to 2009. We can see that the non-linear behavior for homicides and suicides are robust for this 19 years period, and for the traffic accidents the exponent remain close to 1.0.

US data:



Density of public and private facilities:



$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

 **Left plot:** ambulatory hospitals in the U.S.

 **Right plot:** public schools in the U.S.



Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition
between public and
private at $\alpha \simeq 0.8$.

Note: * indicates
analysis is at
state/province level;
otherwise county
level.



Public versus private facilities: evidence

The PoCverse

Scaling

84 of 114

Scaling-at-large

Allometry

Biology

Physics

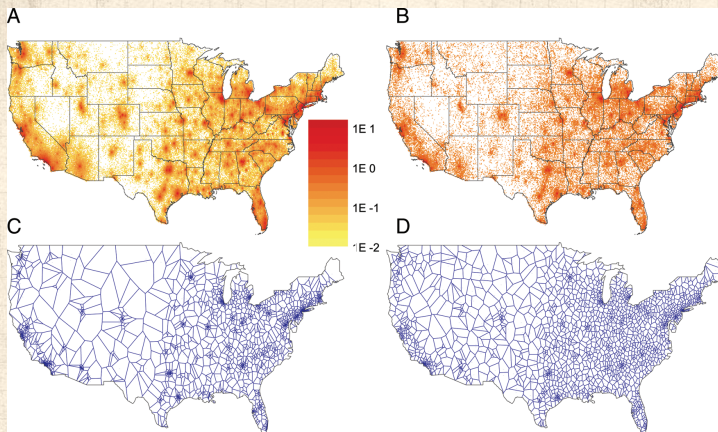
People

Money

Language

Technology

References



A, C: ambulatory hospitals in the U.S.; **B, D:** public schools in the U.S.; **A, B:** data; **C, D:** Voronoi diagram from model simulation.





“Pattern in escalations in insurgent and terrorist activity”

Johnson et al.,
Science, **333**, 81–84, 2011. ^[16]

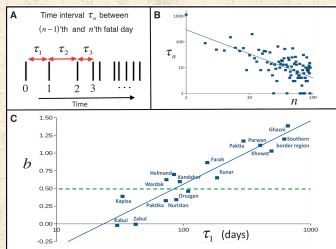


Fig. 1. (A) Schematic timeline of successive fatal days shown as vertical bars. τ_i is the time interval between the first two fatal days, labeled 0 and 1. **(B)** Successive time intervals τ_n between days with IED fatalities in the Afghanistan province of Kandahar (squares). On this log-log plot, the best-fit power-law progress curve is by definition a straight (blue) line with slope $-b$ (b is an escalation rate). **(C)** The solid blue line shows best linear fit through progress-curve parameter values τ_1 and b for individual Afghanistan provinces (blue squares) for all hostile military fatalities attributed to insurgent activity. The green dashed line shows value $b = 0.5$, which is the situation in which there are no correlations. The subset of fatalities recorded in casualties as “southern Afghanistan” is shown as a separate region because of their likely connection to operations near the Pakistan border.

Escalation: $\tau_n \sim \tau_1 n^{-b}$

b = scaling exponent
(escalation rate)

Interevent time τ_n between
fatal attacks $n - 1$ and n
(binned by days)

Learning curves for
organizations ^[37]

More later on size
distributions ^[8, 17, 6]





Irregular verbs

Cleaning up the code that is English:



“Quantifying the evolutionary dynamics of language” ↗

Lieberman et al.,

Nature, ~~449~~, 713–716, 2007. [20]



Exploration of how verbs with irregular conjugation gradually become regular over time.



Comparison of verb behavior in Old, Middle, and Modern English.

The PoCVerse

Scaling

87 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Irregular verbs

The PoCVerse

Scaling

88 of 114

Scaling-at-large

Allometry

Biology

Physics

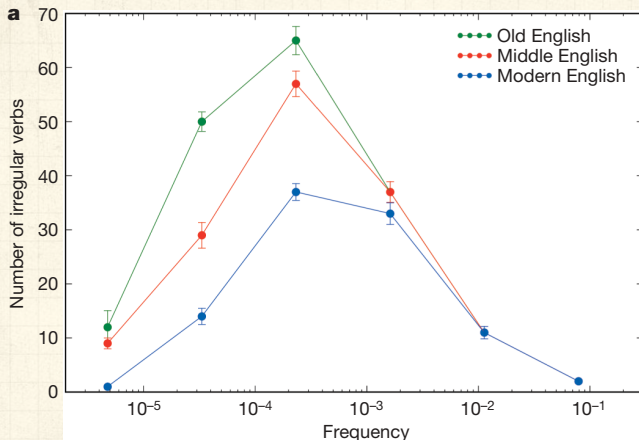
People

Money

Language

Technology

References



Universal tendency towards regular conjugation



Rare verbs tend to be regular in the first place



Irregular verbs

The PoCverse

Scaling

89 of 114

Scaling-at-large

Allometry

Biology

Physics

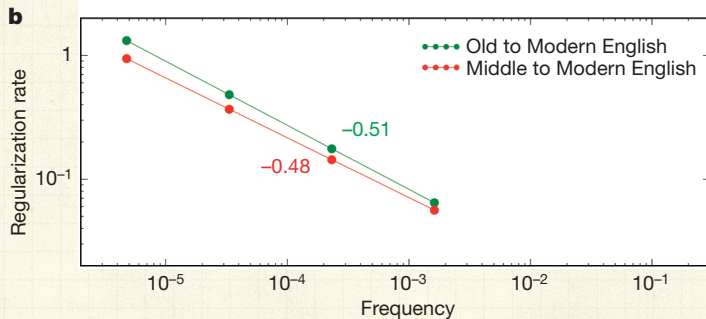
People

Money

Language

Technology

References



Rates are relative.



Irregular verbs

The PoCverse

Scaling

89 of 114

Scaling-at-large

Allometry

Biology

Physics

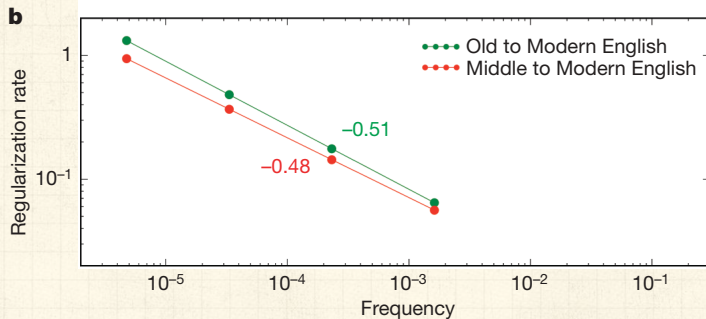
People

Money

Language

Technology

References



Rates are relative.



The **more common** a verb is, the **more resilient** it is to change.



Irregular verbs

Table 1 | The 177 irregular verbs studied

Frequency	Verbs	Regularization (%)	Half-life (yr)
10 ⁻¹ –1	be, have	0	38,800
10 ⁻² –10 ⁻¹	come, do, find, get, give, go, know, say, see, take, think	0	14,400
10 ⁻³ –10 ⁻²	begin, break, bring, buy, choose, draw, drink, drive, eat, fall, fight, forget, grow, hang, help , hold, leave, let, lie, lose, reach , rise, run, seek, set, shake, sit, sleep, speak, stand, teach, throw, understand, walk , win, work , write	10	5,400
10 ⁻⁴ –10 ⁻³	arise, bake , bear, beat, bind, bite, blow, bow , burn, burst, carve , chew , climb , cling, creep, dare , dig, drag , flee, float , flow , fly, fold , freeze, grind, leap, lend, lock , melt , reckon , ride, rush , shape , shine, shoot, shrink, sigh , sing, sink, slide, slip , smoke , spin, spring, starve , steal, step , stretch , strike, stroke , suck , swallow , swear, sweep, swim, swing, tear, wake, wash , weave, weep, weigh , wind, yell , yield	43	2,000
10 ⁻⁵ –10 ⁻⁴	bark , bellow, bid, blend , braid, brew, cleave, cringe, crow, dive, drip, fare, fret, glide, gnaw, grip, heave, knead, low, milk, mourn, mow, prescribe , redde n, reek, row, scrape, seethe , shear, shed, shove , slay, slit, smite , sow, span, spurn , sting, stink, strew, stride, swell, tread , uproot, wade , warp, wax, wield , wring, writhe	72	700
10 ⁻⁶ –10 ⁻⁵	bide , chide, delve, flay, hew, rue, shrive, slink, snip , spew, sup, wreak	91	300

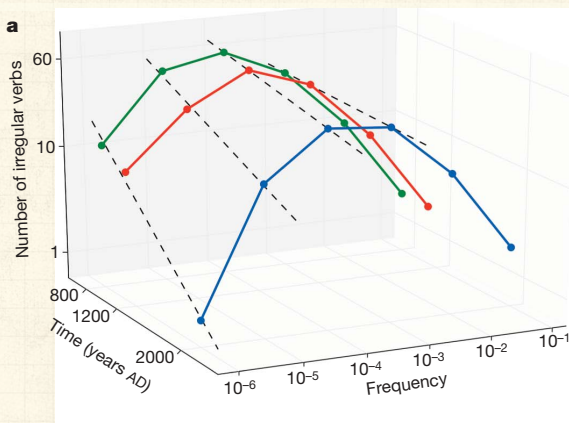
177 Old English irregular verbs were compiled for this study. These are arranged according to frequency bin, and in alphabetical order within each bin. Also shown is the percentage of verbs in each bin that have regularized. The half-life is shown in years. Verbs that have regularized are indicated in red. As we move down the list, an increasingly large fraction of the verbs are red; the frequency-dependent regularization of irregular verbs becomes immediately apparent.



Red = regularized



Estimates of half-life for regularization ($\propto f^{1/2}$)



'Wed' is next to go.

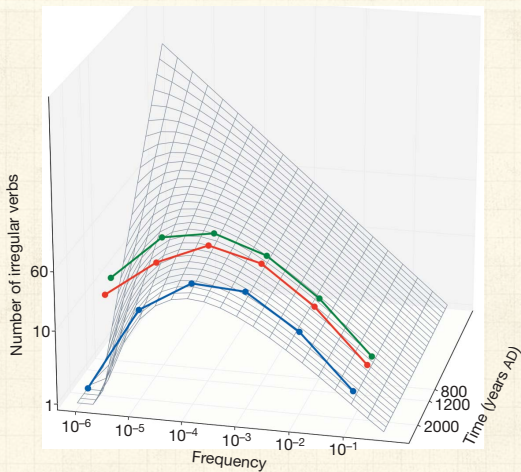


-ed is the winning rule...



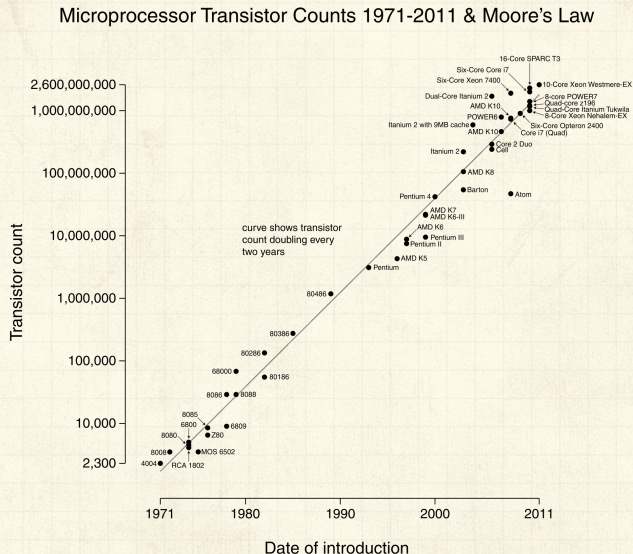
But 'snuck' is sneaking up on sneaked. [\[29\]](#)





Projecting back in time to proto-Zipf story of many tools.



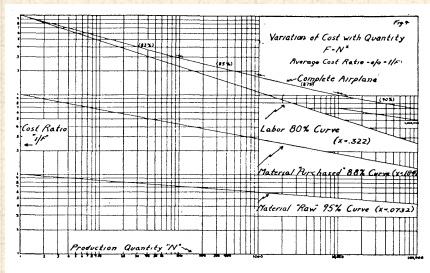
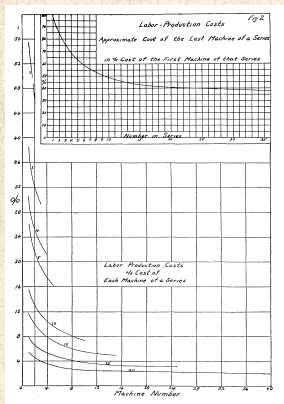




“Factors affecting the costs of airplanes” ↗

T. P. Wright,

Journal of Aeronautical Sciences, **10**, 302–328, 1936. [37]



Power law decay of cost with number of planes produced.



“The present writer started his studies of the variation of cost with quantity in 1922.”

The PoCverse

Scaling

94 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

The PoCSverse

Scaling

95 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

 y_t = stuff unit cost; x_t = total amount of stuff made.

The PoCVerse

Scaling

95 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

 y_t = stuff unit cost; x_t = total amount of stuff made.

 Wright’s Law, cost decreases as a power of total stuff made: ^[37]

$$y_t \propto x_t^{-w}.$$



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

 y_t = stuff unit cost; x_t = total amount of stuff made.

 Wright’s Law, cost decreases as a power of total stuff made: ^[37]

$$y_t \propto x_t^{-w}.$$

 Moore’s Law , framed as cost decrease connected with doubling of transistor density every two years: ^[30]

$$y_t \propto e^{-mt}.$$



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

 y_t = stuff unit cost; x_t = total amount of stuff made.

 Wright’s Law, cost decreases as a power of total stuff made: ^[37]

$$y_t \propto x_t^{-w}.$$

 Moore’s Law , framed as cost decrease connected with doubling of transistor density every two years: ^[30]

$$y_t \propto e^{-mt}.$$

 Sahal’s observation that Moore’s law gives rise to Wright’s law if stuff production grows exponentially: ^[32]

$$x_t \propto e^{gt}.$$



Scaling laws for technology production:

 “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. ^[31]

 y_t = stuff unit cost; x_t = total amount of stuff made.

 Wright’s Law, cost decreases as a power of total stuff made: ^[37]

$$y_t \propto x_t^{-w}.$$

 Moore’s Law , framed as cost decrease connected with doubling of transistor density every two years: ^[30]

$$y_t \propto e^{-mt}.$$

 Sahal’s observation that Moore’s law gives rise to Wright’s law if stuff production grows exponentially: ^[32]

$$x_t \propto e^{gt}.$$

 Sahal + Moore gives Wright with $w = m/g$.



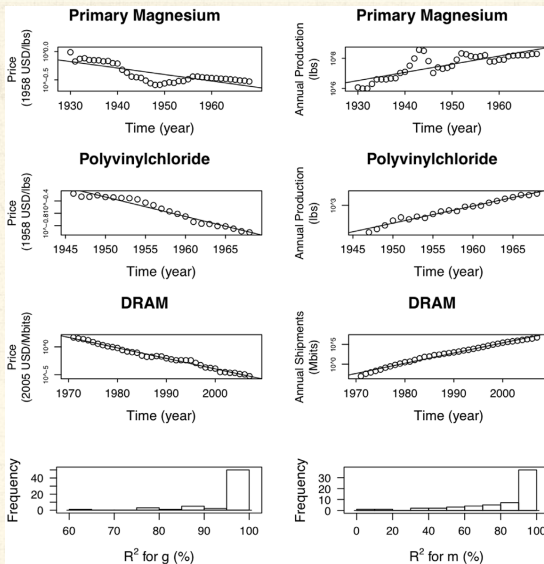


Figure 3. Three examples showing the logarithm of price as a function of time in the left column and the logarithm of production as a function of time in the right column, based on industry-wide data. We have chosen these examples to be representative: The top row contains an example with one of the worst fits, the second row an example with an intermediate goodness of fit, and the third row one of the best examples. The fourth row of the figure shows histograms of R^2 values for fitting g and m for the 62 datasets.

doi:10.1371/journal.pone.0052669.g003



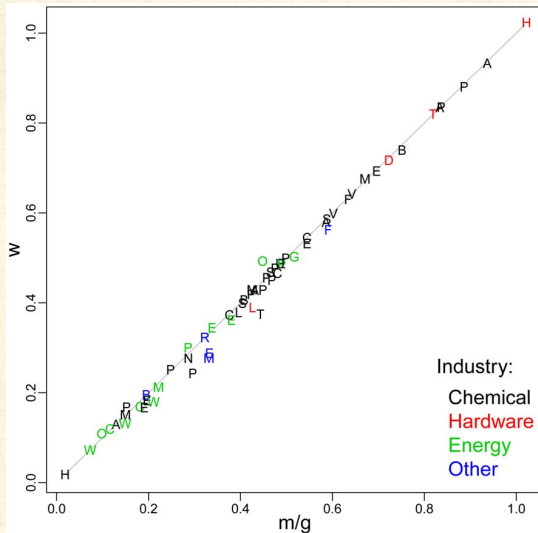
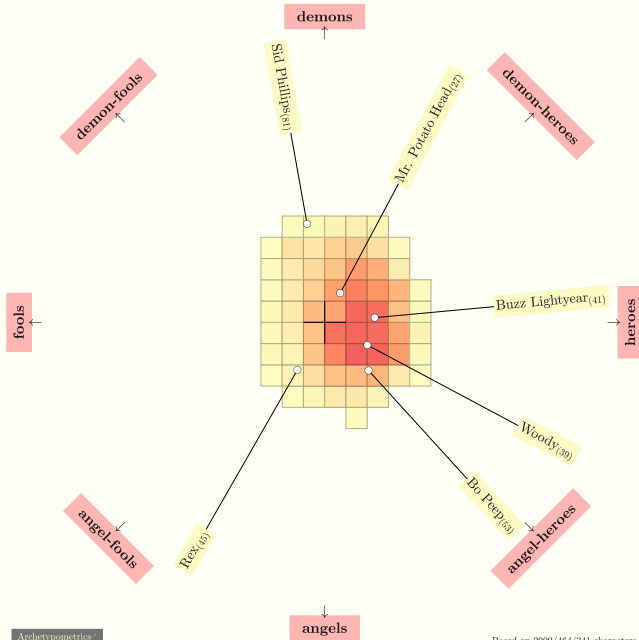


Figure 4. An illustration that the combination of exponentially increasing production and exponentially decreasing cost are equivalent to Wright's law. The value of the Wright parameter w is plotted against the prediction m/g based on the Sahal formula, where m is the exponent of cost reduction and g the exponent of the increase in cumulative production.
doi:10.1371/journal.pone.0052669.g004



ousiogram planes:

	1-2	1-3	1-4	1-5	1-6
2-1		2-3	2-4	2-5	2-6
3-1	3-2		3-4	3-5	3-6
4-1	4-2	4-3		4-5	4-6
5-1	5-2	5-3	5-4		5-6
6-1	6-2	6-3	6-4	6-5	

Toy Story and Moore's law:

The PoCVerse

Scaling

99 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

‘When the group moved to California to become part of Lucasfilm, we got close to making a computer-animated movie again in the mid-1980s – this time about a monkey with godlike powers but a missing prefrontal cortex. We had a sponsor, a story treatment, and a marketing survey. We were prepared to make a screen test: Our hot young animator John Lasseter had sketched numerous studies of the hero monkey and had the sponsor salivating over a glass-dragon protagonist.’



⁸“How Pixar Used Moore's Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Toy Story and Moore's law:

The PoCVerse

Scaling

100 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

“But when it came time to harden the deal and run the numbers for the contracts, I discovered to my dismay that computers were still too slow: The projected production cost was too high and the computation time way too long. We had to back out of the deal. This time, we did know enough detail to correctly apply Moore's Law – and it told us that we had to wait another five years to start making the first movie. And sure enough, five years later Disney approached us to make Toy Story.”



⁸“How Pixar Used Moore's Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Toy Story and Moore's law:

The PoCSverse

Scaling

101 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

‘We implement each step to see if it actually works, then gain the courage, the insight, and the engineering mastery to proceed to the next step.

Moore's Law told us that the new company we were starting, Pixar, had to bide its time—building hardware instead of making movies.’



⁸“How Pixar Used Moore's Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Toy Story and Moore's law:

The PoCverse

Scaling

102 of 114

Scaling-at-large

Allometry

Biology

Physics

People

Money

Language

Technology

References

Rhetoric of maybeness with hook to “More is different” [?]

‘That’s the reason for expressing Moore’s Law in orders of magnitude rather than factors of 10. The latter form is merely arithmetic, but the former implies an intellectual challenge. We use “order of magnitude” to imply a change so great that it requires new thought processes, new conceptualizations: It’s not simply more, it’s different.’



⁸“How Pixar Used Moore’s Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Shell of the nut:



Scaling is a fundamental feature of complex systems.



Shell of the nut:



Scaling is a fundamental feature of complex systems.



Basic distinction between isometric and allometric scaling.



Shell of the nut:

-  Scaling is a fundamental feature of complex systems.
-  Basic distinction between isometric and allometric scaling.
-  Powerful envelope-based approach: Dimensional analysis.



Shell of the nut:

-  Scaling is a fundamental feature of complex systems.
-  Basic distinction between isometric and allometric scaling.
-  Powerful envelope-based approach: Dimensional analysis.
-  “Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist .



Shell of the nut:



Scaling is a fundamental feature of complex systems.



Basic distinction between isometric and allometric scaling.



Powerful envelope-based approach: Dimensional analysis.



“Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist .

[insert your own adjective] physicist ↗.



Tricksiness: A wide variety of mechanisms give rise to scalings.⁹

scalings.⁹

Shell of the nut:

-  Scaling is a fundamental feature of complex systems.
-  Basic distinction between isometric and allometric scaling.
-  Powerful envelope-based approach: Dimensional analysis.
-  “Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist .
-  **Tricksiness:** A wide variety of mechanisms give rise to scalings.⁹

⁹It’s not your great-great-great-grandparents’ normal distribution



Shell of the nut:

-  Scaling is a fundamental feature of complex systems.
-  Basic distinction between isometric and allometric scaling.
-  Powerful envelope-based approach: Dimensional analysis.
-  “Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist .
-  **Tricksiness:** A wide variety of mechanisms give rise to scalings.⁹
-  Some mechanisms are common, some are rare.¹⁰

⁹It’s not your great-great-great-grandparents’ normal distribution



Shell of the nut:

- Scaling is a fundamental feature of complex systems.
- Basic distinction between isometric and allometric scaling.
- Powerful envelope-based approach: Dimensional analysis.
- “Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist .
- Tricksiness:** A wide variety of mechanisms give rise to scalings.⁹
- Some mechanisms are common, some are rare.¹⁰

⁹It’s not your great-great-great-grandparents’ normal distribution

¹⁰To be understood: The scaling story of scaling-making mechanisms



References I

- [1] J. L. Aragón, G. G. Naumis, M. Bai, M. Torres, and P. K. Maini.
Turbulent luminance in impassioned van Gogh paintings.
J. Math. Imaging Vis., 30:275–283, 2008. pdf ↗
- [2] G. I. Barenblatt.
Scaling, self-similarity, and intermediate asymptotics,
volume 14 of Cambridge Texts in Applied Mathematics.
Cambridge University Press, 1996.
- [3] L. M. A. Bettencourt.
The origins of scaling in cities.
Science, 340:1438–1441, 2013. pdf ↗
- [4] L. M. A. Bettencourt, J. Lobo, D. Helbing, C. Kühnert, and
G. B. West.
Growth, innovation, scaling, and the pace of life in cities.
Proc. Natl. Acad. Sci., 104(17):7301–7306, 2007. pdf ↗



References II

- [5] L. M. A. Bettencourt, J. Lobo, D. Strumsky, and G. B. West.
Urban scaling and its deviations: Revealing the structure of
wealth, innovation and crime across cities.
[PLoS ONE](#), 5:e13541, 2010. pdf ↗
- [6] J. C. Bohorquez, S. Gourley, A. R. Dixon, M. Spagat, and
N. F. Johnson.
Common ecology quantifies human insurgency.
[Nature](#), 462:911–914, 2009. pdf ↗
- [7] E. Buckingham.
On physically similar systems: Illustrations of the use of
dimensional equations.
[Phys. Rev.](#), 4:345–376, 1914. pdf ↗
- [8] A. Clauset, M. Young, and K. S. Gleditsch.
On the Frequency of Severe Terrorist Events.
[Journal of Conflict Resolution](#), 51(1):58–87, 2007. pdf ↗



References III

- [9] E. Durkheim.
Suicide: A study in sociology.
Free Press, 2005.
Reissue edition (February 1, 1997).
- [10] G. Galilei.
Dialogues Concerning Two New Sciences.
Kessinger Publishing, 2010.
Translated by Henry Crew and Alfonso De Salvio.
- [11] J. H. Gayford, D. J. Irschick, J. Martin, A. Chin, and J. L. Rummer.
The geometry of life: Testing the scaling of whole-organism surface area and volume using sharks.
Royal Society Open Science, 12(6):242205, 2025. pdf ↗



References IV

- [12] M. R. Hirt, W. Jetz, B. C. Rall, and U. Brose.
A general scaling law reveals why the largest animals are not the fastest.
[Nature Ecology & Evolution](#), 1:1116, 2017. pdf ↗
- [13] R. E. Horton.
Erosional development of streams and their drainage basins; hydrophysical approach to quantitative morphology.
[Bulletin of the Geological Society of America](#), 56(3):275–370, 1945. pdf ↗
- [14] H. E. Hurst.
Long term storage capacity of reservoirs.
[Transactions of the American Society of Civil Engineers](#), 116:770–808, 1951.



- [15] J. S. Huxley and G. Teissier.
Terminology of relative growth.
Nature, 137:780–781, 1936. pdf ↗
- [16] N. Johnson, S. Carran, J. Botner, K. Fontaine, N. Laxague,
P. Nuetzel, J. Turnley, and B. Tivnan.
Pattern in escalations in insurgent and terrorist activity.
Science, 333:81–84, 2011. pdf ↗
- [17] N. F. Johnson, M. Spagat, J. A. Restrepo, O. Becerra, J. C.
Bohorquez, N. Suarez, E. M. Restrepo, and R. Zarama.
Universal patterns underlying ongoing wars and terrorism,
2006. pdf ↗



- [18] A. N. Kolmogorov.

The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers.

[Proceedings of the USSR Academy of Sciences](#), 30:299–303, 1941.

- [19] S. Levin.

The problem of pattern and scale in ecology.

[Ecology](#), 73(6):1943–1967, 1992.

[.pdf](#) 

- [20] E. Lieberman, J.-B. Michel, J. Jackson, T. Tang, and M. A. Nowak.

Quantifying the evolutionary dynamics of language.

[Nature](#), 449:713–716, 2007. [pdf](#) 



References VII

- [21] R. H. MacArthur and E. O. Wilson.
An equilibrium theory of insular zoogeography.
Evolution, 17:373–387, 1963. [pdf](#) 
- [22] B. B. Mandelbrot.
How long is the coast of britain? statistical self-similarity and
fractional dimension.
Science, 156(3775):636–638, 1967. [pdf](#) 
- [23] B. B. Mandelbrot.
Fractals: Form, Chance, and Dimension.
Freeman, San Francisco, 1977.
- [24] B. B. Mandelbrot.
The Fractal Geometry of Nature.
Freeman, San Francisco, 1983.



References VIII

- [25] T. McMahon.
Size and shape in biology.
[Science](#), 179:1201–1204, 1973. pdf ↗
- [26] T. A. McMahon and J. T. Bonner.
On Size and Life.
Scientific American Library, New York, 1983.
- [27] H. P. M. Melo, A. A. Moreira, É. Batista, H. A. Makse, and J. S. Andrade.
Statistical signs of social influence on suicides.
[Scientific Reports](#), 4:6239, 2014. pdf ↗
- [28] N. Meyer-Vernet and J.-P. Rospars.
How fast do living organisms move: Maximum speeds from bacteria to elephants and whales.
[American Journal of Physics](#), pages 719–722, 2015. pdf ↗

The PoCVerse
Scaling
111 of 114
Scaling-at-large
Allometry
Biology
Physics
People
Money
Language
Technology
References



References IX

- [29] J.-B. Michel, Y. K. Shen, A. P. Aiden, A. Veres, M. K. Gray, T. G. B. Team, J. P. Pickett, D. Hoiberg, D. Clancy, P. Norvig, J. Orwant, S. Pinker, M. A. Nowak, and E. A. Lieberman. Quantitative analysis of culture using millions of digitized books.
[Science](#), 2010. pdf ↗
- [30] G. E. Moore.
Cramming more components onto integrated circuits.
[Electronics Magazine](#), 38:114–117, 1965.
- [31] B. Nagy, J. D. Farmer, Q. M. Bui, and J. E. Trancik.
Statistical basis for predicting technological progress.
[PloS one](#), 8(2):e52669, 2013. pdf ↗
- [32] D. Sahal.
A theory of progress functions.
[AIIE Transactions](#), 11:23–29, 1979.

The PoCVerse
Scaling
112 of 114
Scaling-at-large
Allometry
Biology
Physics
People
Money
Language
Technology
References



References X

- [33] S. Savaglio and V. Carbone.
Scaling in athletic world records.
[Nature](#), 404:244, 2000. pdf ↗
- [34] A. Shingleton.
Allometry: The study of biological scaling.
[Nature Education Knowledge](#), 1:2, 2010.
- [35] A. J. Tatem, C. A. Guerra, P. M. Atkinson, and S. I. Hay.
Athletics: Momentous sprint at the 2156 Olympics?
[Nature](#), 431(7008):525–525, 2004. pdf ↗
- [36] C. Tomasetti and B. Vogelstein.
Variation in cancer risk among tissues can be explained by the
number of stem cell divisions.
[Science](#), 347:78–81, 2015. pdf ↗

The PoCVerse
Scaling
113 of 114
Scaling-at-large
Allometry
Biology
Physics
People
Money
Language
Technology
References



[37] T. P. Wright.

Factors affecting the costs of airplanes.

Journal of Aeronautical Sciences, 10:302–328, 1936. pdf ↗

[38] P. J. Yang, J. Pham, J. Choo, and D. L. Hu.

Duration of urination does not change with body size.

Proceedings of the National Academy of Sciences,
111:11932–11937, 2014. pdf ↗

[39] K. Zhang and T. J. Sejnowski.

A universal scaling law between gray matter and white matter
of cerebral cortex.

Proceedings of the National Academy of Sciences,
97:5621–5626, 2000. pdf ↗

