

Scaling—a Plenitude of Power Laws

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Principles of Complex Systems,
Vols. 1, 2, 3D, 4 Fouerever, V for Vendetta

Deliverator: Calla Beauregard
Showrunner: Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
University of Vermont | Santa Fe Institute

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Outline

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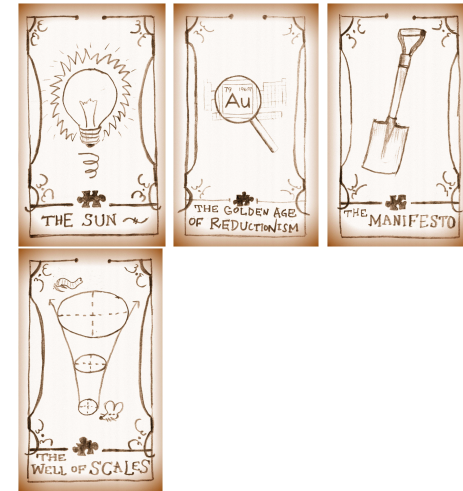
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Archival object:



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Scalingarama

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of **scaling**.

Outline—All about scaling:

- Basic definitions.
- Examples.

Possibly later:

- Advances in measuring your power-law relationships.
- Scaling in blood and river networks.
- The Unsolved Allometry Theoridices.

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Definitions

A **power law** relates two variables x and y as follows:

$$y = cx^\alpha$$

- α is the **scaling exponent** (or just exponent)
- α can be any number in principle but we will find various restrictions.
- c is the **prefactor** (which can be important!)

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Definitions

- The **prefactor** c must **balance dimensions**.
- Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$

- Using $[\cdot]$ to indicate dimension, then

$$[c] = [\ell]/[v^{1/4}] = L/L^{3/4} = L^{1/4}.$$

- More on this later with the Buckingham π theorem.

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Looking at data

- Power-law relationships are linear in log-log space:

$$y = cx^\alpha$$

$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

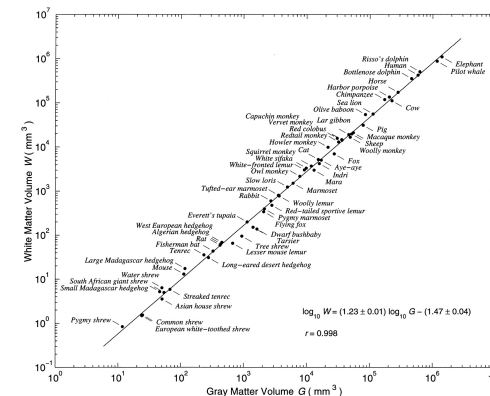
with slope equal to α , the scaling exponent.

- Much searching for straight lines on log-log or double-logarithmic plots.
- Good practice: **Always, always, always use base 10**.
- Yes, the **Dozenalists**  are right, 12 would be better.
- But: hands.¹ And social pressure.
- Talk only about orders of magnitude (powers of 10).

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A beautiful, heart-warming example:

- G = volume of gray matter: **‘computing elements’**
- W = volume of white matter: **‘wiring’**
- $W \sim cG^{1.23}$



 from Zhang & Sejnowski, PNAS (2000) [39]

¹Probably an accident of evolution—debated.

Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

- G = Volume of gray matter (cortex/processors)
- W = Volume of white matter (wiring)
- T = Cortical thickness (wiring)
- S = Cortical surface area
- L = Average length of white matter fibers
- p = density of axons on white matter/cortex interface

A rough understanding:

- $G \sim ST$ (convolutions are okay)
- $W \sim \frac{1}{2}pSL$
- $G \sim L^3$
- Eliminate S and L to find $W \propto G^{4/3}/T$

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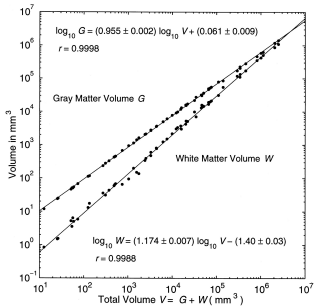
Why is $\alpha \simeq 1.23$?

A rough understanding:

- We are here: $W \propto G^{4/3}/T$
- Observe weak scaling $T \propto G^{0.10 \pm 0.02}$
- Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.
- $\Rightarrow W \propto G^{4/3}/T \propto G^{1.23 \pm 0.02}$

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Tricksiness:



- With $V = G + W$, some power laws must be approximations.
- Measuring exponents is a hairy business...

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Disappointing deviations from scaling:



Per George Carlin
Yes, should be the median.
#painful
Image from here

The koala, a few roos short in the top paddock:

- Very small brains relative to body size.
- Wrinkle-free, smooth.
- Not many algorithms needed:
 - Only eat eucalyptus leaves (no water) (Will not eat leaves picked and presented to them)
 - Move to the next tree.
 - Sleep.
 - Defend themselves if needed (tree-climbing crocodiles, humans).
 - Occasionally make more koalas.

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Good scaling:

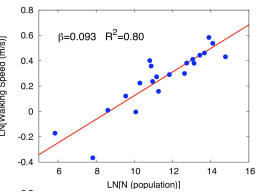
General rules of thumb:

- High quality:** scaling persists over three or more orders of magnitude for **each variable**.
- Medium quality:** scaling persists over three or more orders of magnitude for **only one variable** and at least one for the **other**.
- Very dubious:** scaling 'persists' over less than an order of magnitude for **both variables**.

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Unconvincing scaling:

Average walking speed as a function of city population:



- Two problems:
- use of natural log, and
 - minute variation in dependent variable.

from Bettencourt et al. (2007) [4]; otherwise totally great—more later.

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Definitions

Power laws are the signature of **scale invariance**:

Scale invariant 'objects' look the 'same' when they are appropriately rescaled.

- Objects = geometric shapes, time series, functions, relationships, distributions,...
- 'Same' might be 'statistically the same'
- To **rescale** means to change the units of measurement for the relevant variables

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Scale invariance

Our friend $y = cx^\alpha$:

- If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,
- then
$$r^\alpha y' = c(rx')^\alpha$$
$$\Rightarrow y' = cr^\alpha x'^\alpha r^{-\alpha}$$
$$\Rightarrow y' = cx'^\alpha$$

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Scale invariance

Compare with $y = ce^{-\lambda x}$:

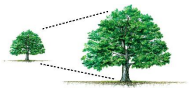
- If we rescale x as $x = rx'$, then
$$y = ce^{-\lambda rx'}$$
- Original form cannot be recovered.
- Scale matters** for the exponential.

More on $y = ce^{-\lambda x}$:

- Say $x_0 = 1/\lambda$ is the **characteristic scale**.
- For $x \gg x_0$, y is small, while for $x \ll x_0$, y is large.

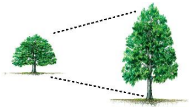
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Isometry:



Dimensions scale linearly with each other.

Allometry:



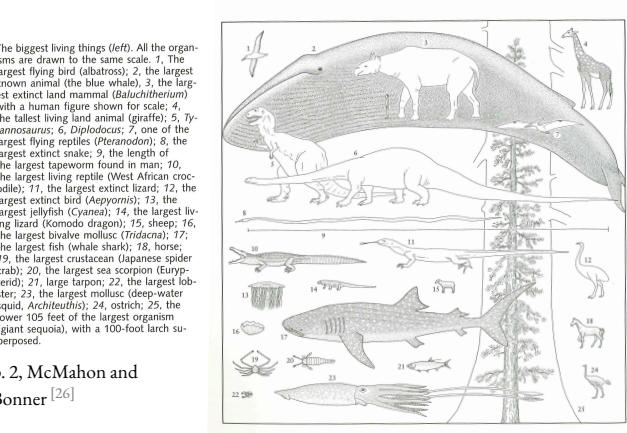
Dimensions scale nonlinearly.

Allometry:

Refers to differential growth rates of the parts of a living organism's body part or process.

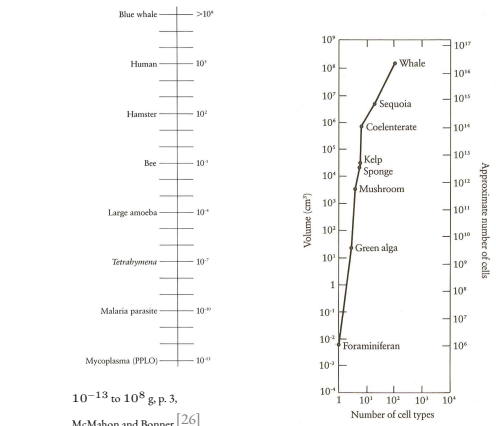
First proposed by Huxley and Teissier, Nature, 1936
“Terminology of relative growth” [15, 34]

The many scales of life:



p. 2, McMahon and Bonner [26]

Size range (in grams) and cell differentiation:



10⁻¹³ to 10⁸ g, p. 3, McMahon and Bonner [26]

Definitions

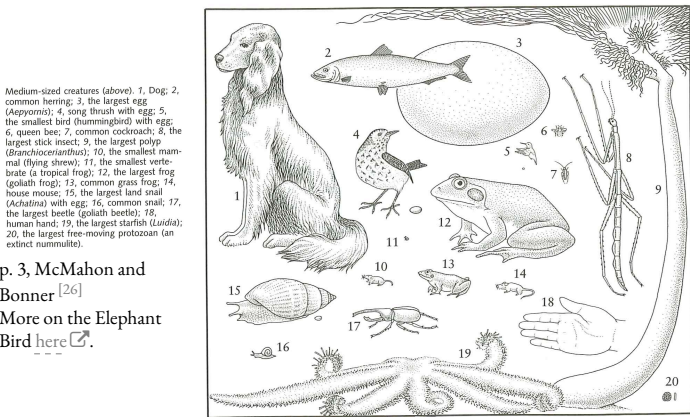
Isometry versus Allometry:

- Iso-metry = ‘same measure’
- Allo-metry = ‘other measure’

We use allometric scaling to refer to both:

- Nonlinear scaling of a dependent variable on an independent one (e.g., $y \propto x^{1/3}$)
- The relative scaling of correlated measures (e.g., white and gray matter).

The many scales of life:



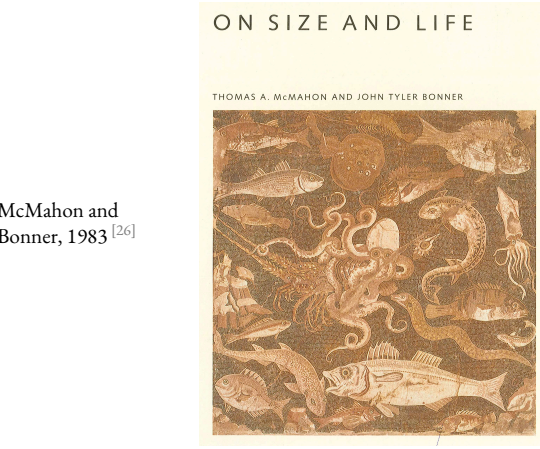
p. 3, McMahon and Bonner [26]
More on the Elephant Bird here .

Titanotheres horns: $L_{\text{horn}} \sim L_{\text{skull}}^4$



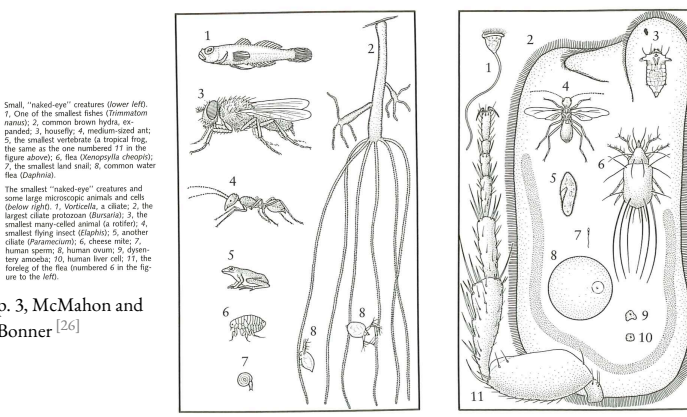
p. 36, McMahon and Bonner [26]; a bit dubious.

An interesting, earlier treatise on scaling:



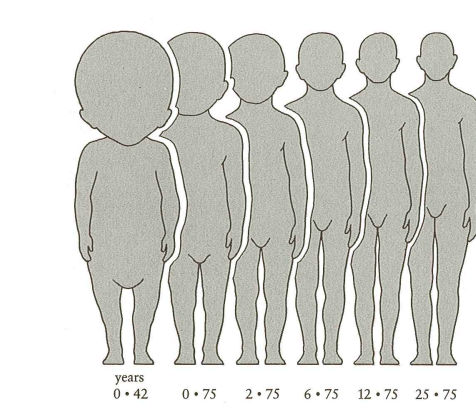
McMahon and Bonner, 1983 [26]

The many scales of life:



p. 3, McMahon and Bonner [26]

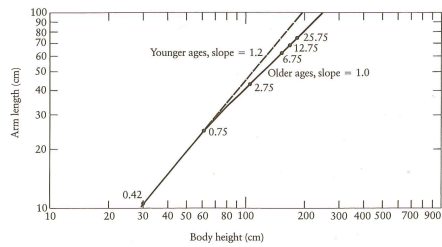
Non-uniform growth:



p. 32, McMahon and Bonner [26]

Non-uniform growth—arm length versus height:

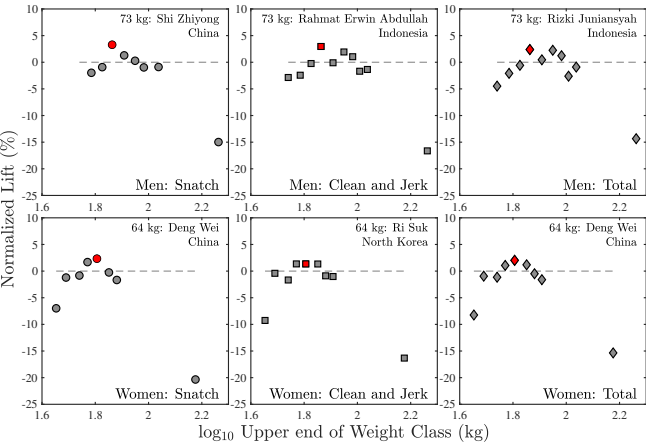
Good example of a **break in scaling**:



A **crossover** in scaling occurs around a height of 1 metre.

p. 32, McMahon and Bonner [26]

The “best” overall lifters:



$P = c M^\alpha$

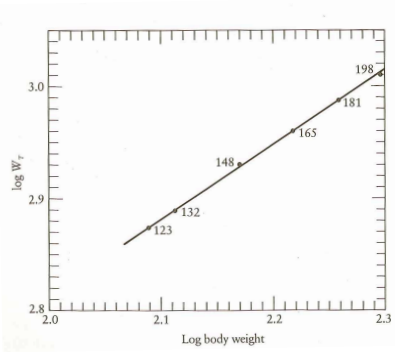
Prefactor *c* depends on **body plan** and **body temperature**:

Birds	39–41 °C
Eutherian Mammals	36–38 °C
Marsupials	34–36 °C
Monotremes	30–31 °C



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Weightlifting: $M_{\text{world record}} \propto M_{\text{lifter}}^{2/3}$



Idea: Power $\sim$ cross-sectional area of isometric lifters.
But modern data suggests an exponent of 1/2.

p. 53, McMahon and Bonner [26]

Stories—The Fraction Assassin:²



²bonk bonk*

What one might expect:

$\alpha = 2/3$ because ...

Dimensional analysis suggests an energy balance surface law:

$P \propto S \propto V^{2/3} \propto M^{2/3}$

Assumes isometric scaling (not quite the spherical cow).

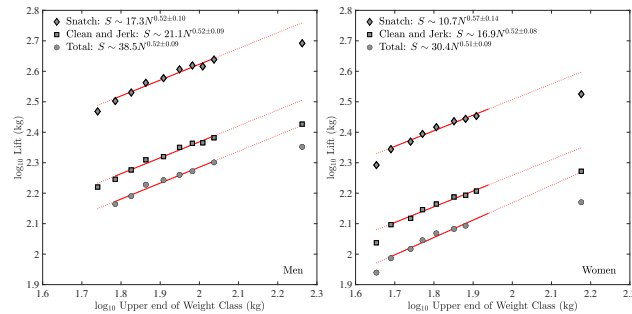
Lognormal fluctuations:
Gaussian fluctuations in log *P* around log *cM*^α.

Stefan-Boltzmann law for radiated energy:

$\frac{dE}{dt} = \sigma \epsilon S T^4 \propto S$

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Evidence for a 1/2 scaling exponent for weightlifting:



Li Wenwen’s gold medal joy in Paris: [Enjoy](#) (at 2:25 with bonus Australian commentary).

Animal power

Fundamental biological and ecological constraint:

$P = c M^\alpha$

P = basal metabolic rate

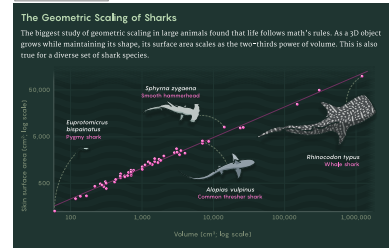
M = organismal body mass



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Sharks are super-duper isometric:

“The geometry of life: Testing the scaling of whole-organism surface area and volume using sharks”
Gayford et al.,
Royal Society Open Science, **12**, 242205, 2025. [11]



Quanta Magazine [article](#): “Shark Data Suggests Animals Scale Like Geometric Objects”

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The prevailing belief of the Church of Quarterology:

$\alpha = 3/4$

$P \propto M^{3/4}$

Huh?

The prevailing belief of the Church of Quarterology:

Most obvious concern:

$3/4 - 2/3 = 1/12$

- ☿ An exponent higher than 2/3 points suggests a fundamental inefficiency in biology.
- ☿ Organisms must somehow be running ‘hotter’ than they need to balance heat loss.

Related putative scalings:

Wait! There’s more!:

- ☿ number of capillaries $\propto M^{3/4}$
- ☿ time to reproductive maturity $\propto M^{1/4}$
- ☿ heart rate $\propto M^{-1/4}$
- ☿ cross-sectional area of aorta $\propto M^{3/4}$
- ☿ population density $\propto M^{-3/4}$

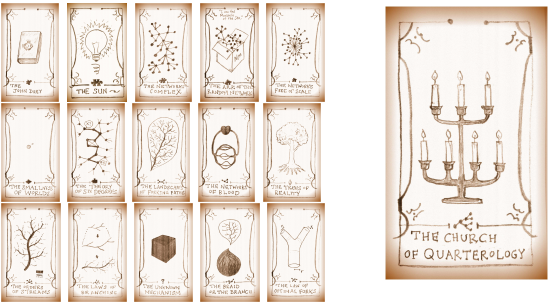
The great ‘law’ of heartbeats:

Assuming:

- ☿ Average lifespan $\propto M^\beta$
- ☿ Average heart rate $\propto M^{-\beta}$
- ☿ Irrelevant but perhaps $\beta = 1/4$.

Then:

- ☿ Average number of heart beats in a lifespan $\simeq (\text{Average lifespan}) \times (\text{Average heart rate})$
 $\propto M^{\beta-\beta}$
 $\propto M^0$
- ☿ Number of heartbeats per life time is independent of organism size!
- ☿ ≈ 1.5 billion....



Ecology—Species-area law:

Allegedly (data is messy): [21, 19]



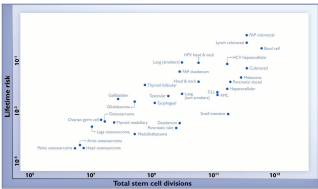
“An equilibrium theory of insular zoogeography”
MacArthur and Wilson,
Evolution, 17, 373–387, 1963. [21]

- ☿ $N_{\text{species}} \propto A^\beta$
- ☿ According to physicists—on islands: $\beta \approx 1/4$.
- ☿ Also—on continuous land: $\beta \approx 1/8$.

Cancer:



“Variation in cancer risk among tissues can be explained by the number of stem cell divisions”
Tomasetti and Vogelstein,
Science, 347, 78–81, 2015. [36]



“How fast do living organisms move: Maximum speeds from bacteria to elephants and whales”
Meyer-Vernet and Rospars,
American Journal of Physics, 83, 719–722, 2015. [28]

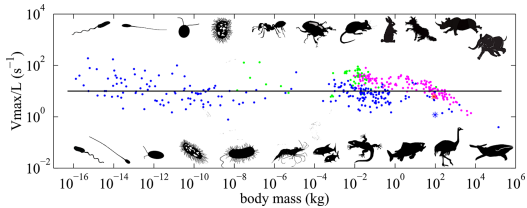


Fig. 1. Maximum relative speed versus body mass for 202 running species (157 mammals plotted in magenta and 45 non-mammals plotted in green), 127 swimming species and 91 micro-organisms (plotted in blue). The sources of the data are given in Ref. 16. The solid line is the maximum relative speed (Fig. 13b) estimated in Sec. III. The human world records are plotted as asterisks (upper for running and lower for swimming). Some examples of organisms of various masses are sketched in black (drawings by François Meyer).



“A general scaling law reveals why the largest animals are not the fastest”
Hirt et al.,
Nature Ecology & Evolution, 1, 1116, 2017. [12]

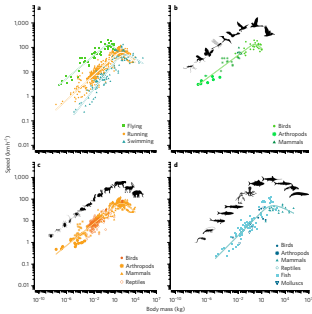


Figure 2 | Empirical data and time-dependent model fits for the allometric scaling of maximum speed. a, Comparison of scaling for the different locomotion modes (Flying, running, swimming). b, c, Taxonomic differences are illustrated separately for flying (b, n=55), running (c, n=488) and swimming (d, n=129) animals. Overall model fit: $R^2=0.8813$. The residual variation does not exhibit a signature of taxonomy (only a weak effect of thermoregulation, see Methods).

Roughly: $p \sim r^{2/3}$ where p = life time probability and r = rate of stem cell replication.



“A general scaling law reveals why the largest animals are not the fastest”
Hirt et al.,
Nature Ecology & Evolution, **1**, 1116, 2017. [12]

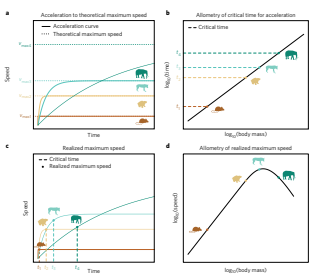


Figure 11 Concept of time-dependent and mass-dependent maximum speed of animals. a. Acceleration of animals follows a saturation curve (solid line) representing the theoretical maximum speed (dashed line) depending on body mass (color scale). b. The time available for acceleration increases with body mass following a power law. c. d. This critical time determines the realized maximum speed (d), yielding a hump-shaped increase of maximum speed with body mass (d).



“Athletics: Momentous sprint at the 2156 Olympics?”
Tatem et al.,
Nature, **431**, 525–525, 2004. [35]

Linear extrapolation for the 100 metres:

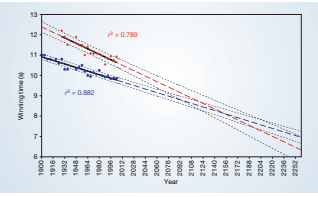


Figure 1 The winning Olympic 100 metre sprint times for men (blue points) and women (red points), with superimposed best-fit linear regression lines (solid black lines) and coefficients of determination. The regression lines are extrapolated (dashed blue and red lines for men and women, respectively) and 95% confidence intervals (dotted black lines) based on the available points are superimposed. The projections intersected before the 2156 Olympics, when the winning women's 100 metre sprint time of 0.173 s will be faster than the men's at 0.181 s.

Tatem: “If I’m wrong anyone is welcome to come and question me about the result after the 2156 Olympics.”

From How do wombats poop cubes? Scientists get to the bottom of the mystery, Science, 2021/01/27:

“That just leaves one mystery: why wombats evolved cubic poop in the first place.

Hu speculates that because the animals climb up on rocks and logs to mark their territory, the flat-sided [poops] aren’t as likely to roll off from these high perches.

...

In the meantime, Hu also thinks this knowledge could help researchers raising wombats in captivity.

“Sometimes their [poops] aren’t as cubic as the [wild] ones,” he says.

The squarer the poop, the healthier the wombat.’

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Theoretical story:

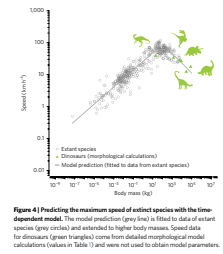


Figure 4 Predicting the maximum speed of extant species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extrapolated to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 3) and were not used to obtain model parameters.

- Maximum speed increases with size:
 $v_{\max} = aM^b$
- Takes a while to get going:
 $v(t) = v_{\max}(1 - e^{-kt})$
- $k \sim F_{\max}/M \sim cM^{d-1}$
Literature: $0.75 \lesssim d \lesssim 0.94$
- Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$
Literature: $0.76 \lesssim g \lesssim 1.27$
- $v_{\max} = aM^b(1 - e^{-hM^f})$
- $i = d - 1 + g$ and $h = cf$

Note: [28] not cited.

“Duration of urination does not change with body size”
Yang et al., Proceedings of the National Academy of Sciences, **111**, 11932–11937, 2014. [38]

- 32 mammals at Zoo Atlanta
- Figs. 1 and 2 are NSFTCR³
- $M = 3 \times 10^1 \text{ g}$ to $8 \times 10^6 \text{ g}$
- For $\geq 3 \times 10^3 \text{ g}$, $T \sim M^{1/6}$
- Duration $\sim 21 \pm 13$ seconds
- Smaller mammals: $T \sim M^0$
- Duration ~ 0.02 to 2 seconds

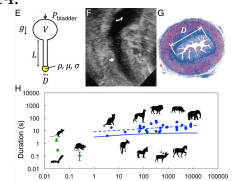


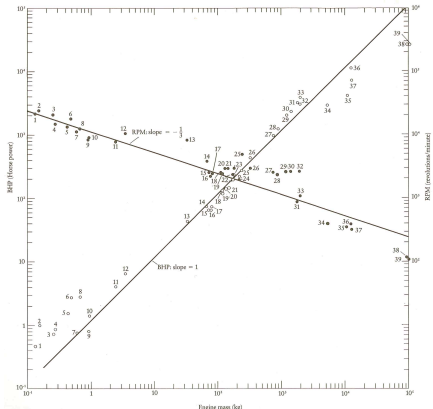
Table 1. Measured allometric relationships for the urinary system of animals

Variable	Units	Best fit	R^2	N
Duration of urination	T	$8.2 \text{ s M}^{0.167}$	0.2	32
Urethral length	L	$35 \text{ mm M}^{0.167}$	0.9	47
Urethral diameter	D	$2.5 \text{ mm M}^{0.167}$	0.9	22
Shape factor	σ	$0.2 \text{ M}^{0.167}$	0.5	5
Bladder capacity	V	$4.5 \text{ mL M}^{0.167}$	0.9	9
Bladder pressure	P_{bladder}	$5.2 \text{ kPa M}^{0.167}$	0.02	8
Flow rate for urination	Q_{ur}	$1.6 \text{ mL s}^{-1} \text{ M}^{0.167}$	0.9	66
Flow rate for males	Q_{m}	$0.3 \text{ mL s}^{-1} \text{ M}^{0.167}$	0.9	15

Body mass M given in kilograms. Duration of urination corresponds to animals heavier than 3 kg. Urethral length and diameter, shape factor, bladder capacity, bladder pressure, and flow rates correspond to animals heavier than 0.02 kg.

³Not Safe For The Class Room

Engines:

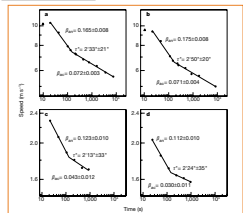


BHP = brake horse power

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“Scaling in athletic world records”
Savaglio and Carbone,
Nature, **404**, 244, 2000. [33]



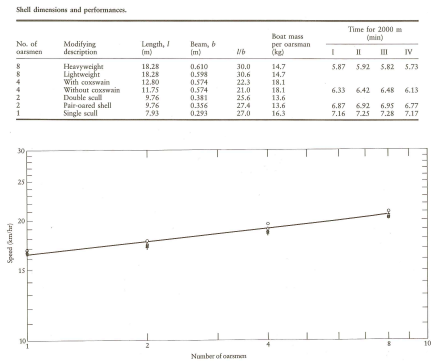
The allometry of nails:

A buckling instability?:

- Physics/Engineering result: Columns buckle under a load which depends on d^4/ℓ^2 .
- To drive nails in, posit resistive force $\propto$ nail circumference $= \pi d$.
- Match forces independent of nail size: $d^4/\ell^2 \propto d$.
- Leads to $d \propto \ell^{2/3}$.
- Argument made by Galileo^[10] in 1638 in “Discourses on Two New Sciences.” Also, see here.
- Another smart person’s contribution: Euler, 1757
- Also see McMahon, “Size and Shape in Biology,” Science, 1973.^[25]

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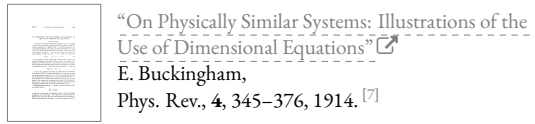
Rowing: Speed $\propto$ (number of rowers)^{1/9}



- Very weak scaling and size variation but it’s theoretically explainable ...

Dimensional Analysis:

The Buckingham π theorem⁵:



As captured in the 1990s in the MIT physics library:



⁵Stigler’s Law of Eponymy applies yet again. See here. More later.

A little formalism:

- Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).
- Consider $\pi_i = q_1^{x_1} q_2^{x_2} \dots q_n^{x_n}$.
- We (desperately) want to find all sets of powers x_j that create dimensionless quantities.
- Dimensions: want $[\pi_i] = [q_1]^{x_1} [q_2]^{x_2} \dots [q_n]^{x_n} = 1$.
- For the platypus pendulum we have $[q_1] = L, [q_2] = M, [q_3] = LT^{-2}$, and $[q_4] = T$, with dimensions $d_1 = L, d_2 = M$, and $d_3 = T$.
- So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.
- We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.
- We now need: $x_1 + x_3 = 0, x_2 = 0$, and $-2x_3 + x_4 = 0$.
- Time for matrixology ...

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Dimensional Analysis:⁶

Fundamental equations cannot depend on units:

- System involves n related quantities with some unknown equation $f(q_1, q_2, \dots, q_n) = 0$.
- Geometric ex.: area of a square, side length ℓ : $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.
- Rewrite as a relation of $p \leq n$ independent dimensionless parameters⁶ where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$

- e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.
- Another example: $F = ma \Rightarrow F/ma - 1 = 0$.
- Plan: solve problems using only backs of envelopes.

⁶Length is a dimension, furlongs and smoots are units

Well, of course there are matrices:

- Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.
- Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.
- Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1 = \text{const.}$
- In general: Create a matrix $\mathbf{A}$ where ij th entry is the power of dimension i in the j th variable, and solve by row reduction to find basis null vectors.
- We (you) find: $\pi_1 = \ell/g\tau^2 = \text{const.}$ Upshot: $\tau \propto \sqrt{\ell}$.

Insert assignment question

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Physics:

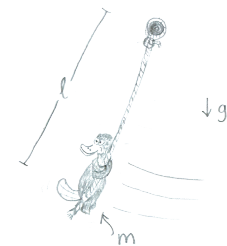
Scaling in elementary laws of physics:

- Inverse-square law of gravity and Coulomb’s law:
$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$
- Force is diminished by expansion of space away from source.
- The square is $d - 1 = 3 - 1 = 2$, the dimension of a sphere’s surface.
- We’ll see a gravity law applies for a range of human phenomena.

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Example:

Simple pendulum:



- Idealized mass/platypus swinging forever.
- Four quantities:
 - Length ℓ ,
 - mass m ,
 - gravitational acceleration g , and
 - pendulum’s period τ .

- Variable dimensions: $[\ell] = L, [m] = M, [g] = LT^{-2}$, and $[\tau] = T$.
- Turn over your envelopes and find some π ’s.



“Scaling, self-similarity, and intermediate asymptotics”⁷ by G. I. Barenblatt (1996).^[2]

G. I. Taylor, magazines, and classified secrets:



Self-similar blast wave:

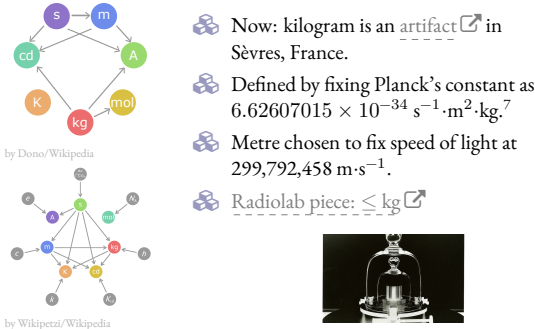
- Radius: $[R] = L$,
Time: $[t] = T$,
Density of air: $[\rho] = M/L^3$,
Energy: $[E] = ML^2/T^2$.
- Four variables, three dimensions.
- One dimensionless variable:
 $E = \text{constant} \times \rho R^5/t^2$.
- Scaling: Speed decays as $1/R^{3/2}$.

Related: Radiolab’s Elements on the Cold War, the Bomb Pulse, and the dating of cell age (33:30).

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Sorting out base units of fundamental measurement:

SI base units were redefined in 2019: [↗](#)



- Now: kilogram is an artifact [↗](#) in Sèvres, France.
- Defined by fixing Planck's constant as $6.62607015 \times 10^{-34} \text{ s}^{-1} \cdot \text{m}^2 \cdot \text{kg}^{-1}$.
- Metre chosen to fix speed of light at $299,792,458 \text{ m} \cdot \text{s}^{-1}$.
- Radiolab piece: $\leq \text{kg}$ [↗](#)



⁷Not without some arguing ...

Advances in turbulence:

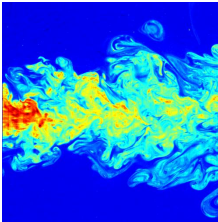
In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: ^[18]

$E(k) = C\epsilon^{2/3}k^{-5/3}$

- $E(k)$ = energy spectrum function.
- ϵ = rate of energy dissipation.
- $k = 2\pi/\lambda$ = wavenumber.
- Energy is distributed across all modes, decaying with wave number.
- No internal characteristic scale to turbulence.
- Stands up well experimentally and there has been no other advance of similar magnitude.

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Turbulence:



Big whirls have little whirls
That heed on their velocity,
And little whirls have littler whirls
And so on to viscosity.
— Lewis Fry Richardson [↗](#)

- Image from [here](#) [↗](#).
- Jonathan Swift (1733): “Big fleas have little fleas upon their backs to bite ’em, And little fleas have lesser fleas, and so, ad infinitum.” The Siphonaptera. [↗](#)

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“Turbulent luminance in impassioned van Gogh paintings” [↗](#)
Áragón et al.,
J. Math. Imaging Vis., **30**, 275–283, 2008. ^[1]

- Examined the probability pixels a distance R apart share the same luminance.
- “Van Gogh painted perfect turbulence” [↗](#) by Phillip Ball, July 2006.
- Apparently not observed in other famous painter’s works or when van Gogh was stable.
- Oops: Small ranges and natural log used.

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Advances in turbulence:

“The Geometry of Nature”: [↗](#)



- “Anomalous” scaling of lengths, areas, volumes relative to each other.
- The enduring question: how do self-similar geometries form?

- Robert E. Horton [↗](#): Self-similarity of river (branching) networks (1945). ^[13]
- Harold Hurst [↗](#)—Roughness of time series (1951). ^[14]
- Lewis Fry Richardson [↗](#)—Coastlines (1961).
- Benoît B. Mandelbrot [↗](#)—Introduced the term “Fractals” and explored them everywhere, 1960s on. ^[22, 23, 24]

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Scaling in Cities:



“Growth, innovation, scaling, and the pace of life in cities” [↗](#)
Bettencourt et al.,
Proc. Natl. Acad. Sci., **104**, 7301–7306, 2007. ^[4]

- Quantified levels of
 - Infrastructure
 - Wealth
 - Crime levels
 - Disease
 - Energy consumption
- as a function of city size N (population).

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⁸Note to self: Make millions with the “Fractal Diet”

Scaling in Cities:

Table 1. Scaling exponents for urban indicators vs. city size

Y	β	95% CI	Adj- R^2	Observations	Country-year
New patents	1.27	[1.25, 1.29]	0.72	331	U.S. 2001
Inventors	1.25	[1.22, 1.27]	0.76	331	U.S. 2001
Private R&D employment	1.34	[1.29, 1.39]	0.92	266	U.S. 2002
“Supercreative” employment	1.15	[1.11, 1.18]	0.89	287	U.S. 2003
R&D establishments	1.19	[1.14, 1.22]	0.77	287	U.S. 1997
R&D employment	1.26	[1.18, 1.43]	0.93	295	China 2002
Total wages	1.12	[1.09, 1.13]	0.96	361	U.S. 2002
Total bank deposits	1.08	[1.03, 1.11]	0.91	267	U.S. 1996
GDP	1.15	[1.06, 1.23]	0.96	295	China 2002
GDP	1.26	[1.09, 1.46]	0.64	196	EU 1999–2003
GDP	1.13	[1.03, 1.23]	0.94	37	Germany 2003
Total electrical consumption	1.07	[1.03, 1.11]	0.88	392	Germany 2002
New AIDS cases	1.23	[1.18, 1.29]	0.76	93	U.S. 2002–2003
Serious crimes	1.16	[1.11, 1.18]	0.89	287	U.S. 2003
Total housing	1.00	[0.99, 1.01]	0.99	316	U.S. 1990
Total employment	1.01	[0.99, 1.02]	0.98	331	U.S. 2001
Household electrical consumption	1.00	[0.94, 1.06]	0.88	377	Germany 2002
Household electrical consumption	1.05	[0.89, 1.22]	0.91	295	China 2002
Household water consumption	1.01	[0.89, 1.11]	0.96	295	China 2002
Gasoline stations	0.77	[0.74, 0.81]	0.93	318	U.S. 2001
Gasoline sales	0.79	[0.73, 0.80]	0.94	318	U.S. 2001
Length of electrical cables	0.87	[0.82, 0.92]	0.75	380	Germany 2002
Road surface	0.83	[0.74, 0.92]	0.87	29	Germany 2002

Data sources are shown in [SI Text](#). CI, confidence interval; Adj- R^2 , adjusted R^2 ; GDP, gross domestic product.

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Scaling in Cities:

Intriguing findings:

- Global supply costs scale **sublinearly** with N ($\beta < 1$).
 - Returns to scale for infrastructure.
- Total individual costs scale **linearly** with N ($\beta = 1$)
 - Individuals consume similar amounts independent of city size.
- Social quantities scale **superlinearly** with N ($\beta > 1$)
 - Creativity (# patents), wealth, disease, crime, ...

Density doesn’t seem to matter...

- Surprising given that across the world, we observe two orders of magnitude variation in area covered by agglomerations [↗](#) of fixed populations.

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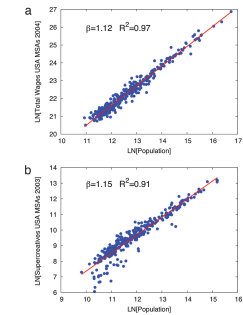


Fig. 1. Examples of scaling relationships. (a) Total wages per MSA in 2004 for the U.S. (blue points) vs. metropolitan population. (b) Supercreative employment per MSA in 2003, for the U.S. (blue points) vs. metropolitan population. Best-fit scaling relations are shown as solid lines.

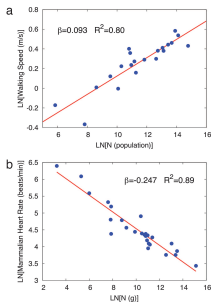


Fig. 2. The pace of urban life increases with city size in contrast to the pace of biological life, which decreases with organism size. (a) Scaling of walking speed vs. population for cities around the world. (b) Heart rate vs. the size (mass) of organisms.



“Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities”
Bettencourt et al.,
PLoS ONE, 5, e13541, 2010. [5]

Comparing city features across populations:

- Cities = Metropolitan Statistical Areas (MSAs)
- Story: Fit scaling law and examine residuals
- Does a city have more or less crime than expected when normalized for population?
- Same idea as Encephalization Quotient (EQ).

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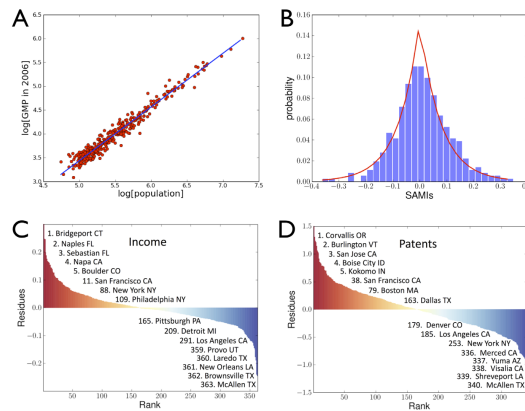


Figure 1. Urban Agglomeration effects result in per capita nonlinear scaling of urban metrics. Subtracting these effects produces a truly local measure of urban dynamics and a reference scale for ranking cities. a) A typical superlinear scaling law (solid line) Gross Metropolitan Product of US MSAs in 2006 (red dots) vs. population; the slope of the solid line has exponent, $\beta = 1.26$ (95% CI [1.101, 1.440]). b) Histogram showing frequency of residuals (SAMs, see Eq. (2)), the statistics of residuals is well described by a Laplace distribution (red line). Scale independent ranking (SAMs) for US MSAs by c) personal income and d) patenting (red denotes above average performance, blue below). For more details see Text S1, Table S1 and Figure S1. doi:10.1371/journal.pone.0013541.g001

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A possible theoretical explanation?



“The origins of scaling in cities”
Luís M. A. Bettencourt,
Science, 340, 1438–1441, 2013. [3]

#sixthology

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Non-simple scaling for death:



“Statistical signs of social influence on suicides”
Melo et al.,
Scientific Reports, 4, 6239, 2014. [27]

- Bettencourt *et al.*'s initial work suggested social phenomena would follow superlinear scaling (wealth, crime, disease)
- Homicide, traffic, and suicide [9] all tied to social context in complex, different ways.
- For cities in Brazil, Melo *et al.* show:
 - Homicide appears to follow superlinear scaling ($\beta = 1.24 \pm 0.01$)
 - Traffic accident deaths appear to follow linear scaling ($\beta = 0.99 \pm 0.02$)
 - Suicide appears to follow sublinear scaling. ($\beta = 0.84 \pm 0.02$)

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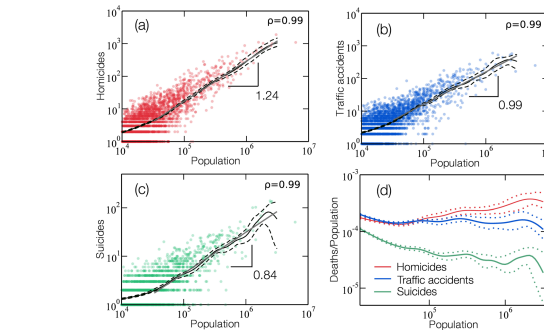


Figure 1 | Scaling relations for homicides, traffic accidents, and suicides for the year of 2009 in Brazil. The small circles show the total number of deaths by (a) homicides (red), (b) traffic accidents (blue), and (c) suicides (green) vs the population of each city. Each graph represents only one urban indicator, and the solid gray line indicate the best fit for a power-law relation, using OLS regression, between the average total number of deaths and the city size (population). To reduce the fluctuations we also performed a Nadaraya-Watson kernel regression [15]. The dashed lines show the 95% confidence band for the Nadaraya-Watson kernel regression. The ordinary least-squares (OLS) fit to the Nadaraya-Watson kernel regression applied to the data on homicides in (a) reveals an allometric exponent $\beta = 1.24 \pm 0.01$, with a 95% confidence interval estimated by bootstrap. This is compatible with previous results obtained for U.S. that also indicate a super-linear scaling relation with population and an exponent $\beta = 1.16$. Using the same procedure, we find $\beta = 0.99 \pm 0.02$ and 0.84 ± 0.02 for the numbers of deaths in traffic accidents (b) and suicides (c), respectively. The values of the Pearson correlation coefficients ρ associated with these scaling relations are shown in each plot. This non-linear behavior observed for homicides and suicides certainly reflects the complexity of human social relations and strongly suggests that the topology of the social network plays an important role on the rate of these events. (d) The solid lines show the Nadaraya-Watson kernel regression rate of deaths (total number of deaths divided by the population of a city) for each urban indicator, namely, homicides (red), traffic accidents (blue), and suicides (green). The dashed lines represent the 95% confidence bands. While the rate of fatal traffic accidents remains approximately invariant, the rate of homicides systematically increases, and the rate of suicides decreases with population. doi:10.1371/journal.pone.0013541.g001

Dynamics (Brazil):

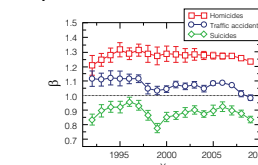
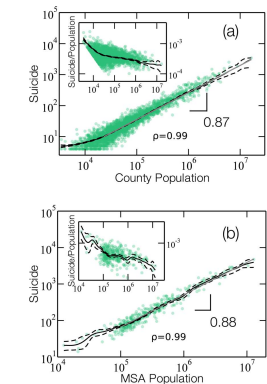


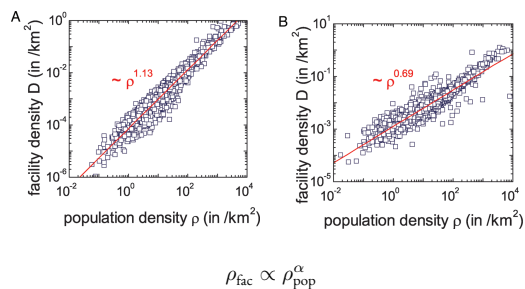
Figure 2 | Temporal evolution of allometric exponent β for homicides (red squares), deaths in traffic accidents (blue circles), and suicides (green diamonds). Time evolution of the power-law exponent β for each behavioral urban indicator in Brazil from 1992 to 2009. We can see that the non-linear behavior for homicides and suicides are robust for this 19 years period, and for the traffic accidents the exponent remains close to 1.0. doi:10.1371/journal.pone.0013541.g002

US data:



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Density of public and private facilities:



$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.

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Public versus private facilities: evidence

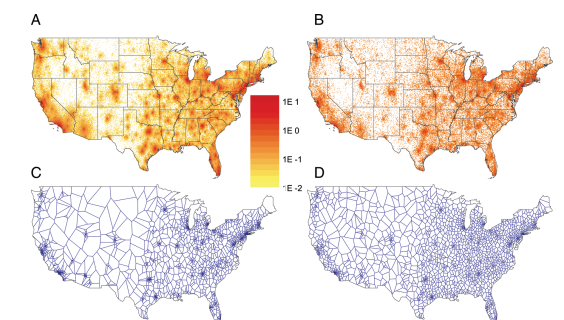
US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87
SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \approx 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

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Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

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“Pattern in escalations in insurgent and terrorist activity”
Johnson et al.,
Science, **333**, 81–84, 2011. [16]

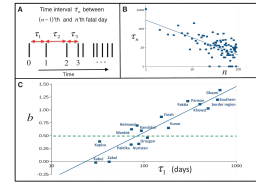
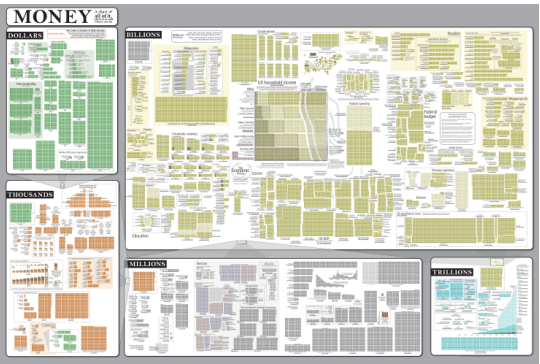


Fig. 1. **AB** Schematic timeline of successive fatal days shown as vertical bars. τ_n is the time interval between the first two fatal days, labeled 0 and 1. **BC** Successive time intervals τ_n between days with 100 fatalities in the Afghanistan province of Kandahar reported. On this log-log plot, the best-fit power-law program curve is by definition a straight line with slope $-b$ (it is an escalation rate). **CD** The solid blue line shows how time τ_n through time intervals τ_n and τ_{n+1} for individual days. The green dashed line shows values $b \approx 0.5$, which is the situation in which there are no correlations. The value of fatalities recorded in variables as “Number of fatalities” is shown as a separate region because of their likely connection to operations near the Paktia border.

- Escalation: $\tau_n \sim \tau_1 n^{-b}$
- b = scaling exponent (escalation rate)
- Interevent time τ_n between fatal attacks $n-1$ and n (binned by days)
- Learning curves for organizations [37]
- More later on size distributions [8, 17, 6]

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Explore the original zoomable and interactive version here:
<http://xkcd.com/980/>

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Irregular verbs

Cleaning up the code that is English:



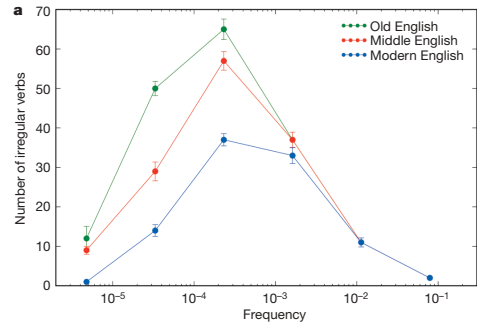
“Quantifying the evolutionary dynamics of language”
Lieberman et al.,
Nature, **449**, 713–716, 2007. [20]



- Exploration of how verbs with irregular conjugation gradually become regular over time.
- Comparison of verb behavior in Old, Middle, and Modern English.

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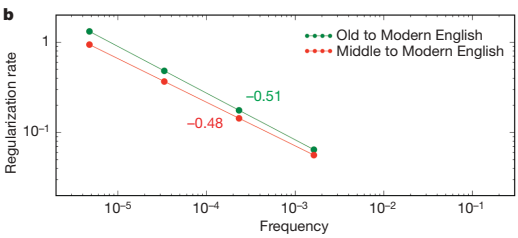
Irregular verbs



- Universal tendency towards regular conjugation
- Rare verbs tend to be regular in the first place

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Irregular verbs



- Rates are relative.
- The more common a verb is, the more resilient it is to change.

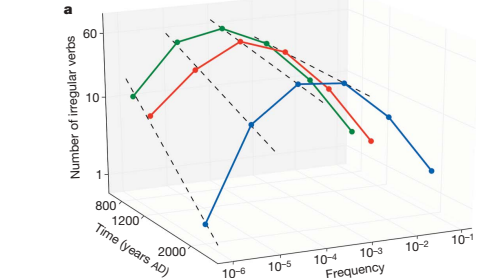
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Irregular verbs

Frequency	Verbs	Regularization (%)	Half-life (yr)
10^{-1} – 1	be, have come, do, find, get, give, go, know, say, see, take, think begin, break, bring, buy, choose, draw, drink, drive, eat, fall, fight, forget, grow, hang, help, hold, leave, let, lie, lose, reach, rise, run, seek, set, shake, sit, sleep, speak, stand, teach, throw, understand, walk, win, work, write	0	38,800
10^{-4} – 10^{-1}	right, forget, grow, hang, help, hold, leave, let, lie, lose, reach, rise, run, seek, set, shake, sit, sleep, speak, stand, teach, throw, understand, walk, win, work, write	10	14,400
10^{-4} – 10^{-3}	arise, back, bear, beat, bind, bite, blow, bow, burn, burst, carve, chew, climb, cling, creep, dare, dig, drag, flee, float, flow, fly, fold, freeze, grind, leap, lend, lock, melt, reckon, ride, run, shape, shine, shoot, shrink, sigh, sing, sink, slide, slip, smoke, spin, spring, starve, steal, step, stretch, strike, stroke, suck, swallow, swear, sweep, swim, swing, tear, velvet, wash, waste, weep, weigh, wind, yet, yield	43	2,000
10^{-4} – 10^{-4}	bark, below, bid, blend, braid, brew, cleave, cinge, crow, dive, dig, fare, feel, glide, grin, grip, heave, knead, low, milk, mourn, move, prescribe, redden, reek, row, scrape, seethe, shear, shed, shove, slay, sift, smile, sow, span, spurn, sting, stir, stirre, strow, swell, sweat, uproot, waste, warp, wax, wield, wring, writhe	72	700
10^{-4} – 10^{-5}	bide, chide, deliver, fly, hew, run, shrive, slink, snip, spew, stop, wreak	91	300

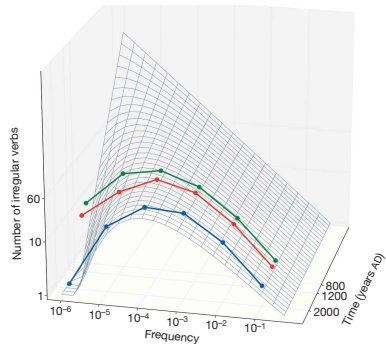
177 Old English irregular verbs were compiled for this study. These are arranged according to frequency bins, and in alphabetical order within each bin. Also shown is the percentage of verbs in each bin that have regularized. The half-life is shown in years. Verbs that have regularized are indicated in red. As we move down the list, an increasingly large fraction of the verbs are red: the frequency-dependent regularization of irregular verbs becomes immediately apparent.

- Red = regularized
- Estimates of half-life for regularization ($\propto f^{1/2}$)



- ‘Wed’ is next to go.
- ed is the winning rule...
- But ‘snuck’ is sneaking up on sneaked. [29]

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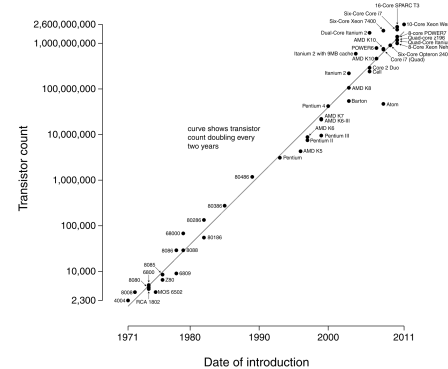


- Projecting back in time to proto-Zipf story of many tools.

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Moore’s Law:

Microprocessor Transistor Counts 1971–2011 & Moore’s Law

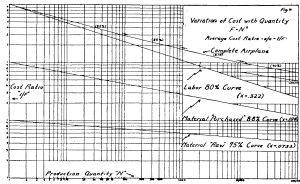
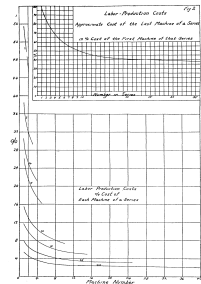


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“Factors affecting the costs of airplanes”

T. P. Wright,
Journal of Aeronautical Sciences, **10**, 302–328, 1936. [37]



- Power law decay of cost with number of planes produced.
- “The present writer started his studies of the variation of cost with quantity in 1922.”

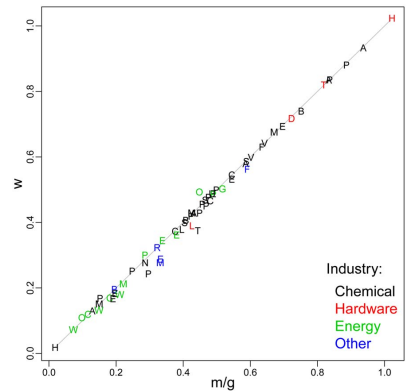


Figure 4. An illustration that the combination of exponentially increasing production and exponentially decreasing cost are equivalent to Wright’s law. The value of the Wright parameter v is plotted against the prediction m/g based on the Sahal formula, where m is the exponent of cost reduction and g the exponent of the increase in cumulative production.

Toy Story and Moore’s law:

“But when it came time to harden the deal and run the numbers for the contracts, I discovered to my dismay that computers were still too slow: The projected production cost was too high and the computation time way too long. We had to back out of the deal. This time, we did know enough detail to correctly apply Moore’s Law – and it told us that we had to wait another five years to start making the first movie. And sure enough, five years later Disney approached us to make Toy Story.”

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Scaling laws for technology production:

- “Statistical Basis for Predicting Technological Progress” Nagy et al., PLoS ONE, 2013. [31]
- y_t = stuff unit cost; x_t = total amount of stuff made.
- Wright’s Law, cost decreases as a power of total stuff made: [37]

$$y_t \propto x_t^{-w}.$$

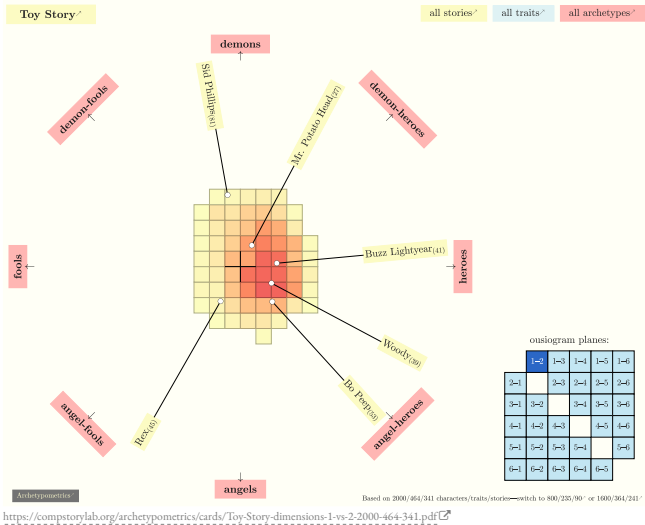
- Moore’s Law, framed as cost decrease connected with doubling of transistor density every two years: [30]

$$y_t \propto e^{-mt}.$$

- Sahal’s observation that Moore’s law gives rise to Wright’s law if stuff production grows exponentially: [32]

$$x_t \propto e^{gt}.$$

- Sahal + Moore gives Wright with $w = m/g$.



Toy Story and Moore’s law:

“We implement each step to see if it actually works, then gain the courage, the insight, and the engineering mastery to proceed to the next step. Moore’s Law told us that the new company we were starting, Pixar, had to bide its time—building hardware instead of making movies.”

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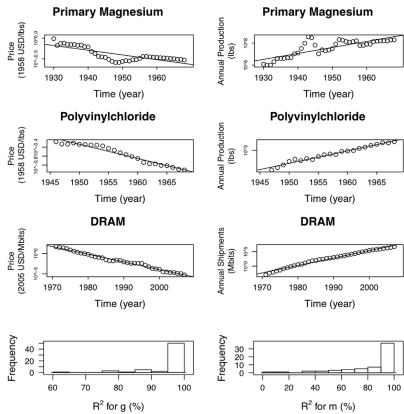


Figure 3. Three examples showing the logarithm of price as a function of time in the left column and the logarithm of production as a function of time in the right column, based on industry-wide data. We have chosen these examples to be representative. The top row contains an example with one of the worst fits, the second row an example with an intermediate goodness of fit, and the third row one of the best examples. The fourth row of the figure shows histograms of R^2 values for fitting y and m for the 62 datasets.

Toy Story and Moore’s law:

“When the group moved to California to become part of Lucasfilm, we got close to making a computer-animated movie again in the mid-1980s – this time about a monkey with godlike powers but a missing prefrontal cortex. We had a sponsor, a story treatment, and a marketing survey. We were prepared to make a screen test: Our hot young animator John Lasseter had sketched numerous studies of the hero monkey and had the sponsor salivating over a glass-dragon protagonist.”

⁸“How Pixar Used Moore’s Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Toy Story and Moore’s law:

Rhetoric of maybeness with hook to “More is different” [2]

“That’s the reason for expressing Moore’s Law in orders of magnitude rather than factors of 10. The latter form is merely arithmetic, but the former implies an intellectual challenge. We use “order of magnitude” to imply a change so great that it requires new thought processes, new conceptualizations: It’s not simply more, it’s different.”

⁸“How Pixar Used Moore’s Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

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⁸“How Pixar Used Moore’s Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

⁸“How Pixar Used Moore’s Law to Predict the Future,” Wired, 2013/04/17 <https://www.wired.com/2013/04/how-pixar-used-moores-law-to-predict-the-future/>

Shell of the nut:

- Scaling is a fundamental feature of complex systems.
- Basic distinction between isometric and allometric scaling.
- Powerful envelope-based approach: Dimensional analysis.
- “Oh yeah, well that’s just dimensional analysis” said the [insert your own adjective] physicist.
- Trickiness: A wide variety of mechanisms give rise to scalings.⁹
- Some mechanisms are common, some are rare.¹⁰

⁹It’s not your great-great-great-grandparents’ normal distribution
¹⁰To be understood: The scaling story of scaling-making mechanisms

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