

Random Networks

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2025–2026

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



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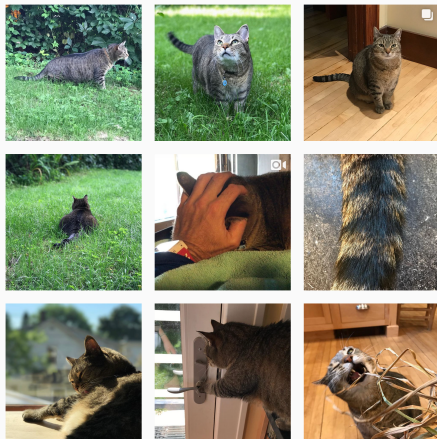
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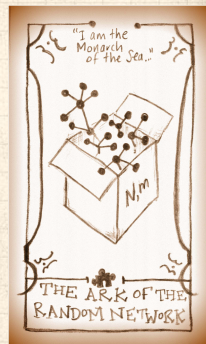
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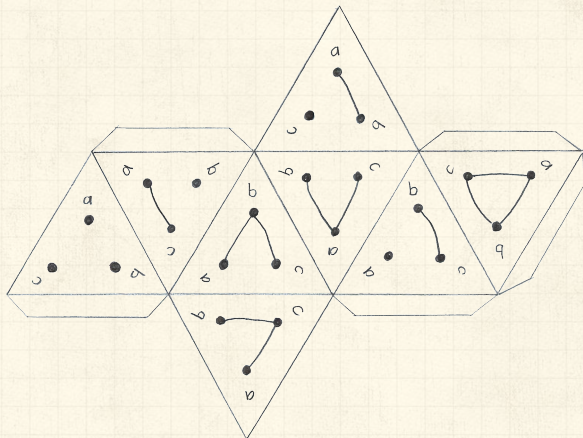
Some important models:

1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models (p^*).





Random network generator for $N = 3$:



Get your own exciting generator [here](#) .



As $N \nearrow$, polyhedral die rapidly becomes a ball...

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Pure, abstract random networks:

 Consider set of all networks with N labelled nodes and m edges.

 Standard random network = one **randomly chosen** network from this set.

 To be clear: each network is **equally** probable.

 Sometimes equiprobability is a good assumption, but it is always an assumption.

 Known as Erdős-Rényi random networks or **ER graphs**.



Random networks—basic features:

 Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

 Limit of $m = 0$: empty graph.

 Limit of $m = \binom{N}{2}$: complete or fully-connected graph.

 Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N(N-1)}.$$

 Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.

 Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.

 **Real world:** links are usually costly so real networks are almost always **sparse**.

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How to build standard random networks:



Given N and m .



Two probabilistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p .



Useful for theoretical work.

2. Take N nodes and add exactly m links by selecting edges without replacement.



Algorithm: Randomly choose a pair of nodes i and j , $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.



Best for adding relatively small numbers of links (most cases).



1 and 2 are effectively equivalent for large N .



Random networks

A few more things:

 For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

 So the expected or **average degree** is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \cancel{N} p \frac{1}{\cancel{N}} N(N-1) = p(N-1).$$

 Which is what it should be...

 If we keep $\langle k \rangle$ constant then $p \propto 1/N \rightarrow 0$ as $N \rightarrow \infty$.



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Next slides:

Example realizations of random networks



$N = 500$



Vary m , the number of edges from 100 to 1000.



Average degree $\langle k \rangle$ runs from 0.4 to 4.



Look at full network plus the largest component.



Random networks: examples for $N=500$

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$m = 100$
 $\langle k \rangle = 0.4$



$m = 200$
 $\langle k \rangle = 0.8$



$m = 230$
 $\langle k \rangle = 0.92$



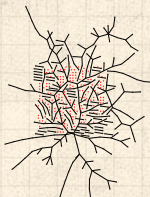
$m = 240$
 $\langle k \rangle = 0.96$



$m = 250$
 $\langle k \rangle = 1$



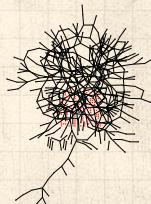
$m = 260$
 $\langle k \rangle = 1.04$



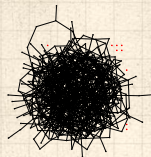
$m = 280$
 $\langle k \rangle = 1.12$



$m = 300$
 $\langle k \rangle = 1.2$



$m = 500$
 $\langle k \rangle = 2$



$m = 1000$
 $\langle k \rangle = 4$



Random networks: largest components

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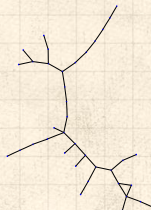
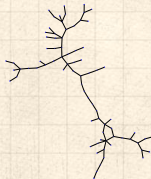
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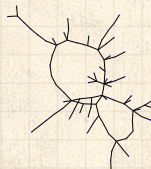
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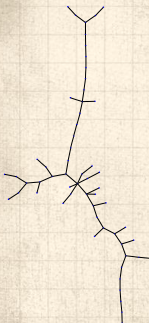
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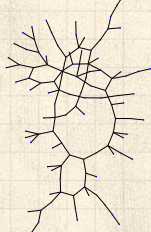
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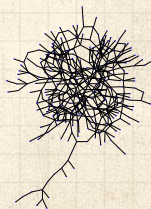
$m = 250$
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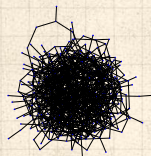
$m = 200$
 $\langle k \rangle = 0.8$



$m = 300$



$m = 500$
 $\langle k \rangle = 2$



$m = 1000$
 $\langle k \rangle = 4$



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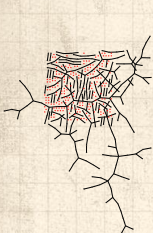
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$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



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 $\langle k \rangle = 1$



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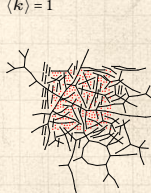
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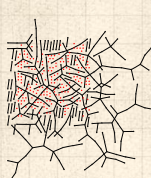
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$m = 250$
 $\langle k \rangle = 1$



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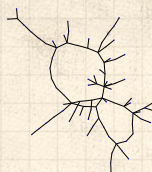
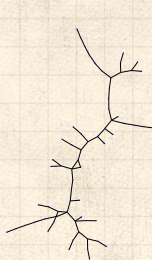
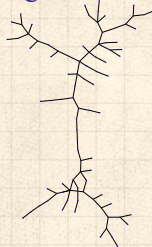
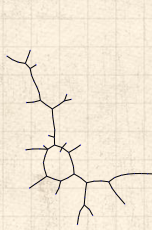
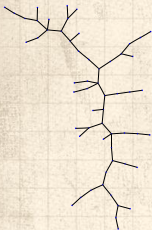
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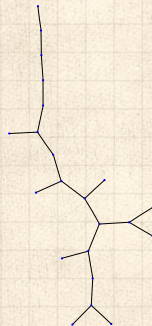
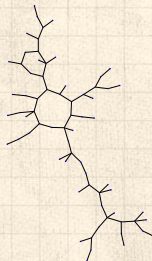
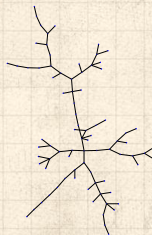
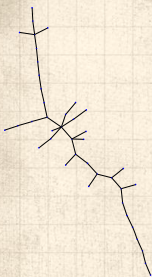
$m = 250$
 $\langle k \rangle = 1$

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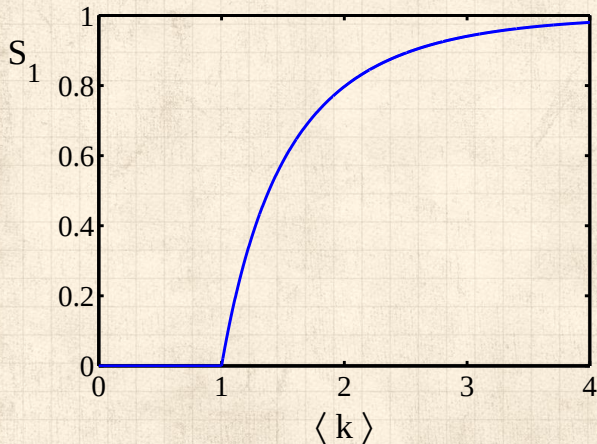
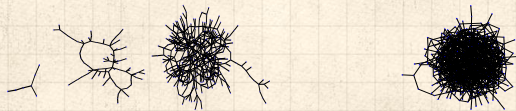
$m = 250$
 $\langle k \rangle = 1$

$m = 250$

$m = 250$

$m = 250$

Giant component



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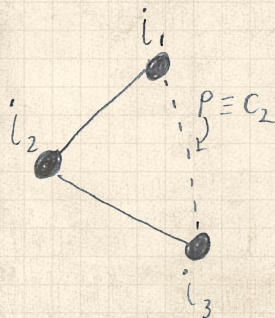


Clustering in random networks:

For construction method 1, what is the clustering coefficient for a finite network?

Consider triangle/triple clustering coefficient: [7]

$$C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$$



Recall: C_2 = probability that two friends of a node are also friends.

Or: C_2 = probability that a triple is part of a triangle.

For standard random networks, we have simply that

$$C_2 = p.$$



Clustering in random networks:



So for large random networks ($N \rightarrow \infty$), clustering drops to zero.



Key structural feature of random networks is that they locally look like
pure branching networks



No small loops.



Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k .
- Consider method 1 for constructing random networks: each possible link is realized with probability p .
- Now consider one node: there are ' $N - 1$ choose k ' ways the node can be connected to k of the other $N - 1$ nodes.
- Each connection occurs with probability p , each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution ↗:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$



Limiting form of $P(k; p, N)$:



Our degree distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$



What happens as $N \rightarrow \infty$?



We must end up with the normal distribution right?



If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \rightarrow \infty$.



But we want to keep $\langle k \rangle$ fixed...



So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N-1) = \text{constant}$.

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

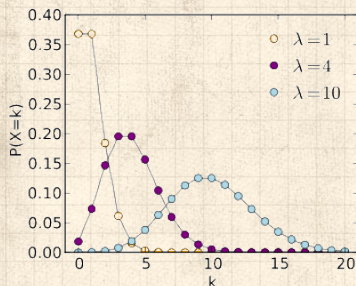


This is a Poisson distribution  with mean $\langle k \rangle$.



Poisson basics:

$$P(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$



$\lambda > 0$



$k = 0, 1, 2, 3, \dots$



Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.



e.g.:
phone calls/minute,
horse-kick deaths.



'Law of small numbers'



Poisson basics:



Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$



Checking:

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle} = 1$$

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Poisson basics:



Mean degree: we must have

$$\langle k \rangle = \sum_{k=0}^{\infty} k P(k; \langle k \rangle).$$



Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} k P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} k \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^k}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^i}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle} = \langle k \rangle \end{aligned}$$



In CocoNuTs, we find a different, crazier way of doing this...



Poisson basics:

 The **variance** of degree distributions for random networks turns out to be **very important**.

 Using calculation similar to one for finding $\langle k \rangle$ we find the **second moment** to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

 Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

 So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.

 Note: This is a special property of Poisson distribution and can trip us up...



- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k .
- Also known as the **configuration model**. [7]
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
 2. Examining mechanisms that lead to networks with certain degree distributions.

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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

-  $N = 1000$.
-  $P_k \propto k^{-\gamma}$ for $k \geq 1$.
-  Set $P_0 = 0$ (no isolated nodes).
-  Vary exponent γ between 2.10 and 2.91.
-  Again, look at full network plus the largest component.
-  Apart from degree distribution, wiring is random.

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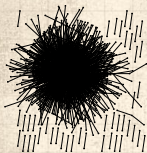
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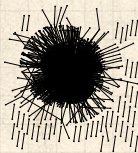
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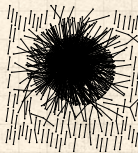
$\gamma = 2.1$
 $\langle k \rangle = 3.448$



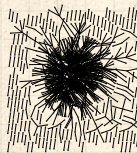
$\gamma = 2.19$
 $\langle k \rangle = 2.986$



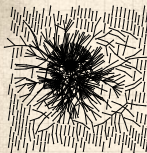
$\gamma = 2.28$
 $\langle k \rangle = 2.306$



$\gamma = 2.37$
 $\langle k \rangle = 2.504$



$\gamma = 2.46$
 $\langle k \rangle = 1.856$



$\gamma = 2.55$
 $\langle k \rangle = 1.712$



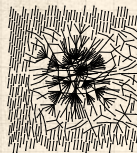
$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$



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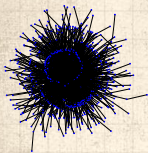
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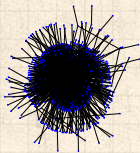
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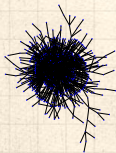
References



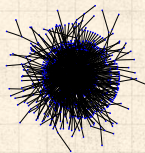
$\gamma = 2.1$
 $\langle k \rangle = 3.448$



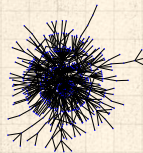
$\gamma = 2.19$
 $\langle k \rangle = 2.986$



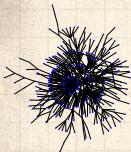
$\gamma = 2.28$
 $\langle k \rangle = 2.306$



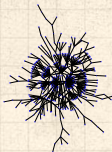
$\gamma = 2.37$
 $\langle k \rangle = 2.504$



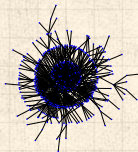
$\gamma = 2.46$
 $\langle k \rangle = 1.856$



$\gamma = 2.55$
 $\langle k \rangle = 1.712$



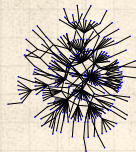
$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$



Generalized random networks:

-  Arbitrary degree distribution P_k .
-  Create (unconnected) nodes with degrees sampled from P_k .
-  Wire nodes together randomly.
-  Create ensemble to test deviations from randomness.

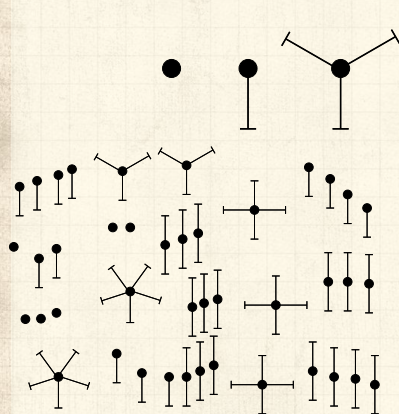


Building random networks: Stubs

Phase 1:



Idea: start with a soup of unconnected nodes with **stubs** (half-edges):



Randomly select stubs (not nodes!) and connect them.



Must have an even number of stubs.



Initially allow **self**- and **repeat** connections.

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Building random networks: First rewiring

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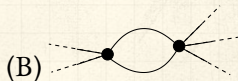
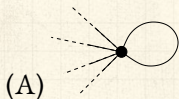
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Phase 2:

 Now find any (A) self-loops and (B) repeat edges and **randomly rewire** them.

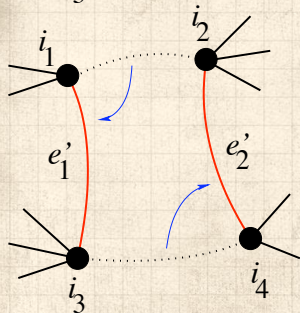
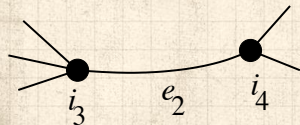
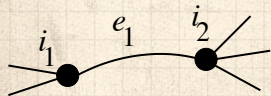


 **Being careful:** we can't change the degree of any node, so we can't simply move links around.

 **Simplest solution:** randomly rewire **two edges** at a time.



General random rewiring algorithm



Randomly choose **two edges**.
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.



Rewire one end of each edge.



Node degrees **do not change**.



Works if e_1 is a self-loop or repeated edge.



Same as finding on/off/on/off 4-cycles.
and rotating them.



Sampling random networks

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Phase 2:

 Use rewiring algorithm to remove all self and repeat loops.

Phase 3:

 **Randomize network** wiring by applying rewiring algorithm liberally.

 **Rule of thumb:** # Rewirings $\simeq 10 \times$ # edges ^[5].



Random sampling

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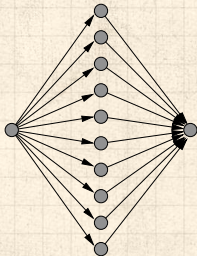


Problem with only joining up stubs is **failure** to randomly sample from all possible networks.



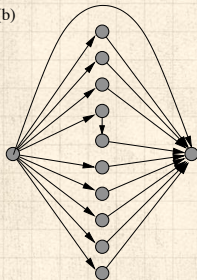
Example from Milo et al. (2003) [5]:

(a)



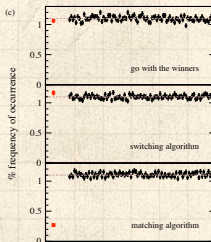
1 configuration

(b)



90 configurations

(c)



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What if we have P_k instead of N_k ?



Must now create nodes before start of the construction algorithm.



Generate N nodes by sampling from degree distribution P_k .



Easy to do exactly numerically since k is discrete.



Note: not all P_k will always give nodes that can be wired together.



Network motifs

- ❏ Idea of **motifs**^[8] introduced by Shen-Orr, Alon et al. in 2002.
- ❏ Looked at gene expression within full context of **transcriptional regulation networks**.
- ❏ Specific example of Escherichia coli.
- ❏ Directed network with 577 interactions (edges) and 424 operons (nodes).
- ❏ Used network randomization to produce ensemble of alternate networks with same degree frequency N_k .
- ❏ Looked for **certain subnetworks (motifs)** that appeared more or less often than expected

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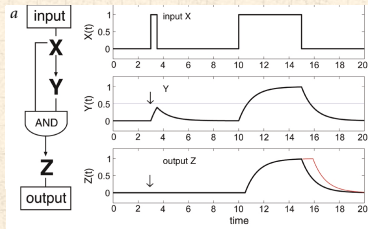
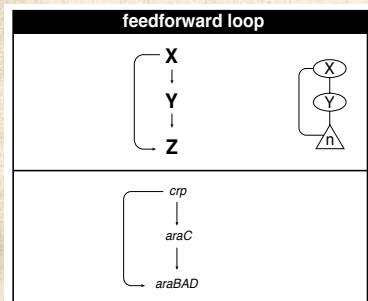
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 Z only turns on in response to sustained activity in X .

 Turning off X rapidly turns off Z .

 Analogy to elevator doors.



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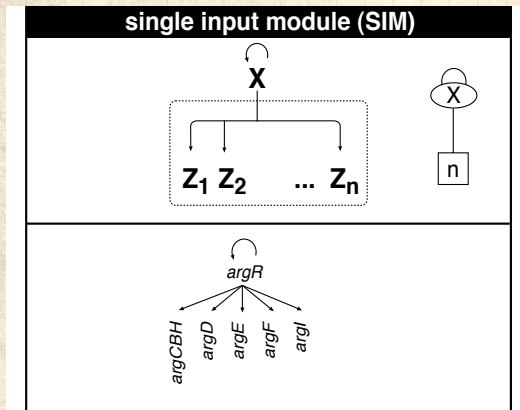
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Master switch.



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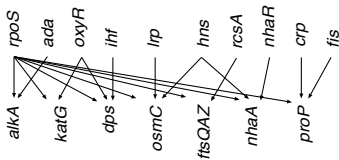
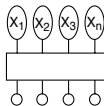
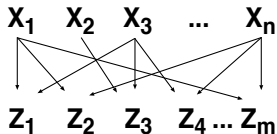
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dense overlapping regulons (DOR)



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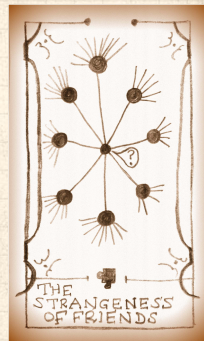
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 Note: selection of motifs to test is reasonable but nevertheless ad-hoc.

 For more, see work carried out by Wiggins *et al.* at Columbia.





The edge-degree distribution:

 The degree distribution P_k is fundamental for our description of many complex networks

 Again: P_k is the degree of **randomly chosen node**.

 A second very important distribution arises from **choosing randomly on edges** rather than on nodes.

 Define Q_k to be the probability the node at a **random end** of a **randomly chosen edge** has degree k .

 Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto kP_k$$

 Normalized form:

$$Q_k = \frac{kP_k}{\sum_{k'=0}^{\infty} k'P_{k'}} = \frac{kP_k}{\langle k \rangle}.$$

 **Big deal:** Rich-get-richer mechanism is built into this selection process.



The edge-degree distribution:



For networks, Q_k is also the probability that a friend (neighbor) of a random node has k friends.



Useful variant on Q_k :

R_k = probability that a friend of a random node has k other friends.



$$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0} (k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$



Equivalent to friend having degree $k+1$.



Natural question: what's the expected number of other friends that one friend has?



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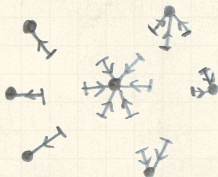
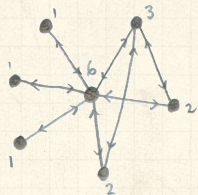
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Probability of randomly selecting a node of degree k by choosing from nodes:

$$P_1 = 3/7, P_2 = 2/7, P_3 = 1/7, \\ P_6 = 1/7.$$



Probability of landing on a node of degree k after randomly selecting an edge and then randomly choosing one direction to travel:

$$Q_1 = 3/16, Q_2 = 4/16, \\ Q_3 = 3/16, Q_6 = 6/16.$$



Probability of finding # outgoing edges = k after randomly selecting an edge and then randomly choosing one direction to travel:

$$R_0 = 3/16, R_1 = 4/16, \\ R_2 = 3/16, R_5 = 6/16.$$



The edge-degree distribution:



Given R_k is the probability that a friend has k other friends, then the average number of **friends' other friends** is

$$\langle k \rangle_R = \sum_{k=0}^{\infty} k R_k = \sum_{k=0}^{\infty} k \frac{(k+1) P_{k+1}}{\langle k \rangle}$$

$$= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1) P_{k+1}$$

$$= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} ((k+1)^2 - (k+1)) P_{k+1}$$

(where we have sneakily matched up indices)

$$= \frac{1}{\langle k \rangle} \sum_{j=0}^{\infty} (j^2 - j) P_j \quad (\text{using } j = k+1)$$

$$= \frac{1}{\langle k \rangle} (\langle k(k-1) \rangle)$$



The edge-degree distribution:

 Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k(k-1) \rangle)$, is true for **all** random networks, **independent of degree distribution**.

 For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

 Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k \rangle^2 + \langle k \rangle - \langle k \rangle) = \langle k \rangle$$

 Again, neatness of results is a special property of the Poisson distribution.

 So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...



The edge-degree distribution:



In fact, R_k is rather special for pure random networks ...



Substituting

$$P_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

into

$$R_k = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

we have

$$R_k = \frac{(k+1)}{\langle k \rangle} \frac{\langle k \rangle^{(k+1)}}{(k+1)!} e^{-\langle k \rangle} = \frac{\cancel{(k+1)}}{\cancel{\langle k \rangle}} \frac{\langle k \rangle^{(k+1)}}{(k+1)k!} e^{-\langle k \rangle}$$

$$= \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \equiv P_k.$$



#samesies.

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Two reasons why this matters

Reason #1:

 Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

 Key: Average depends on the **1st and 2nd moments** of P_k and **not just the 1st moment**.

 Three peculiarities:

1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
2. If P_k has a **large second moment**, then $\langle k_2 \rangle$ will be big.
(e.g., in the case of a power-law distribution)
3. Your friends really are different from you... ^[4, 6]
4. See also: class size paradoxes (nod to: Gelman)

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Two reasons why this matters

More on peculiarity #3:

 A node's average # of friends: $\langle k \rangle$

 Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$

 Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2} \right) \geq \langle k \rangle$$

 So only if everyone has the same degree (variance= $\sigma^2 = 0$) can a node be the same as its friends.

 Intuition: For networks, the more connected a node, the more likely it is to be chosen as a friend.



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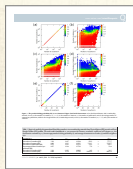
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“Generalized friendship paradox in complex networks: The case of scientific collaboration” 

Eom and Jo,

Nature Scientific Reports, **4**, 4603, 2014. ^[3]

Your friends really are ~~monsters~~ #winners:¹



Go on, hurt me: Friends have more coauthors, citations, and publications.



Other horrific studies: your connections on Twitter have more followers than you, are happier than you ^[1], more sexual partners than you, ...



The hope: Maybe they have more enemies and diseases too.



Research possibility: The Frenemy Paradox.



¹Some press [here](#)  [MIT Tech Review].

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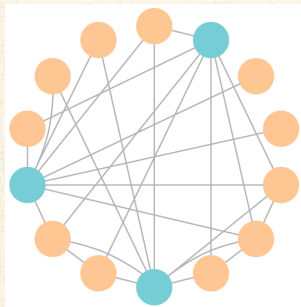
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Related disappointment:



Nodes see their friends' color choices.



Which color is more popular?¹



Again: thinking in edge space changes everything.

¹[https:](https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/)

[//www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/](https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/)



Two reasons why this matters

(Big) Reason #2:

 $\langle k \rangle_R$ is **key** to understanding how well random networks are connected together.

 e.g., we'd like to know what's the size of the largest component within a network.

 As $N \rightarrow \infty$, does our network have a **giant component**?

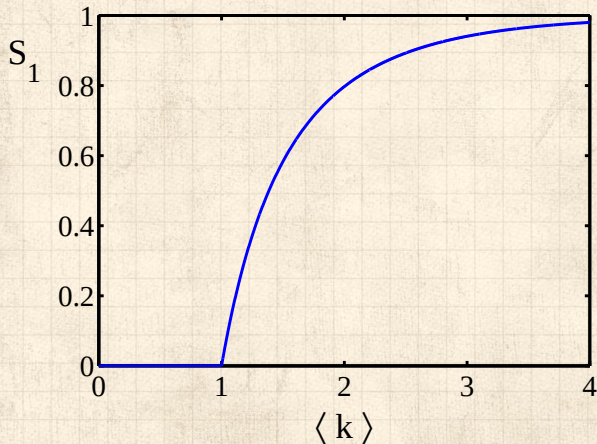
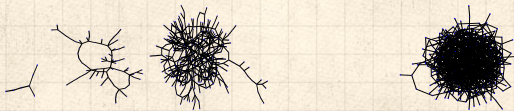
 **Defn:** Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.

 **Defn:** Giant component = component that comprises a non-zero fraction of a network as $N \rightarrow \infty$.

 Note: Component = Cluster



Giant component



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Structure of random networks

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Giant component:

 A giant component exists if when we follow a random edge, we are likely to hit a node with **at least 1** other outgoing edge.

 Equivalently, expect exponential growth in node number as we move out from a random node.

 All of this is the same as requiring $\langle k \rangle_R > 1$.

 **Giant component condition** (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

 Again, see that the second moment is an essential part of the story.

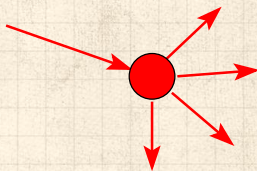
 Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$



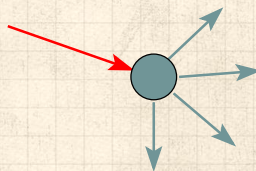
Spreading on Random Networks

For random networks, we know local structure is pure branching.

Successful spreading is $\therefore$ contingent on **single edges** infecting nodes.
Success



Failure:



Focus on **binary** case with edges and nodes either infected or not.

First big question: for a given network and contagion process, can global spreading from a single seed occur?



Global spreading condition



We need to find: [2]

R = the average # of infected edges that one random infected edge brings about.



Call **R** the **gain ratio**.



Define B_{k1} as the probability that a node of degree k is infected by a single infected edge.



$$\begin{aligned}
 \mathbf{R} = & \sum_{k=0}^{\infty} \underbrace{\frac{k P_k}{\langle k \rangle}}_{\text{prob. of connecting to a degree } k \text{ node}} \cdot \underbrace{(k-1)}_{\substack{\# \text{ outgoing} \\ \text{infected} \\ \text{edges}}} \cdot \underbrace{B_{k1}}_{\substack{\text{Prob. of} \\ \text{infection}}} \\
 & + \sum_{k=0}^{\infty} \underbrace{\frac{k P_k}{\langle k \rangle}}_{\text{prob. of connecting to a degree } k \text{ node}} \cdot \underbrace{0}_{\substack{\# \text{ outgoing} \\ \text{infected} \\ \text{edges}}} \cdot \underbrace{(1 - B_{k1})}_{\substack{\text{Prob. of} \\ \text{no infection}}}
 \end{aligned}$$



Global spreading condition

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 Our global spreading condition is then:

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet B_{k1} > 1.$$

 **Case 1–Rampant spreading:** If $B_{k1} = 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) = \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

 **Good:** This is just our giant component condition again.



Global spreading condition



Case 2—Simple disease-like: If $B_{k1} = \beta < 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet \beta > 1.$$



A fraction $(1-\beta)$ of edges do not transmit infection.



Analogous phase transition to giant component case but **critical value** of $\langle k \rangle$ is **increased**.



Aka bond percolation ↗.



Resulting degree distribution $\tilde{P}_k$:

$$\tilde{P}_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$



Giant component for standard random networks:

Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.

Determine condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.

When $\langle k \rangle < 1$, all components are finite.

Fine example of a continuous phase transition .

We say $\langle k \rangle = 1$ marks the critical point of the system.



Random networks with skewed P_k :

🧱 e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \geq 1$, then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=1}^{\infty} x^{2-\gamma} dx$$

$$\propto x^{3-\gamma} \Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

🧱 So giant component **always exists** for these kinds of networks.

🧱 Cutoff scaling is k^{-3} : if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.

🧱 How about $P_k = \delta_{kk_0}$?



Giant component

And how big is the largest component?

- Define S_1 as the **size of the largest component**.
- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let's find S_1 with a back-of-the-envelope argument.
- Define δ as the probability that a randomly chosen node **does not** belong to the largest component.
- Simple connection: $\delta = 1 - S_1$.
- Dirty trick:** If a randomly chosen node is not part of the largest component, then none of its neighbors are.
- So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

- Substitute in Poisson distribution...



Giant component



Carrying on:

$$\begin{aligned}\delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.\end{aligned}$$



Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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We can figure out some limits and details for

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$



First, we can write $\langle k \rangle$ in terms of S_1 :

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$



As $\langle k \rangle \rightarrow 0$, $S_1 \rightarrow 0$.



As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$.



Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.



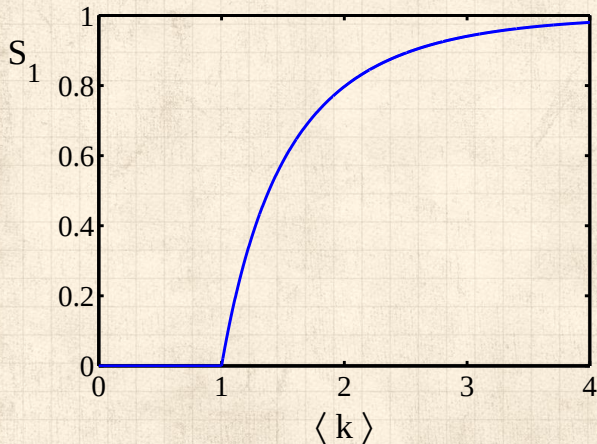
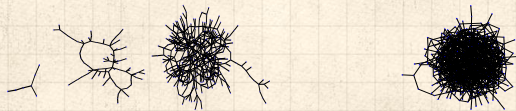
Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.



Really a transcritical bifurcation. ^[9]



Giant component



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Turns out we were lucky...

- Our dirty trick **only works for** ER random networks.
- The problem:** We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
- We need a separate probability δ' for the chance that an edge **leads to** the giant (infinite) component.
- We can sort many things out with **sensible probabilistic arguments...**
- More detailed investigations will profit from a spot of **Generatingfunctionology**.^[10]
- CocoNuTs: We figure out the final size and complete dynamics.



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