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# Random Networks

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Principles of Complex Systems, Vols. 1, 2, & 3D  
CSYS/MATH 6701, 6713, & a pretend number, 2025–2026

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Center  
Santa Fe Institute | University of Vermont



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## Sealie & Lambie Productions

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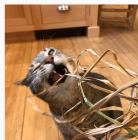
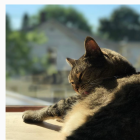
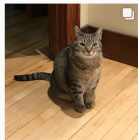
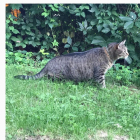
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## Special Guest Executive Producer



 On Instagram at [pratchett\\_the\\_cat](https://www.instagram.com/pratchett_the_cat) 

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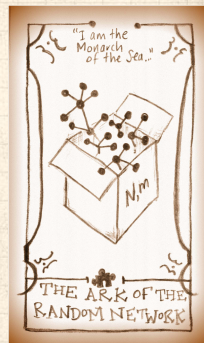
## References

## Some important models:

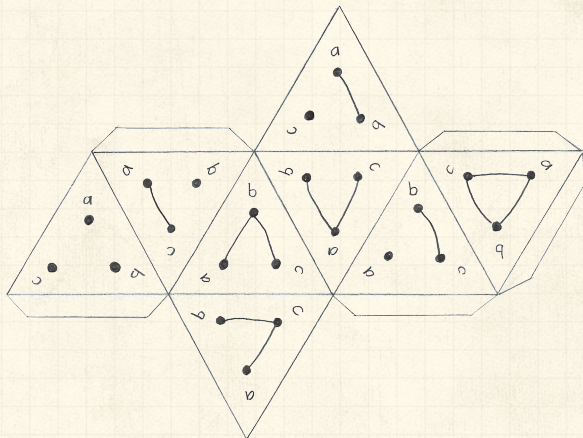
1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models ( $p^*$ ).







## Random network generator for $N = 3$ :



Get your own exciting generator [here](#) .



As  $N \nearrow$ , polyhedral die rapidly becomes a ball...

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# Random networks

## Pure, abstract random networks:



Consider set of all networks with  $N$  labelled nodes and  $m$  edges.

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# Random networks

## Pure, abstract random networks:

 Consider set of all networks with  $N$  labelled nodes and  $m$  edges.

 Standard random network =  
one **randomly chosen** network from this set.

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- Consider set of all networks with  $N$  labelled nodes and  $m$  edges.
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- To be clear: each network is **equally** probable.

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# Random networks

## Pure, abstract random networks:

- Consider set of all networks with  $N$  labelled nodes and  $m$  edges.
- Standard random network = one **randomly chosen** network from this set.
- To be clear: each network is **equally** probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or **ER** graphs.

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## Random networks—basic features:



Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

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Limit of  $m = \binom{N}{2}$ : complete or fully-connected graph.

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 Limit of  $m = \binom{N}{2}$ : complete or fully-connected graph.

 Number of possible networks with  $N$  labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N(N-1)}.$$

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 Crazy factorial explosion for  $1 \ll m \ll \binom{N}{2}$ .

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 Given  $m$  edges, there are  $\binom{\binom{N}{2}}{m}$  different possible networks.

 Crazy factorial explosion for  $1 \ll m \ll \binom{N}{2}$ .

 **Real world:** links are usually costly so real networks are almost always **sparse**.

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# Random networks

## How to build standard random networks:

 Given  $N$  and  $m$ .

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# Random networks

## How to build standard random networks:



Given  $N$  and  $m$ .



Two probabilistic methods

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# Random networks

## How to build standard random networks:



Given  $N$  and  $m$ .



Two probabilistic methods (we'll see a third later on)

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## How to build standard random networks:



Given  $N$  and  $m$ .



Two probabilistic methods (we'll see a third later on)

1. Connect each of the  $\binom{N}{2}$  pairs with appropriate probability  $p$ .



## How to build standard random networks:



Given  $N$  and  $m$ .



Two probabilistic methods (we'll see a third later on)

1. Connect each of the  $\binom{N}{2}$  pairs with appropriate probability  $p$ .
2. Take  $N$  nodes and add exactly  $m$  links by selecting edges without replacement.



## How to build standard random networks:



Given  $N$  and  $m$ .



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Useful for theoretical work.

2. Take  $N$  nodes and add exactly  $m$  links by selecting edges without replacement.



## How to build standard random networks:



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**Algorithm:** Randomly choose a pair of nodes  $i$  and  $j$ ,  $i \neq j$ , and connect if unconnected; repeat until all  $m$  edges are allocated.





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Given  $N$  and  $m$ .



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Best for adding relatively small numbers of links (most cases).



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Given  $N$  and  $m$ .



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Useful for theoretical work.

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**Algorithm:** Randomly choose a pair of nodes  $i$  and  $j$ ,  $i \neq j$ , and connect if unconnected; repeat until all  $m$  edges are allocated.



Best for adding relatively small numbers of links (most cases).



1 and 2 are effectively equivalent for large  $N$ .



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A few more things:



For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2}$$

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# Random networks

A few more things:



For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

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For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$



So the expected or **average degree** is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

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$$= \frac{2}{N} p \frac{1}{2} N(N-1)$$

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So the expected or **average degree** is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \cancel{N} p \frac{1}{\cancel{2}} \cancel{N}(N-1)$$

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$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

 So the expected or **average degree** is

$$\begin{aligned} \langle k \rangle &= \frac{2 \langle m \rangle}{N} \\ &= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{2}{N} p \frac{1}{2} N(N-1) = p(N-1). \end{aligned}$$

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 So the expected or **average degree** is

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$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \cancel{\frac{2}{N}} p \frac{1}{\cancel{2}} \cancel{N} (N-1) = p(N-1).$$

 Which is what it should be...



# Random networks

## A few more things:

 For method 1, # links is probabilistic:

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 So the expected or **average degree** is

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 Which is what it should be...

 If we keep  $\langle k \rangle$  constant then  $p \propto 1/N \rightarrow 0$  as  $N \rightarrow \infty$ .



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Next slides:

Example realizations of random networks





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Next slides:

Example realizations of random networks

  $N = 500$



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Next slides:

Example realizations of random networks

  $N = 500$

 Vary  $m$ , the number of edges from 100 to 1000.



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## Next slides:

### Example realizations of random networks



$N = 500$



Vary  $m$ , the number of edges from 100 to 1000.



Average degree  $\langle k \rangle$  runs from 0.4 to 4.



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## Next slides:

### Example realizations of random networks



$N = 500$



Vary  $m$ , the number of edges from 100 to 1000.



Average degree  $\langle k \rangle$  runs from 0.4 to 4.



Look at full network plus the largest component.





# Random networks: examples for $N=500$

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$m = 100$   
 $\langle k \rangle = 0.4$



$m = 200$   
 $\langle k \rangle = 0.8$



$m = 230$   
 $\langle k \rangle = 0.92$



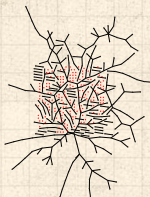
$m = 240$   
 $\langle k \rangle = 0.96$



$m = 250$   
 $\langle k \rangle = 1$



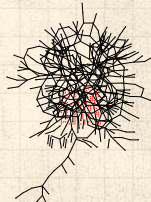
$m = 260$   
 $\langle k \rangle = 1.04$



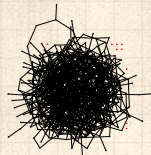
$m = 280$   
 $\langle k \rangle = 1.12$



$m = 300$   
 $\langle k \rangle = 1.2$



$m = 500$   
 $\langle k \rangle = 2$



$m = 1000$   
 $\langle k \rangle = 4$



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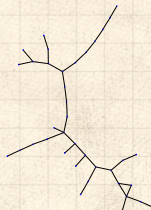
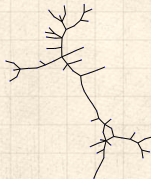
References



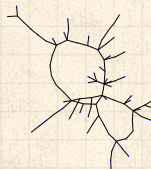
$m = 100$   
 $\langle k \rangle = 0.4$



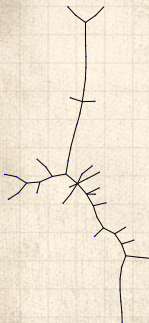
$m = 230$   
 $\langle k \rangle = 0.92$



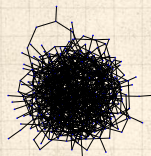
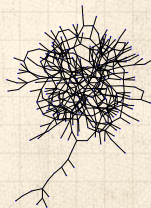
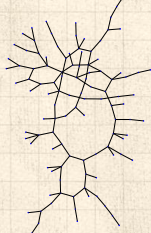
$m = 240$   
 $\langle k \rangle = 0.96$



$m = 250$   
 $\langle k \rangle = 1$



$m = 200$   
 $\langle k \rangle = 0.8$



$m = 1000$   
 $\langle k \rangle = 4$

$m = 300$

$m = 500$   
 $\langle k \rangle = 2$



# Random networks: examples for $N=500$

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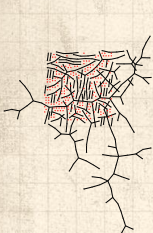
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$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



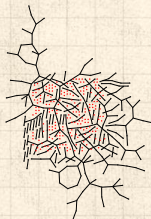
$m = 250$   
 $\langle k \rangle = 1$



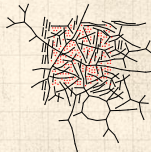
$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



# Random networks: largest components

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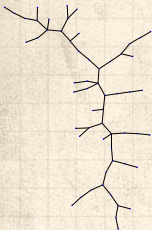
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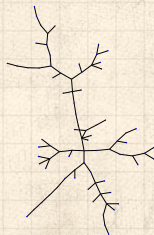
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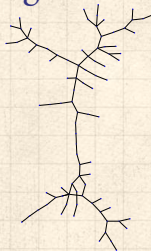
References



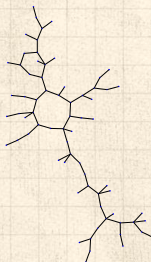
$m = 250$   
 $\langle k \rangle = 1$



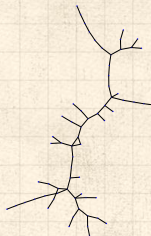
$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



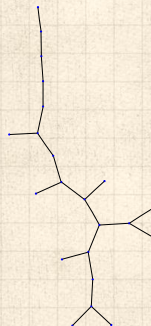
$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$

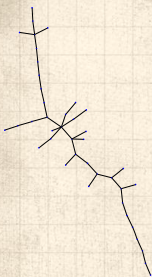


$m = 250$   
 $\langle k \rangle = 1$



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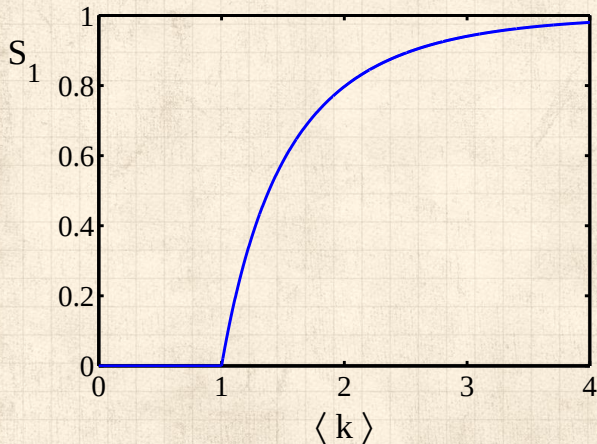
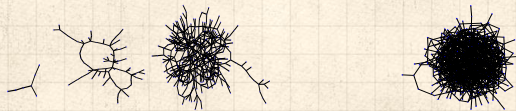
$m = 250$   
 $\langle k \rangle = 1$



$m = 250$   
 $\langle k \rangle = 1$



# Giant component



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# Clustering in random networks:



For construction method 1, what is the clustering coefficient for a finite network?

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# Clustering in random networks:

 For construction method 1, what is the clustering coefficient for a finite network?

 Consider triangle/triple clustering coefficient: [7]

$$C_2 = \frac{3 \times \text{\#triangles}}{\text{\#triples}}$$





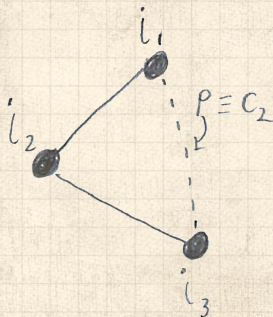
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Recall:  $C_2$  = probability that two friends of a node are also friends.



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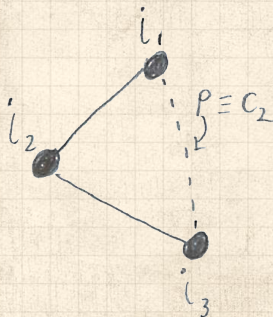
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Or:  $C_2$  = probability that a triple is part of a triangle.

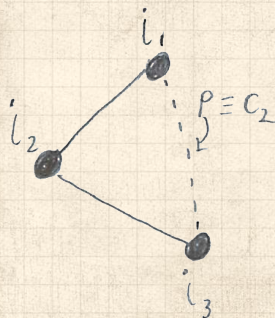


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Recall:  $C_2$  = probability that two friends of a node are also friends.

Or:  $C_2$  = probability that a triple is part of a triangle.

For standard random networks, we have simply that

$$C_2 = p.$$



# Clustering in random networks:



So for large random networks  
( $N \rightarrow \infty$ ), clustering drops to  
zero.

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# Clustering in random networks:



So for large random networks ( $N \rightarrow \infty$ ), clustering drops to zero.



Key structural feature of random networks is that they locally look like  
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# Clustering in random networks:



So for large random networks ( $N \rightarrow \infty$ ), clustering drops to zero.



Key structural feature of random networks is that they locally look like  
pure branching networks



No small loops.

## Pure random networks

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## Degree distribution:



Recall  $P_k$  = probability that a randomly selected node has degree  $k$ .

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## Degree distribution:

 Recall  $P_k$  = probability that a randomly selected node has degree  $k$ .

 Consider method 1 for constructing random networks: each possible link is realized with probability  $p$ .

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## Degree distribution:

-  Recall  $P_k$  = probability that a randomly selected node has degree  $k$ .
-  Consider method 1 for constructing random networks: each possible link is realized with probability  $p$ .
-  Now consider one node: there are ' $N - 1$  choose  $k$ ' ways the node can be connected to  $k$  of the other  $N - 1$  nodes.

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- Each connection occurs with probability  $p$ , each non-connection with probability  $(1 - p)$ .
- Therefore have a binomial distribution ↗:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$





## Limiting form of $P(k; p, N)$ :

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## Limiting form of $P(k; p, N)$ :



Our degree distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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What happens as  $N \rightarrow \infty$ ?

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So examine limit of  $P(k; p, N)$  when  $p \rightarrow 0$  and  $N \rightarrow \infty$  with  $\langle k \rangle = p(N-1) = \text{constant}$ .

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$



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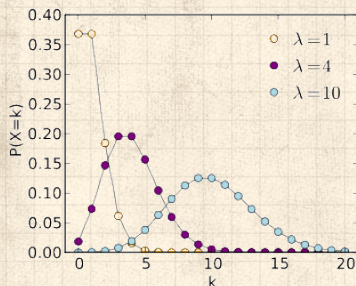
This is a Poisson distribution  with mean  $\langle k \rangle$ .





# Poisson basics:

$$P(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$



$\lambda > 0$



$k = 0, 1, 2, 3, \dots$



Classic use: probability that an event occurs  $k$  times in a given time period, given an average rate of occurrence.



e.g.:  
phone calls/minute,  
horse-kick deaths.



'Law of small numbers'





# Poisson basics:



Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

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# Poisson basics:



Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$



Checking:

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

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Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$



Checking:

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle} = 1$$

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# Poisson basics:



Mean degree: we must have

$$\langle k \rangle = \sum_{k=0}^{\infty} k P(k; \langle k \rangle).$$

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# Poisson basics:



Mean degree: we must have

$$\langle k \rangle = \sum_{k=0}^{\infty} k P(k; \langle k \rangle).$$



Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} k P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} k \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^k}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^i}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle} \end{aligned}$$

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In CocoNuTs, we find a different, crazier way of doing this...

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# Poisson basics:



The **variance** of degree distributions for random networks turns out to be **very important**.

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# Poisson basics:



The **variance** of degree distributions for random networks turns out to be **very important**.



Using calculation similar to one for finding  $\langle k \rangle$  we find the **second moment** to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

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 Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$$



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$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

 So standard deviation  $\sigma$  is equal to  $\sqrt{\langle k \rangle}$ .





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 The **variance** of degree distributions for random networks turns out to be **very important**.

 Using calculation similar to one for finding  $\langle k \rangle$  we find the **second moment** to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

 Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

 So standard deviation  $\sigma$  is equal to  $\sqrt{\langle k \rangle}$ .

 Note: This is a special property of Poisson distribution and can trip us up...



# Neural reboot (NR):

Unrelated: Feline elevation

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# General random networks



So... standard random networks have a Poisson degree distribution

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# General random networks



So... standard random networks have a Poisson degree distribution



Generalize to arbitrary degree distribution  $P_k$ .

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# General random networks



So... standard random networks have a Poisson degree distribution



Generalize to arbitrary degree distribution  $P_k$ .



Also known as the **configuration model**.<sup>[7]</sup>

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# General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution  $P_k$ .
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- Can generalize construction method from ER random networks.

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# General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution  $P_k$ .
- Also known as the **configuration model**.<sup>[7]</sup>
- Can generalize construction method from ER random networks.
- Assign each node a weight  $w$  from some distribution  $P_w$  and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

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- But we'll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.

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# General random networks

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$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
  2. Examining mechanisms that lead to networks with certain degree distributions.



# Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

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# Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

  $N = 1000$ .

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# Random networks: examples

## Coming up:

Example realizations of random networks with power law degree distributions:

  $N = 1000$ .

  $P_k \propto k^{-\gamma}$  for  $k \geq 1$ .

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# Random networks: examples

## Coming up:

Example realizations of random networks with power law degree distributions:

-   $N = 1000$ .
-   $P_k \propto k^{-\gamma}$  for  $k \geq 1$ .
-  Set  $P_0 = 0$  (no isolated nodes).

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# Random networks: examples

## Coming up:

Example realizations of random networks with power law degree distributions:

-   $N = 1000$ .
-   $P_k \propto k^{-\gamma}$  for  $k \geq 1$ .
-  Set  $P_0 = 0$  (no isolated nodes).
-  Vary exponent  $\gamma$  between 2.10 and 2.91.

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-  Set  $P_0 = 0$  (no isolated nodes).
-  Vary exponent  $\gamma$  between 2.10 and 2.91.
-  Again, look at full network plus the largest component.

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# Random networks: examples

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-   $N = 1000$ .
-   $P_k \propto k^{-\gamma}$  for  $k \geq 1$ .
-  Set  $P_0 = 0$  (no isolated nodes).
-  Vary exponent  $\gamma$  between 2.10 and 2.91.
-  Again, look at full network plus the largest component.
-  Apart from degree distribution, wiring is random.

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# Random networks: examples for $N=1000$

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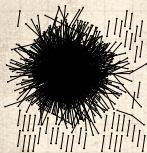
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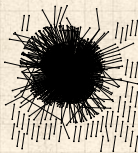
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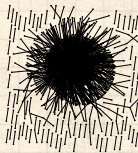
$\gamma = 2.1$   
 $\langle k \rangle = 3.448$



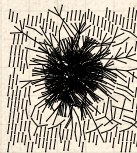
$\gamma = 2.19$   
 $\langle k \rangle = 2.986$



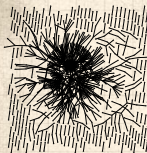
$\gamma = 2.28$   
 $\langle k \rangle = 2.306$



$\gamma = 2.37$   
 $\langle k \rangle = 2.504$



$\gamma = 2.46$   
 $\langle k \rangle = 1.856$



$\gamma = 2.55$   
 $\langle k \rangle = 1.712$



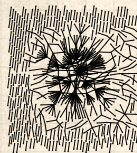
$\gamma = 2.64$   
 $\langle k \rangle = 1.6$



$\gamma = 2.73$   
 $\langle k \rangle = 1.862$



$\gamma = 2.82$   
 $\langle k \rangle = 1.386$



$\gamma = 2.91$   
 $\langle k \rangle = 1.49$





# Random networks: largest components

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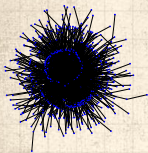
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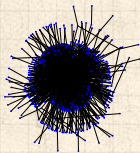
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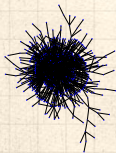
References



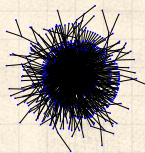
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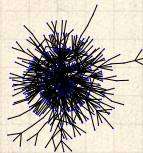
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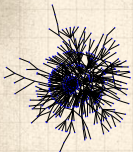
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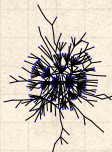
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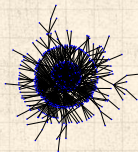
$\gamma = 2.46$   
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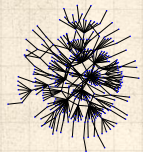
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## Generalized random networks:



Arbitrary degree distribution  $P_k$ .



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### Generalized random networks:

 Arbitrary degree distribution  $P_k$ .

 Create (unconnected) nodes with degrees sampled from  $P_k$ .



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### Generalized random networks:



Arbitrary degree distribution  $P_k$ .



Create (unconnected) nodes with degrees sampled from  $P_k$ .



Wire nodes together randomly.





## Generalized random networks:

-  Arbitrary degree distribution  $P_k$ .
-  Create (unconnected) nodes with degrees sampled from  $P_k$ .
-  Wire nodes together randomly.
-  Create ensemble to test deviations from randomness.

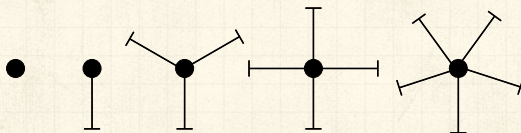


# Building random networks: Stubs

## Phase 1:



**Idea:** start with a soup of unconnected nodes with **stubs** (half-edges):



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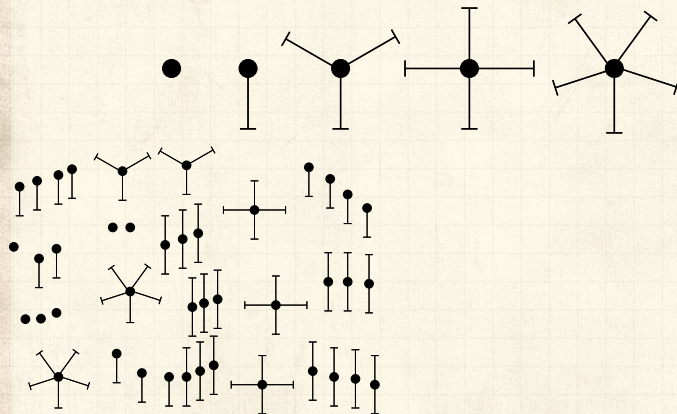


# Building random networks: Stubs

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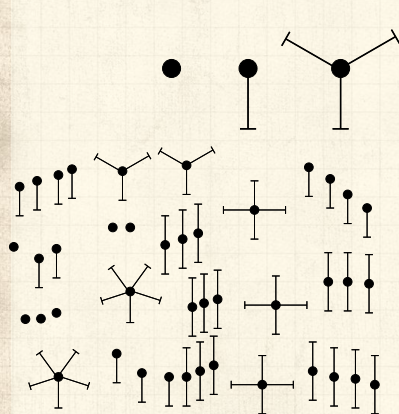


# Building random networks: Stubs

## Phase 1:



**Idea:** start with a soup of unconnected nodes with **stubs** (half-edges):



Randomly select stubs (not nodes!) and connect them.

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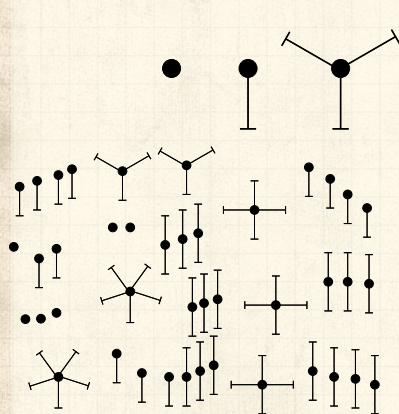


# Building random networks: Stubs

## Phase 1:



**Idea:** start with a soup of unconnected nodes with **stubs** (half-edges):



Randomly select stubs (not nodes!) and connect them.



Must have an even number of stubs.

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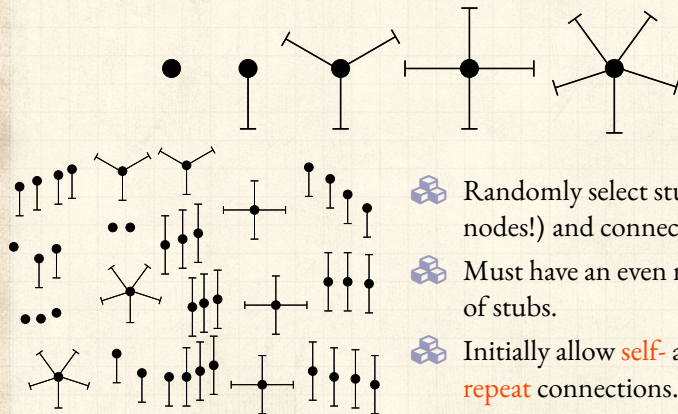


# Building random networks: Stubs

## Phase 1:



**Idea:** start with a soup of unconnected nodes with **stubs** (half-edges):



Randomly select stubs (not nodes!) and connect them.



Must have an even number of stubs.



Initially allow **self**- and **repeat** connections.

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# Building random networks: First rewiring

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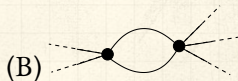
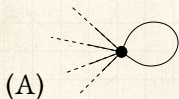
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## Phase 2:



Now find any (A) self-loops and (B) repeat edges and  
**randomly rewire** them.



# Building random networks: First rewiring

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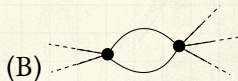
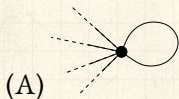
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## Phase 2:



Now find any (A) self-loops and (B) repeat edges and **randomly rewire** them.



**Being careful:** we can't change the degree of any node, so we can't simply move links around.



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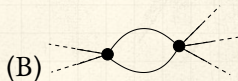
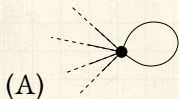
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## Phase 2:

 Now find any (A) self-loops and (B) repeat edges and **randomly rewire** them.

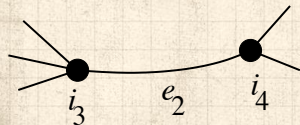
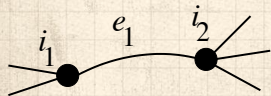


 **Being careful:** we can't change the degree of any node, so we can't simply move links around.

 **Simplest solution:** randomly rewire **two edges** at a time.



# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)

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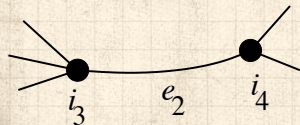
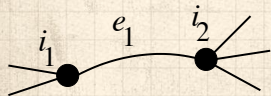
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# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.

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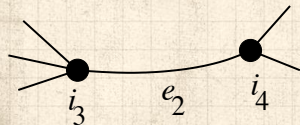
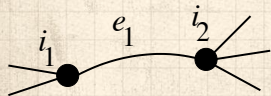
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# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.



Rewire one end of each edge.

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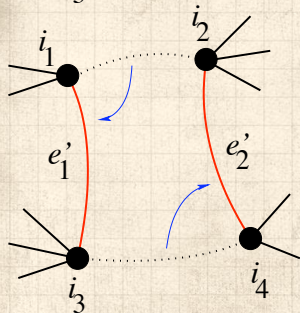
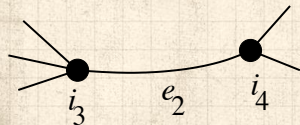
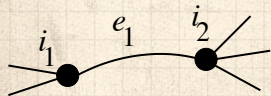
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# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.



Rewire one end of each edge.



Node degrees **do not change**.

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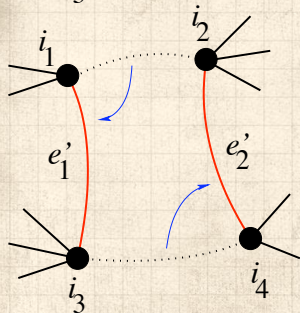
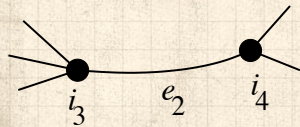
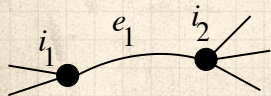
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# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.



Rewire one end of each edge.



Node degrees **do not change**.



Works if  $e_1$  is a self-loop or repeated edge.

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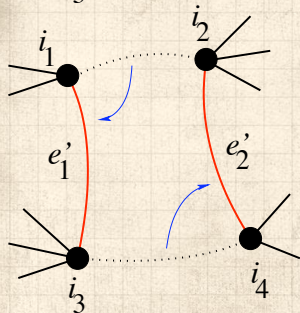
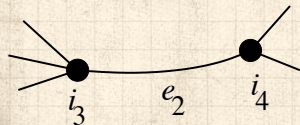
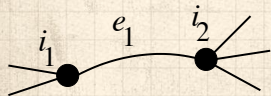
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# General random rewiring algorithm



Randomly choose **two edges**.  
(Or choose problem edge and a random edge)



Check to make sure edges are **disjoint**.



Rewire one end of each edge.



Node degrees **do not change**.



Works if  $e_1$  is a self-loop or repeated edge.



Same as finding on/off/on/off 4-cycles.  
and rotating them.





# Sampling random networks

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Phase 2:



Use rewiring algorithm to remove all self and repeat loops.



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Phase 2:



Use rewiring algorithm to remove all self and repeat loops.

Phase 3:



**Randomize network** wiring by applying rewiring algorithm liberally.



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## Phase 2:

 Use rewiring algorithm to remove all self and repeat loops.

## Phase 3:

 **Randomize network** wiring by applying rewiring algorithm liberally.

 **Rule of thumb:** # Rewirings  $\simeq 10 \times$  # edges <sup>[5]</sup>.



# Random sampling



Problem with only joining up stubs is **failure** to randomly sample from all possible networks.

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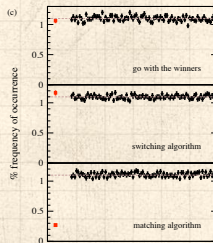
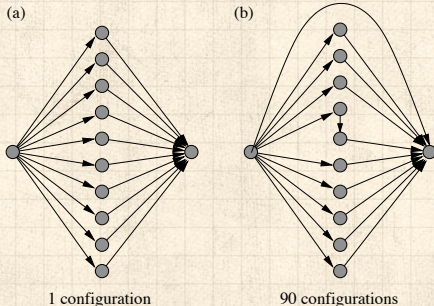
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Problem with only joining up stubs is **failure** to randomly sample from all possible networks.



Example from Milo et al. (2003) [5]:





# Sampling random networks



What if we have  $P_k$  instead of  $N_k$ ?

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What if we have  $P_k$  instead of  $N_k$ ?



Must now create nodes before start of the construction algorithm.



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What if we have  $P_k$  instead of  $N_k$ ?



Must now create nodes before start of the construction algorithm.



Generate  $N$  nodes by sampling from degree distribution  $P_k$ .



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What if we have  $P_k$  instead of  $N_k$ ?



Must now create nodes before start of the construction algorithm.



Generate  $N$  nodes by sampling from degree distribution  $P_k$ .



Easy to do exactly numerically since  $k$  is discrete.



# Sampling random networks

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What if we have  $P_k$  instead of  $N_k$ ?



Must now create nodes before start of the construction algorithm.



Generate  $N$  nodes by sampling from degree distribution  $P_k$ .



Easy to do exactly numerically since  $k$  is discrete.



**Note:** not all  $P_k$  will always give nodes that can be wired together.





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# Network motifs

 Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.

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# Network motifs

 Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.

 Looked at gene expression within full context of transcriptional regulation networks.

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# Network motifs

 Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.

 Looked at gene expression within full context of transcriptional regulation networks.

 Specific example of Escherichia coli.

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# Network motifs

-  Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.
-  Looked at gene expression within full context of transcriptional regulation networks.
-  Specific example of Escherichia coli.
-  Directed network with 577 interactions (edges) and 424 operons (nodes).

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# Network motifs

- ❏ Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.
- ❏ Looked at gene expression within full context of transcriptional regulation networks.
- ❏ Specific example of Escherichia coli.
- ❏ Directed network with 577 interactions (edges) and 424 operons (nodes).
- ❏ Used network randomization to produce ensemble of alternate networks with same degree frequency  $N_k$ .

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-  Idea of **motifs**<sup>[8]</sup> introduced by Shen-Orr, Alon et al. in 2002.
-  Looked at gene expression within full context of **transcriptional regulation networks**.
-  Specific example of Escherichia coli.
-  Directed network with 577 interactions (edges) and 424 operons (nodes).
-  Used network randomization to produce ensemble of alternate networks with same degree frequency  $N_k$ .
-  Looked for **certain subnetworks (motifs)** that appeared more or less often than expected



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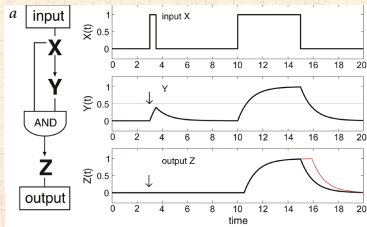
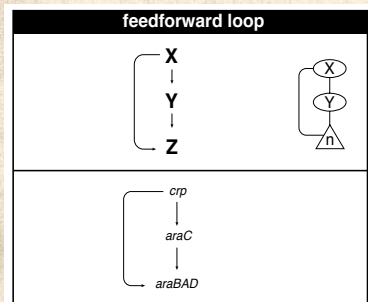
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$Z$  only turns on in response to sustained activity in  $X$ .



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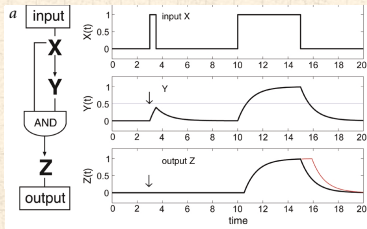
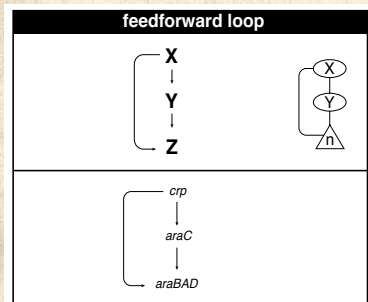
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  $Z$  only turns on in response to sustained activity in  $X$ .

 Turning off  $X$  rapidly turns off  $Z$ .



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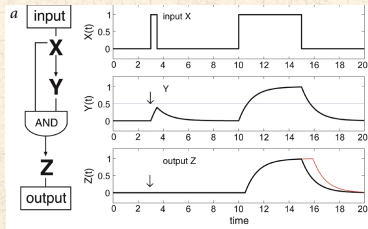
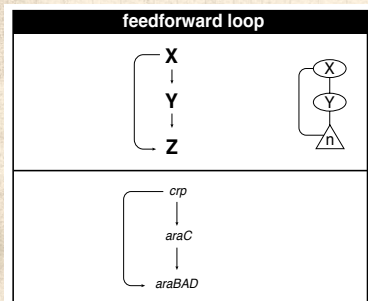
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  $Z$  only turns on in response to sustained activity in  $X$ .

 Turning off  $X$  rapidly turns off  $Z$ .

 Analogy to elevator doors.





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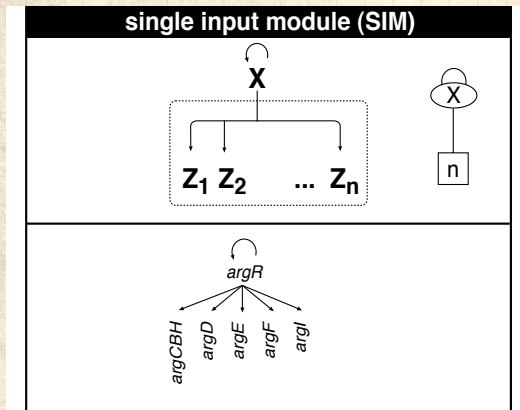
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Master switch.



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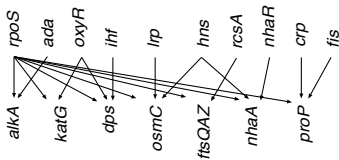
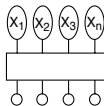
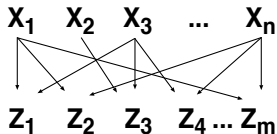
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## dense overlapping regulons (DOR)



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 Note: selection of motifs to test is reasonable but nevertheless ad-hoc.



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 Note: selection of motifs to test is reasonable but nevertheless ad-hoc.

 For more, see work carried out by Wiggins *et al.* at Columbia.



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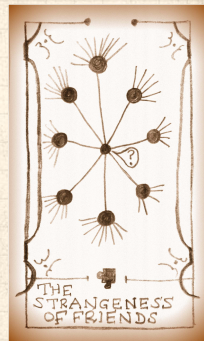
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# The edge-degree distribution:



The degree distribution  $P_k$  is fundamental for our description of many complex networks

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# The edge-degree distribution:



The degree distribution  $P_k$  is fundamental for our description of many complex networks



Again:  $P_k$  is the degree of **randomly chosen node**.

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 **Big deal:** Rich-get-richer mechanism is built into this selection process.

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# The edge-degree distribution:



For networks,  $Q_k$  is also the probability that a friend (neighbor) of a random node has  $k$  friends.

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For networks,  $Q_k$  is also the probability that a friend (neighbor) of a random node has  **$k$  friends**.



Useful variant on  $Q_k$ :

$R_k$  = probability that a friend of a random node has  **$k$  other friends**.



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Equivalent to friend having degree  $k+1$ .



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Equivalent to friend having degree  $k+1$ .



**Natural question:** what's the expected number of other friends that one friend has?





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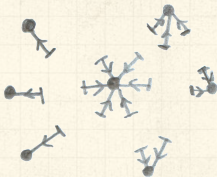
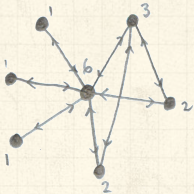
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Probability of randomly selecting a node of degree  $k$  by choosing from nodes:

$$P_1 = 3/7, P_2 = 2/7, P_3 = 1/7, \\ P_6 = 1/7.$$



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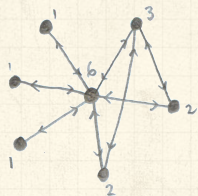
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$$Q_1 = 3/16, Q_2 = 4/16, \\ Q_3 = 3/16, Q_6 = 6/16.$$



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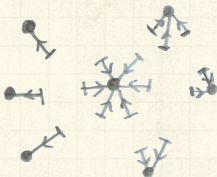
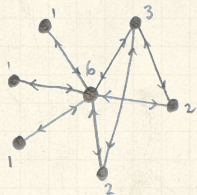
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Probability of finding # outgoing edges =  $k$  after randomly selecting an edge and then randomly choosing one direction to travel:

$$R_0 = 3/16, R_1 = 4/16, \\ R_2 = 3/16, R_5 = 6/16.$$



# The edge-degree distribution:



Given  $R_k$  is the probability that a friend has  $k$  other friends, then the average number of **friends' other friends** is

$$\langle k \rangle_R = \sum_{k=0}^{\infty} k R_k$$

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# The edge-degree distribution:



Note: our result,  $\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k(k-1) \rangle)$ , is true for **all** random networks, **independent of degree distribution**.

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For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$



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 Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k \rangle^2 + \langle k \rangle - \langle k \rangle)$$



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 Again, neatness of results is a special property of the Poisson distribution.

 So friends on average have  $\langle k \rangle$  other friends, and  $\langle k \rangle + 1$  total friends...





# The edge-degree distribution:



In fact,  $R_k$  is rather special for pure random networks ...

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Substituting

$$P_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

into

$$R_k = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

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# Two reasons why this matters

## Reason #1:

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 Average # friends of friends per node is

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 Key: Average depends on the **1st and 2nd moments** of  $P_k$  and **not just the 1st moment.**

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1. We might guess  $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$  but it's actually  $\langle k(k-1) \rangle$ .
2. If  $P_k$  has a **large second moment**, then  $\langle k_2 \rangle$  will be big.

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# Two reasons why this matters

## Reason #1:

 Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

 Key: Average depends on the **1st and 2nd moments** of  $P_k$  and **not just the 1st moment**.

 Three peculiarities:

1. We might guess  $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$  but it's actually  $\langle k(k-1) \rangle$ .
2. If  $P_k$  has a **large second moment**, then  $\langle k_2 \rangle$  will be big.  
(e.g., in the case of a power-law distribution)

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3. Your friends really are different from you... [4, 6]

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4. See also: class size paradoxes (nod to: Gelman)

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# Two reasons why this matters

## More on peculiarity #3:

 A node's average # of friends:  $\langle k \rangle$

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# Two reasons why this matters

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 A node's average # of friends:  $\langle k \rangle$

 Friend's average # of friends:  $\frac{\langle k^2 \rangle}{\langle k \rangle}$

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# Two reasons why this matters

## More on peculiarity #3:

 A node's average # of friends:  $\langle k \rangle$

 Friend's average # of friends:  $\frac{\langle k^2 \rangle}{\langle k \rangle}$

 Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2}$$

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$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2}$$



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 So only if everyone has the same degree (variance=  $\sigma^2 = 0$ )  
can a node be the same as its friends.





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 So only if everyone has the same degree (variance=  $\sigma^2 = 0$ ) can a node be the same as its friends.

 Intuition: For networks, the more connected a node, the more likely it is to be chosen as a friend.





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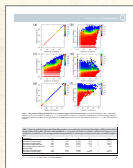
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“Generalized friendship paradox in complex  
networks: The case of scientific collaboration” 

Eom and Jo,

Nature Scientific Reports, **4**, 4603, 2014. <sup>[3]</sup>

Your friends really are ~~monsters~~ #winners:<sup>1</sup>



<sup>1</sup>Some press [here](#)  [MIT Tech Review].

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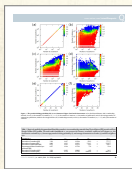
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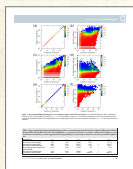
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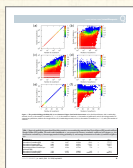
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**The hope:** Maybe they have more enemies and diseases too.



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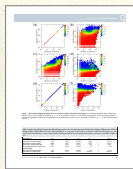
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 **Other horrific studies:** your connections on Twitter have more followers than you, are happier than you <sup>[1]</sup>, more sexual partners than you, ...

 **The hope:** Maybe they have more enemies and diseases too.

 Research possibility: The Frenemy Paradox.



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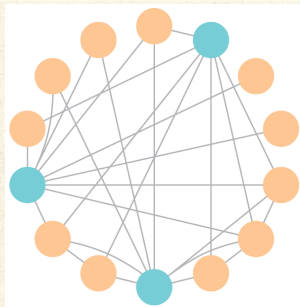
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## Related disappointment:



Nodes see their friends' color choices.



<sup>1</sup>[https:](https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/)

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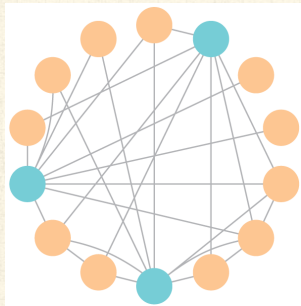
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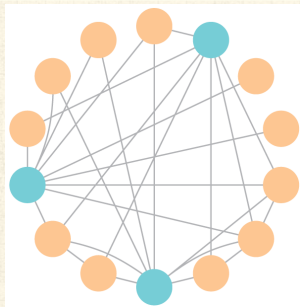
Which color is more popular?<sup>1</sup>

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## Related disappointment:



Nodes see their friends' color choices.



Which color is more popular?<sup>1</sup>



Again: thinking in edge space changes everything.

<sup>1</sup>[https:](https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/)

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# Two reasons why this matters

## (Big) Reason #2:

  $\langle k \rangle_R$  is **key** to understanding how well random networks are connected together.

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# Two reasons why this matters

## (Big) Reason #2:

  $\langle k \rangle_R$  is **key** to understanding how well random networks are connected together.

 e.g., we'd like to know what's the size of the largest component within a network.

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## (Big) Reason #2:

-   $\langle k \rangle_R$  is **key** to understanding how well random networks are connected together.
-  e.g., we'd like to know what's the size of the largest component within a network.
-  As  $N \rightarrow \infty$ , does our network have a **giant component**?

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 As  $N \rightarrow \infty$ , does our network have a **giant component**?

 **Defn:** Component = connected subnetwork of nodes such that  $\exists$  path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.

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 **Defn:** Giant component = component that comprises a non-zero fraction of a network as  $N \rightarrow \infty$ .



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 Note: Component = Cluster





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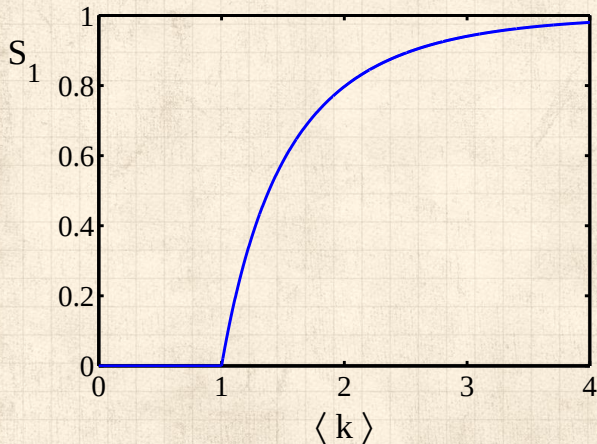
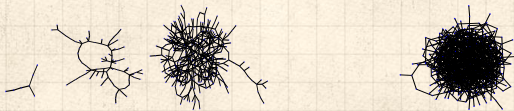
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# Giant component



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# Structure of random networks

## Giant component:



A giant component exists if when we follow a random edge, we are likely to hit a node with **at least 1** other outgoing edge.

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# Structure of random networks

## Giant component:

-  A giant component exists if when we follow a random edge, we are likely to hit a node with **at least 1** other outgoing edge.
-  Equivalently, expect exponential growth in node number as we move out from a random node.

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-  All of this is the same as requiring  $\langle k \rangle_R > 1$ .

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 **Giant component condition** (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$





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 Again, see that the second moment is an essential part of the story.



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 **Giant component condition** (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

 Again, see that the second moment is an essential part of the story.

 Equivalent statement:  $\langle k^2 \rangle > 2\langle k \rangle$



# Spreading on Random Networks



For random networks, we know local structure is pure branching.

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Successful spreading is  $\therefore$  contingent on **single edges** infecting nodes.

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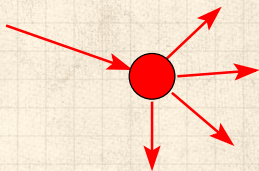
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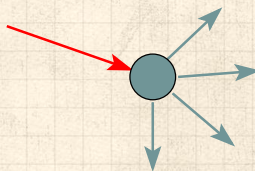


# Spreading on Random Networks

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Success



Failure:



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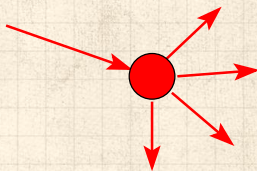
Random friends are strange

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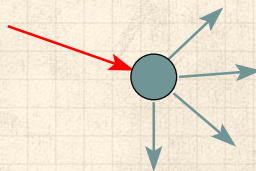
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For random networks, we know local structure is pure branching.

Successful spreading is  $\therefore$  contingent on **single edges** infecting nodes.  
Success



Failure:



Focus on **binary** case with edges and nodes either infected or not.



# Spreading on Random Networks

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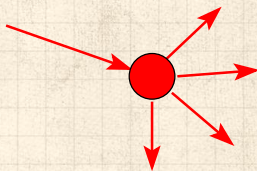
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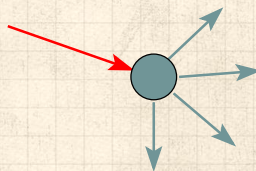
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Focus on **binary** case with edges and nodes either infected or not.

**First big question:** for a given network and contagion process, can global spreading from a single seed occur?



# Global spreading condition



We need to find: <sup>[2]</sup>

**R** = the average # of infected edges that one random infected edge brings about.



Call **R** the **gain ratio**.

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$$+ \sum_{k=0}^{\infty} \frac{\widehat{k P_k}}{\langle k \rangle}$$



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 \end{aligned}$$





# Global spreading condition



Our global spreading condition is then:

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{k P_k}{\langle k \rangle} \bullet (k - 1) \bullet B_{k1} > 1.$$

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Case 1—Rampant spreading:



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 **Good:** This is just our giant component condition again.





# Global spreading condition



## Case 2—Simple disease-like:

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# Global spreading condition



Case 2—Simple disease-like: If  $B_{k1} = \beta < 1$

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Resulting degree distribution  $\tilde{P}_k$ :

$$\tilde{P}_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$



## Giant component for standard random networks:

 Recall  $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$ .

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We say  $\langle k \rangle = 1$  marks the critical point of the system.





## Random networks with skewed $P_k$ :

 e.g, if  $P_k = ck^{-\gamma}$  with  $2 < \gamma < 3, k \geq 1$ , then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

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🧱 How about  $P_k = \delta_{kk_0}$ ?



# Giant component

And how big is the largest component?



Define  $S_1$  as the **size of the largest component**.

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Consider an infinite ER random network with average degree  $\langle k \rangle$ .

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- Substitute in Poisson distribution...

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# Giant component



Carrying on:

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

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# Giant component



Carrying on:

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k$$

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# Giant component



Carrying on:

$$\begin{aligned}\delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}\end{aligned}$$

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Carrying on:

$$\begin{aligned}\delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.\end{aligned}$$

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$$\begin{aligned}\delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.\end{aligned}$$



Now substitute in  $\delta = 1 - S_1$  and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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We can figure out some limits and details for

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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# Giant component



We can figure out some limits and details for

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$



First, we can write  $\langle k \rangle$  in terms of  $S_1$ :

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

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First, we can write  $\langle k \rangle$  in terms of  $S_1$ :

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As  $\langle k \rangle \rightarrow 0$ ,  $S_1 \rightarrow 0$ .





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Notice that at  $\langle k \rangle = 1$ , the critical point,  $S_1 = 0$ .



# Giant component



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Only solvable for  $S_1 > 0$  when  $\langle k \rangle > 1$ .



# Giant component



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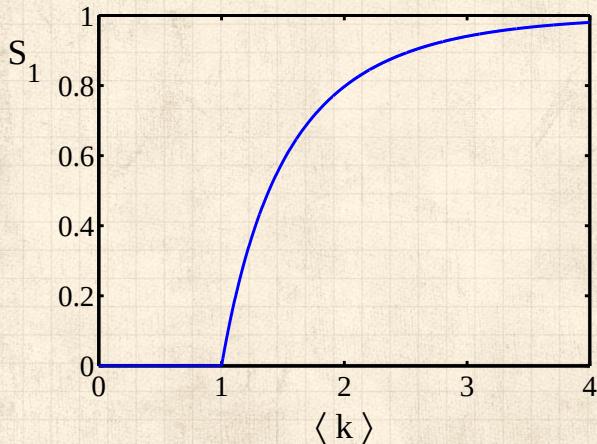
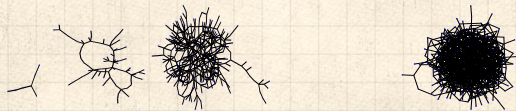
Only solvable for  $S_1 > 0$  when  $\langle k \rangle > 1$ .



Really a transcritical bifurcation. <sup>[9]</sup>



# Giant component



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Turns out we were lucky...



Our dirty trick **only works for** ER random networks.

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Turns out we were lucky...



Our dirty trick **only works for** ER random networks.



**The problem:** We assumed that neighbors have the same probability  $\delta$  of belonging to the largest component.



Turns out we were lucky...



Our dirty trick **only works for** ER random networks.



**The problem:** We assumed that neighbors have the same probability  $\delta$  of belonging to the largest component.



But we know our friends are different from us...



Turns out we were lucky...



Our dirty trick **only works for** ER random networks.



**The problem:** We assumed that neighbors have the same probability  $\delta$  of belonging to the largest component.



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Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .



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Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .



We need a separate probability  $\delta'$  for the chance that an edge **leads to** the giant (infinite) component.





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We can sort many things out with **sensible probabilistic arguments...**



Turns out we were lucky...



Our dirty trick **only works for** ER random networks.



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We can sort many things out with **sensible probabilistic arguments...**



More detailed investigations will profit from a spot of **Generatingfunctionology**.<sup>[10]</sup>



Turns out we were lucky...

- Our dirty trick **only works for** ER random networks.
- The problem:** We assumed that neighbors have the same probability  $\delta$  of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because  $\langle k \rangle = \langle k \rangle_R$ .
- We need a separate probability  $\delta'$  for the chance that an edge **leads to** the giant (infinite) component.
- We can sort many things out with **sensible probabilistic arguments...**

- CocoNuTs: We figure out the final size and complete dynamics.



# Neural reboot (NR):

## Falling maple leaf

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