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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2025–2026

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



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- ## Outline

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 Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- 🧩 Limit of $m = 0$: empty graph.
- 🧩 Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- 🧩 Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N(N-1)}.$$

- Given m edges, there are $\binom{N}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world:** links are usually costly so real networks are almost always **sparse**.

- Get your own exciting generator [here](#).
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

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-  For method 1, # links is probabilistic:

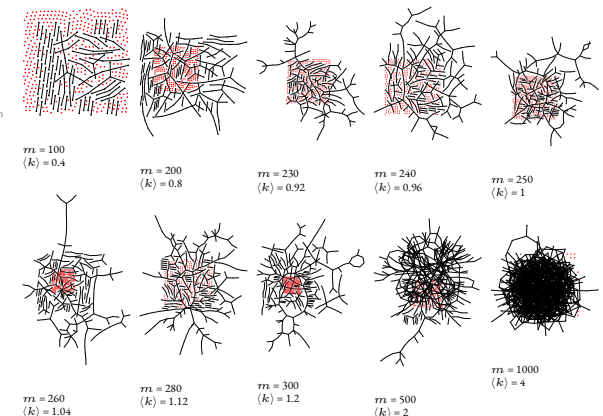
$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

-  So the expected or **average degree** is

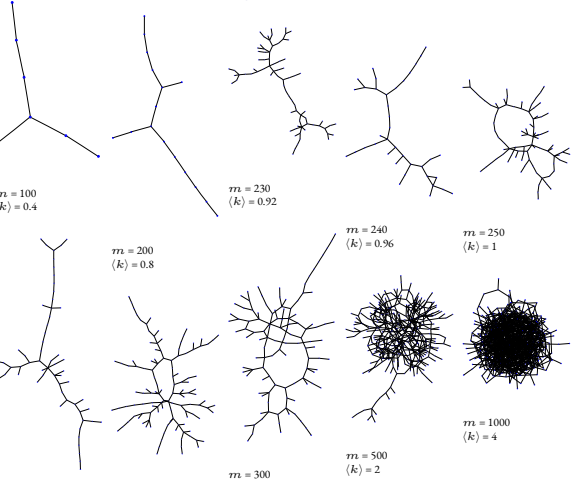
$$\begin{aligned}\langle k \rangle &= \frac{2\langle m \rangle}{N} \\ &= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{2}{N} p \frac{1}{2} N(N-1) = p(N-1).\end{aligned}$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \rightarrow 0$ as $N \rightarrow \infty$.

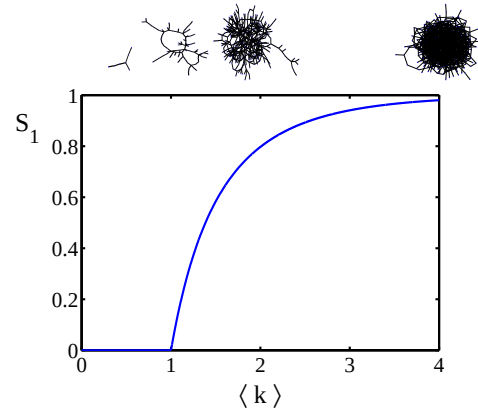
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Random networks: largest components



Giant component



Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k .
- Consider method 1 for constructing random networks: each possible link is realized with probability p .
- Now consider one node: there are ' $N - 1$ choose k ' ways the node can be connected to k of the other $N - 1$ nodes.
- Each connection occurs with probability p , each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution $\square$:

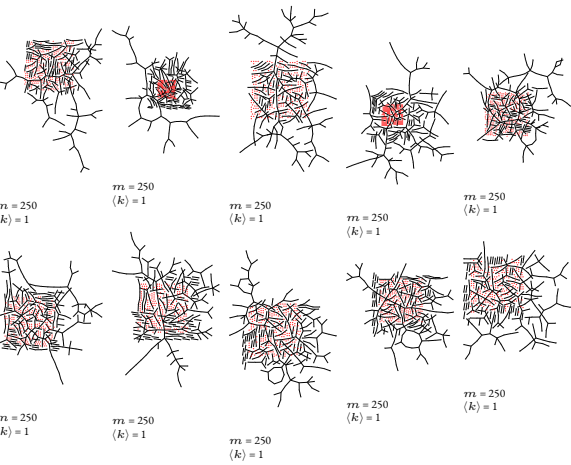
$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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Random networks: examples for N=500



Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient: ^[7]
- $$C_2 = \frac{3 \times \text{\#triangles}}{\text{\#triples}}$$
- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that $C_2 = p$.

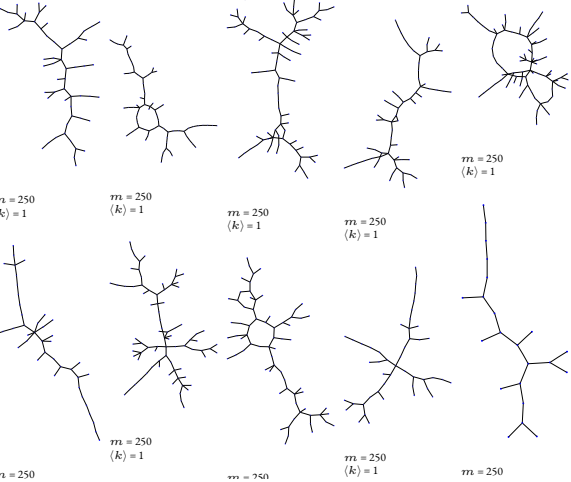
Limiting form of $P(k; p, N)$:

- Our degree distribution: $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \rightarrow \infty$?
- We must end up with the normal distribution right?
- If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \rightarrow \infty$.
- But we want to keep $\langle k \rangle$ fixed...
- So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N - 1) = \text{constant}$.

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-(\langle k \rangle)}$$

This is a Poisson distribution $\square$ with mean $\langle k \rangle$.

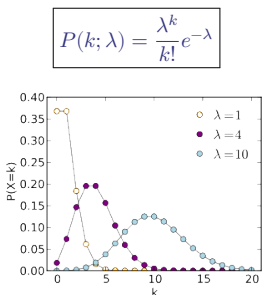
Random networks: largest components



Clustering in random networks:

- So for large random networks ($N \rightarrow \infty$), clustering drops to zero.
- Key structural feature of random networks is that they locally look like pure branching networks
- No small loops.

Poisson basics:



- $\lambda > 0$
- $k = 0, 1, 2, 3, \dots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- 'Law of small numbers'

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Poisson basics:

Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle} = 1 \end{aligned}$$

General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k .
- Also known as the **configuration model**.^[7]
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

But we'll be more interested in

- Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
- Examining mechanisms that lead to networks with certain degree distributions.

Models

Generalized random networks:

- Arbitrary degree distribution P_k .
- Create (unconnected) nodes with degrees sampled from P_k .
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

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Poisson basics:

Mean degree: we must have

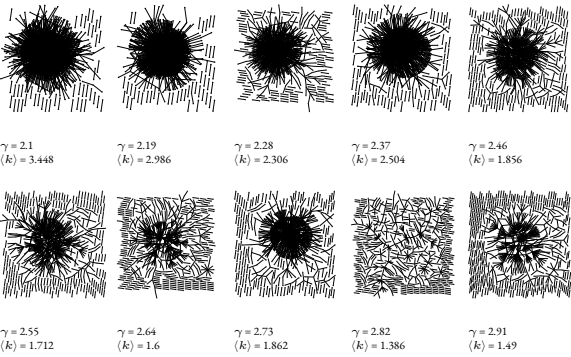
$$\langle k \rangle = \sum_{k=0}^{\infty} k P(k; \langle k \rangle).$$

Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} k P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} k \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^k}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^i}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle} = \langle k \rangle \end{aligned}$$

In CocoNuTs, we find a different, crazier way of doing this...

Random networks: examples for $N=1000$

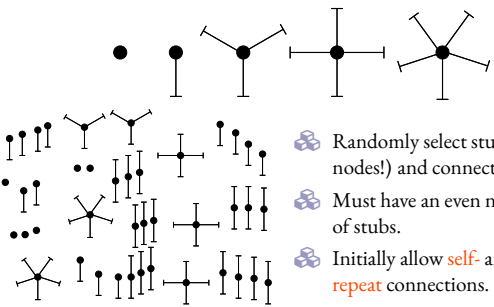


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Building random networks: Stubs

Phase 1:

Idea: start with a soup of unconnected nodes with stubs (half-edges):



- Randomly select stubs (not nodes!) and connect them.
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

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Poisson basics:

- The variance of degree distributions for random networks turns out to be **very important**.
- Using calculation similar to one for finding $\langle k \rangle$ we find the **second moment** to be:

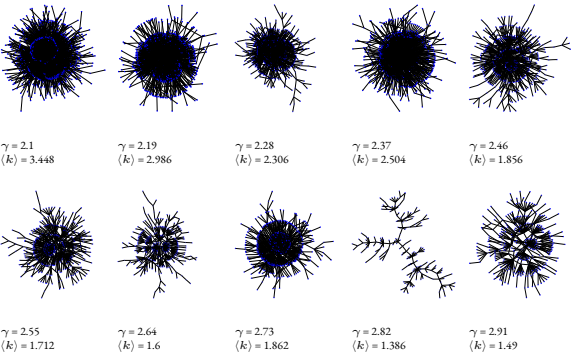
$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

- So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.
- Note: This is a special property of Poisson distribution and can trip us up...

Random networks: largest components

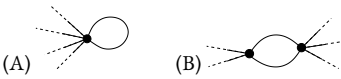


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Building random networks: First rewiring

Phase 2:

Now find any (A) self-loops and (B) repeat edges and **randomly rewire** them.



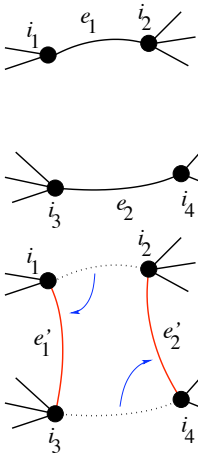
- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire **two edges** at a time.

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General random rewiring algorithm



- Randomly choose **two edges**. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.
- Rewire one end of each edge.
- Node degrees **do not change**.
- Works if e_1 is a self-loop or repeated edge.
- Same as finding on/off/on/off 4-cycles, and rotating them.

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Sampling random networks

- Phase 2:
 - Use rewiring algorithm to remove all self and repeat loops.
- Phase 3:
 - Randomize network** wiring by applying rewiring algorithm liberally.
 - Rule of thumb: # Rewirings $\simeq 10 \times \# \text{ edges}$ [5].

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Random sampling

Problem with only joining up stubs is **failure** to randomly sample from all possible networks.

Example from Milo et al. (2003) [5]:

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Sampling random networks

- What if we have P_k instead of N_k ?
- Must now create nodes before start of the construction algorithm.
- Generate N nodes by sampling from degree distribution P_k .
- Easy to do exactly numerically since k is discrete.
- Note:** not all P_k will always give nodes that can be wired together.

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Network motifs

- Idea of **motifs** [8] introduced by Shen-Orr, Alon et al. in 2002.
- Looked at gene expression within full context of **transcriptional regulation networks**.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k .
- Looked for **certain subnetworks** (motifs) that appeared more or less often than expected

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Network motifs

feedforward loop

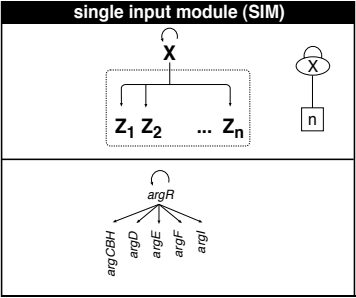
Z only turns on in response to sustained activity in **X**.

Turning off **X** rapidly turns off **Z**.

Analogy to elevator doors.

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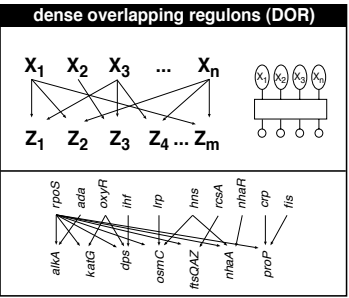
Network motifs



- Master switch.

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Network motifs



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Network motifs

- Note:** selection of motifs to test is reasonable but nevertheless ad-hoc.
- For more, see work carried out by Wiggins *et al.* at Columbia.

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The edge-degree distribution:

- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of **randomly chosen node**.
- A second very important distribution arises from **choosing randomly on edges** rather than on nodes.
- Define Q_k to be the probability the node at a **random end** of a **randomly chosen edge** has degree k .
- Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto k P_k$$

- Normalized form:

$$Q_k = \frac{k P_k}{\sum_{k'=0}^{\infty} k' P_{k'}} = \frac{k P_k}{\langle k \rangle}.$$

- Big deal:** Rich-get-richer mechanism is built into this selection process.

The edge-degree distribution:

- Given R_k is the probability that a friend has k other friends, then the average number of **friends' other friends** is

$$\begin{aligned} \langle k \rangle_R &= \sum_{k=0}^{\infty} k R_k = \sum_{k=0}^{\infty} k \frac{(k+1) P_{k+1}}{\langle k \rangle} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1) P_{k+1} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} ((k+1)^2 - (k+1)) P_{k+1} \end{aligned}$$

(where we have sneakily matched up indices)

$$\begin{aligned} &= \frac{1}{\langle k \rangle} \sum_{j=0}^{\infty} (j^2 - j) P_j \quad (\text{using } j = k+1) \\ &= \frac{1}{\langle k \rangle} (\langle k(k-1) \rangle) \end{aligned}$$

Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the **1st and 2nd moments** of P_k and not just the 1st moment.

- Three peculiarities:

- We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
- If P_k has a **large second moment**, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
- Your friends really are different from you...^[4, 6]
- See also: class size paradoxes (nod to: Gelman)

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The edge-degree distribution:

- For networks, Q_k is also the probability that a friend (neighbor) of a random node has **k friends**.
- Useful variant on Q_k :

R_k = probability that a friend of a random node has **k other friends**.



$$R_k = \frac{(k+1) P_{k+1}}{\sum_{k'=0}^{\infty} (k'+1) P_{k'+1}} = \frac{(k+1) P_{k+1}}{\langle k \rangle}$$

- Equivalent to friend having degree $k+1$.
- Natural question:** what's the expected number of other friends that one friend has?

The edge-degree distribution:

- Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k(k-1) \rangle)$, is true for **all** random networks, **independent of degree distribution**.
- For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k \rangle$$

- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...

Two reasons why this matters

More on peculiarity #3:

- A node's average # of friends: $\langle k \rangle$
- Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2} \right) \geq \langle k \rangle$$

- So only if everyone has the same degree (variance= $\sigma^2 = 0$) can a node be the same as its friends.
- Intuition: For networks, the more connected a node, the more likely it is to be chosen as a friend.

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- Probability of randomly selecting a node of degree k by choosing from nodes:
 $P_1 = 3/7, P_2 = 2/7, P_3 = 1/7, P_6 = 1/7$.
- Probability of landing on a node of degree k after randomly selecting an edge and then randomly choosing one direction to travel:
 $Q_1 = 3/16, Q_2 = 4/16, Q_3 = 3/16, Q_6 = 6/16$.
- Probability of finding # outgoing edges = k after randomly selecting an edge and then randomly choosing one direction to travel:
 $R_0 = 3/16, R_1 = 4/16, R_2 = 3/16, R_5 = 6/16$.

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The edge-degree distribution:

- In fact, R_k is rather special for pure random networks ...
- Substituting

$$P_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

into

$$R_k = \frac{(k+1) P_{k+1}}{\langle k \rangle}$$

we have

$$R_k = \frac{(k+1)}{\langle k \rangle} \frac{\langle k \rangle^{k+1}}{(k+1)!} e^{-\langle k \rangle} = \frac{(k+1)}{\langle k \rangle} \frac{\langle k \rangle^{k+1}}{(k+1)!} e^{-\langle k \rangle}$$

$$= \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \equiv P_k.$$

- #samesies.

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Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

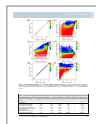
$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the **1st and 2nd moments** of P_k and not just the 1st moment.

- Three peculiarities:

- We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
- If P_k has a **large second moment**, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
- Your friends really are different from you...^[4, 6]
- See also: class size paradoxes (nod to: Gelman)

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“Generalized friendship paradox in complex networks: The case of scientific collaboration”^[3]
Eom and Jo,
Nature Scientific Reports, **4**, 4603, 2014. ^[3]

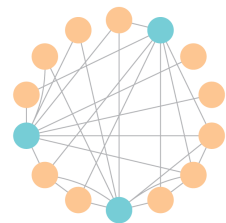
Your friends really are ~~monsters~~ #winners:¹

- Go on, hurt me:** Friends have more coauthors, citations, and publications.
- Other horrific studies:** your connections on Twitter have more followers than you, are happier than you^[1], more sexual partners than you, ...
- The hope:** Maybe they have more enemies and diseases too.
- Research possibility: The Frenemy Paradox.

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¹Some press [here](#) [MIT Tech Review].

Related disappointment:



- Nodes see their friends' color choices.
- Which color is more popular?¹
- Again: thinking in edge space changes everything.

¹https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/

Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with **at least 1** other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition** (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.
- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

Global spreading condition

- Our global spreading condition is then:

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet B_{k1} > 1.$$

- Case 1—Rampant spreading:** If $B_{k1} = 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) = \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- Good:** This is just our giant component condition again.

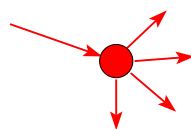
Two reasons why this matters

(Big) Reason #2:

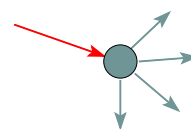
- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \rightarrow \infty$, does our network have a **giant component**?
- Defn:** Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- Defn:** Giant component = component that comprises a non-zero fraction of a network as $N \rightarrow \infty$.
- Note: Component = Cluster

Spreading on Random Networks

- For random networks, we know local structure is pure branching.
- Successful spreading is $\therefore$ contingent on **single edges** infecting nodes.



Failure:



- Focus on **binary** case with edges and nodes either infected or not.
- First big question:** for a given network and contagion process, can global spreading from a single seed occur?

Global spreading condition

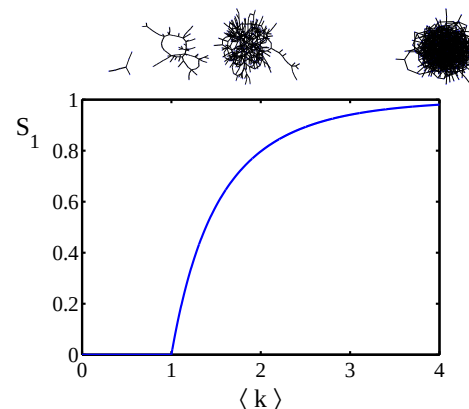
- Case 2—Simple disease-like:** If $B_{k1} = \beta < 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet \beta > 1.$$

- A fraction $(1-\beta)$ of edges do not transmit infection.
- Analogous phase transition to giant component case but **critical value of $\langle k \rangle$ is increased**.
- Aka bond percolation $\square$.
- Resulting degree distribution $\tilde{P}_k$:

$$\tilde{P}_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$

Giant component



Global spreading condition

- We need to find: ^[2]
- $\mathbf{R}$ = the average # of infected edges that one random infected edge brings about.
- Call $\mathbf{R}$ the **gain ratio**.
- Define B_{k1} as the probability that a node of degree k is infected by a single infected edge.

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet \underbrace{(k-1)}_{\substack{\text{prob. of} \\ \text{connecting to} \\ \text{a degree } k \text{ node}}} \bullet \underbrace{B_{k1}}_{\substack{\text{\# outgoing} \\ \text{infected} \\ \text{edges}}} \bullet \underbrace{B_{k1}}_{\substack{\text{Prob. of} \\ \text{infection}}} + \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet \underbrace{0}_{\substack{\text{\# outgoing} \\ \text{infected} \\ \text{edges}}} \bullet \underbrace{(1-B_{k1})}_{\substack{\text{Prob. of} \\ \text{no infection}}}$$

Giant component for standard random networks:

- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Determine condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
- When $\langle k \rangle < 1$, all components are finite.
- Fine example of a continuous phase transition $\square$.
- We say $\langle k \rangle = 1$ marks the critical point of the system.

Random networks with skewed P_k :

e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3, k \geq 1$, then

$$\begin{aligned} \langle k^2 \rangle &= e \sum_{k=1}^{\infty} k^2 k^{-\gamma} \\ &\sim \int_{x=1}^{\infty} x^{2-\gamma} dx \\ &\propto x^{3-\gamma} \Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle). \end{aligned}$$

- So giant component **always exists** for these kinds of networks.
- Cutoff scaling is k^{-3} : if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.
- How about $P_k = \delta_{kk_0}$?

Giant component

- We can figure out some limits and details for $S_1 = 1 - e^{-(k)S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1 :

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- As $\langle k \rangle \rightarrow 0, S_1 \rightarrow 0$.
- As $\langle k \rangle \rightarrow \infty, S_1 \rightarrow 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation. ^[9]

References I

[1] J. Bollen, B. Gonçalves, I. van de Leemput, and G. Ruan. The happiness paradox: Your friends are happier than you. [EPJ Data Science](#), 6:4, 2017. [pdf](#)

[2] P. S. Dodds, K. D. Harris, and J. L. Payne. Direct, physically motivated derivation of the contagion condition for spreading processes on generalized random networks. [Phys. Rev. E](#), 83:056122, 2011. [pdf](#)

[3] Y.-H. Eom and H.-H. Jo. Generalized friendship paradox in complex networks: The case of scientific collaboration. [Nature Scientific Reports](#), 4:4603, 2014. [pdf](#)

[4] S. L. Feld. Why your friends have more friends than you do. [Am. J. of Sociol.](#), 96:1464–1477, 1991. [pdf](#)

Giant component

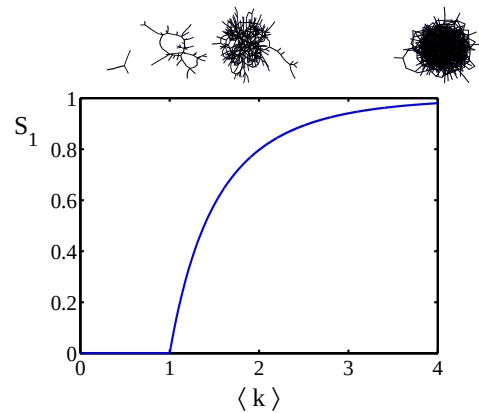
- And how big is the largest component?
- Define S_1 as the **size of the largest component**.
 - Consider an infinite ER random network with average degree $\langle k \rangle$.
 - Let's find S_1 with a back-of-the-envelope argument.
 - Define δ as the probability that a randomly chosen node **does not** belong to the largest component.
 - Simple connection: $\delta = 1 - S_1$.
 - Dirty trick**: If a randomly chosen node is not part of the largest component, then none of its neighbors are.

So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

Substitute in Poisson distribution...

Giant component



References II

[5] R. Milo, N. Kashtan, S. Itzkovitz, M. E. J. Newman, and U. Alon. On the uniform generation of random graphs with prescribed degree sequences, 2003. [pdf](#)

[6] M. E. J. Newman. Ego-centered networks and the ripple effect., [Social Networks](#), 25:83–95, 2003. [pdf](#)

[7] M. E. J. Newman. The structure and function of complex networks. [SIAM Rev.](#), 45(2):167–256, 2003. [pdf](#)

[8] S. S. Shen-Orr, R. Milo, S. Mangan, and U. Alon. Network motifs in the transcriptional regulation network of *Escherichia coli*. [Nature Genetics](#), 31:64–68, 2002. [pdf](#)

Giant component

Carrying on:

$$\begin{aligned} \delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-(k)} \delta^k \\ &= e^{-(k)} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!} \\ &= e^{-(k)} e^{(k) \delta} = e^{-(k)(1-\delta)}. \end{aligned}$$

- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:
- $$S_1 = 1 - e^{-(k)S_1}.$$

Turns out we were lucky...

- Our dirty trick **only works for** ER random networks.
- The problem**: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
- We need a separate probability δ' for the chance that an edge **leads to** the giant (infinite) component.
- We can sort many things out with **sensible probabilistic arguments**...
- More detailed investigations will profit from a spot of **Generatingfunctionology**. ^[10]
- CocoNu Ts: We figure out the final size and complete dynamics.

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[9] S. H. Strogatz. **Nonlinear Dynamics and Chaos**. Addison Wesley, Reading, Massachusetts, 1994.

[10] H. S. Wilf. **Generatingfunctionology**. A K Peters, Natick, MA, 3rd edition, 2006. [pdf](#)

