

Power-Law Mechanisms: Variable Transformation

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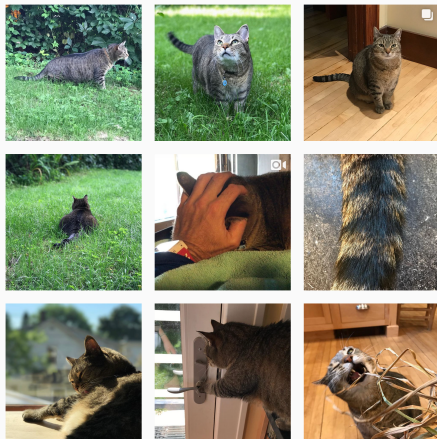
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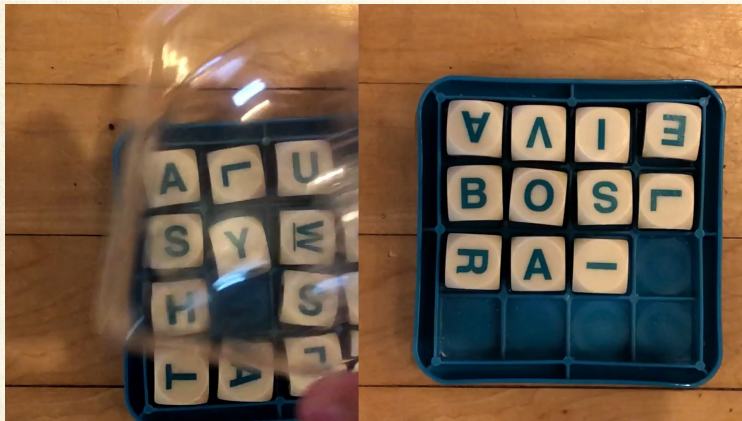
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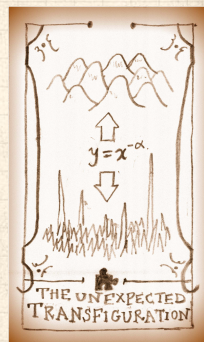
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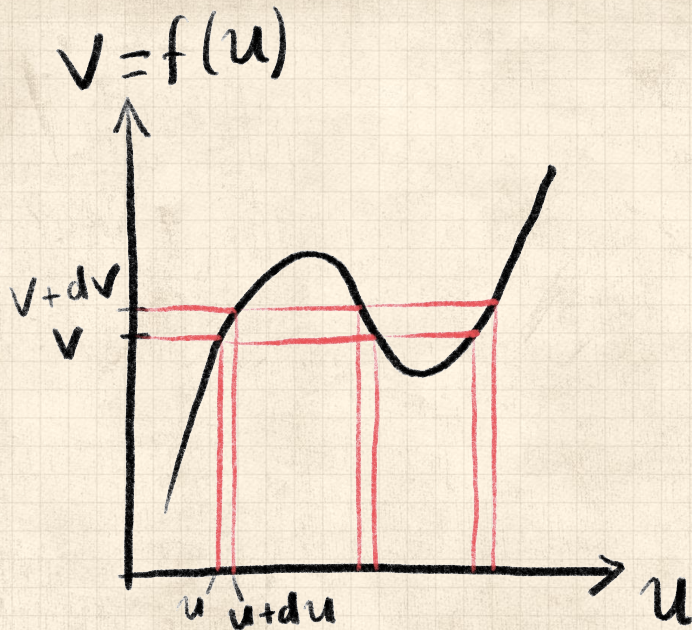
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The Boggoracle Speaks:







Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

 Random variable X with known distribution P_x

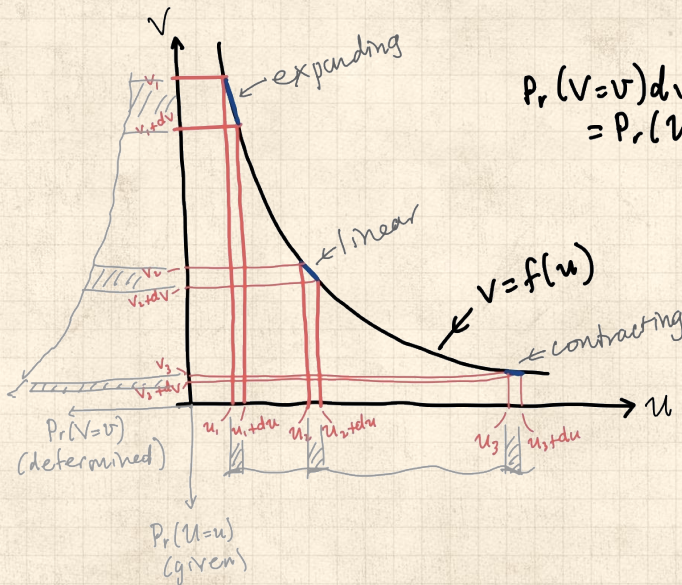
 Second random variable Y with $y = f(x)$.

$$\begin{aligned} \text{ } P_Y(y)dy &= \\ &= \sum_{x|f(x)=y} P_X(x)dx \\ &= \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$

 Often easier to do by hand...



$$P_r(V=v)dv = P_r(U=u)du$$



Example



Assume relationship between x and y is 1-1.



Power-law relationship between variables:

$$y = cx^{-\alpha}, \alpha > 0$$



Look at y large and x small



$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\overbrace{\left(\frac{y}{c} \right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

 If $P_x(x) \rightarrow \text{non-zero constant as } x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

 If $P_x(x) \rightarrow x^\beta \text{ as } x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$



Exponentials arise from randomness (easy) ...



More later when we cover robustness.



Gravity

- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtsmark who was thinking about electrostatics and plasma ^[1].
- Again, the humans naming things after humans, poorly.¹



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¹Stigler's Law of Eponymy 

Matter is concentrated in stars: ^[2]

 F is distributed unevenly

 Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$

 Assume stars are distributed randomly in space (oops?)

 Assume only one star has significant effect at $\vec{x}$.

 Law of gravity:

$$F \propto r^{-2}$$

 invert:

$$r \propto F^{-\frac{1}{2}}$$

 Connect differentials: $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



$$= F^{-1-3/2}dF$$



$$= F^{-5/2}dF.$$



Gravity:

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$$P_F(F)dF = F^{-5/2}dF$$



$$\gamma = 5/2$$



Mean is finite.



Variance = ∞ .



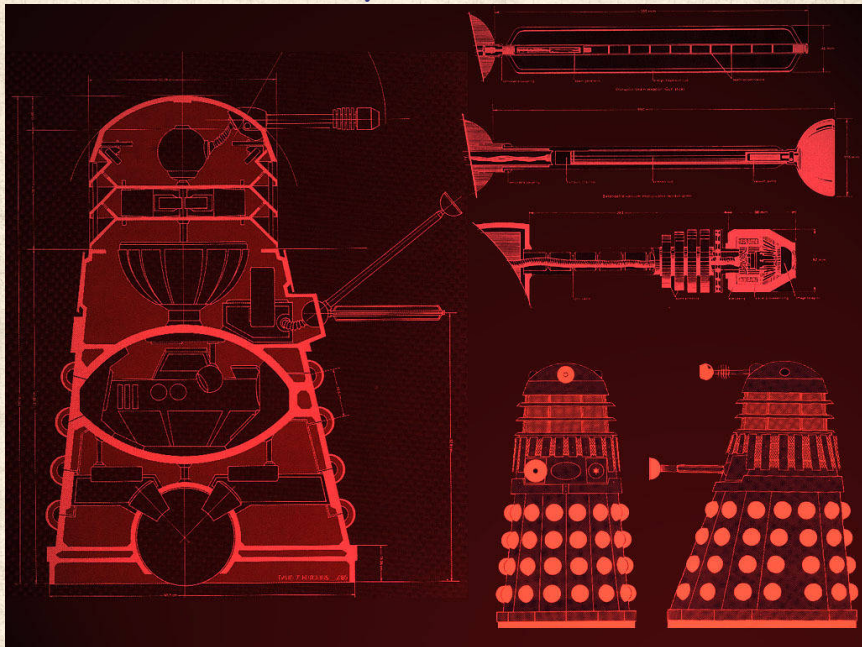
A **wild** distribution.



Upshot: Random sampling of space usually safe
but can end badly...



□ Todo: Build Dalek army.



Extreme Caution!

 PLIPLO = Power law in, power law out

 Explain a power law as resulting from another unexplained power law.

 Yet another homunculus argument ...

 Don't do this!!! (slap, slap)

 MIWO = Mild in, Wild out is the stuff.

 In general: We need mechanisms!



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References

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- [2] D. Sornette.
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