

Power-Law Mechanisms: Variable Transformation

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The PoCVerse
Variable
Transformation
1 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLIO

References



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The PoCverse
Variable
Transformation
2 of 22

Variable
transformation

Basics

Holtzmark's Distribution

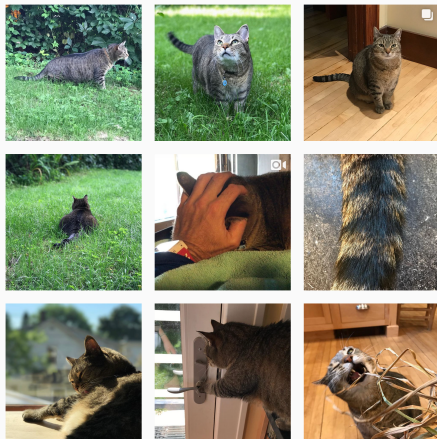
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The PoCVerse
Variable
Transformation
3 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLLO

References



Outline

The PoCSverse
Variable
Transformation
4 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

Variable transformation

Basics

Holtmark's Distribution

PLIPLO

References



Variable
transformation

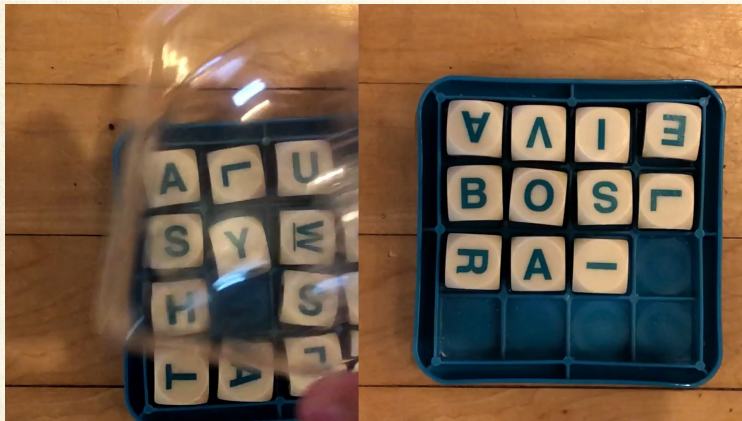
Basics

Holtmark's Distribution

PLIPL0

References

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Outline

The PoCverse
Variable
Transformation
7 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

Variable transformation

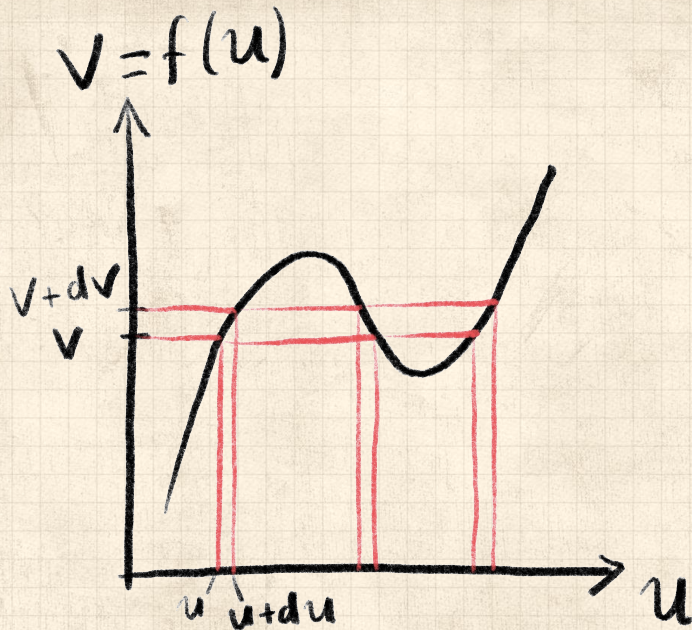
Basics

Holtmark's Distribution

PLIPLO

References





Variable Transformation

Understand power laws as arising from:

The PoCverse
Variable
Transformation
9 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).

The PoCverse
Variable
Transformation
9 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

The PoCverse
Variable
Transformation
9 of 22

Variable
transformation

Basics

Holtzmark's Distribution

PLIPLO

References



Variable Transformation

The PoCverse
Variable
Transformation
9 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPL0

References

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.



Random variable X with known distribution P_x



Variable Transformation

The PoCserve
Variable
Transformation
9 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

 Random variable X with known distribution P_x

 Second random variable Y with $y = f(x)$.



Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

 Random variable X with known distribution P_x

 Second random variable Y with $y = f(x)$.

$$\begin{aligned} \text{ } P_Y(y)dy &= \\ \sum_{x|f(x)=y} P_X(x)dx &= \\ \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$



Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

 Random variable X with known distribution P_x

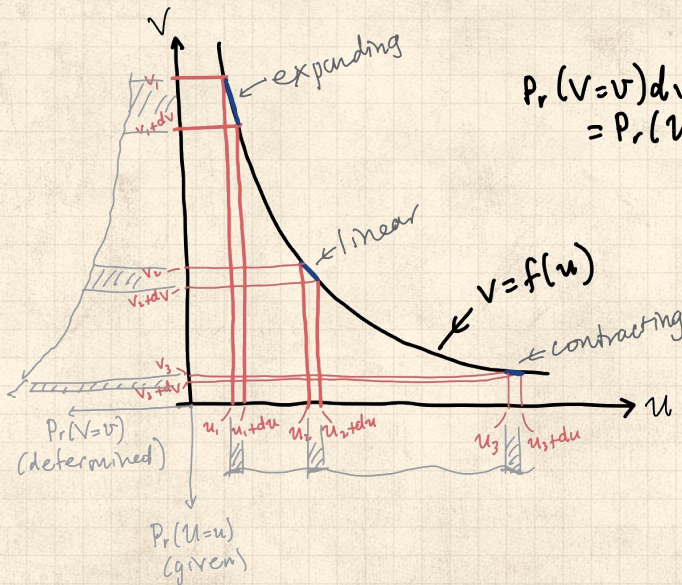
 Second random variable Y with $y = f(x)$.

$$\begin{aligned} \text{ } P_Y(y)dy &= \\ &\sum_{x|f(x)=y} P_X(x)dx \\ &= \\ &\sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$

 Often easier to do by hand...



$$P_r(V=v)dv = P_r(U=u)du$$



Example

The PoCverse
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References



Example



Assume relationship between x and y is 1-1.

The PoC Sverse
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLIO

References



Example



Assume relationship between x and y is 1-1.



Power-law relationship between variables:

$$y = cx^{-\alpha}, \alpha > 0$$

The PoCverse
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtzmark's Distribution

PLIPLD

References



Example



Assume relationship between x and y is 1-1.



Power-law relationship between variables:

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Look at y large and x small

The PoCServe
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtzmark's Distribution

PLIPLO

References



Example



Assume relationship between x and y is 1-1.



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Look at y large and x small



$$dy = d(cx^{-\alpha})$$

The PoCverse
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



Example



Assume relationship between x and y is 1-1.



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Look at y large and x small



$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

The PoCverse
Variable
Transformation
11 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



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$$\text{invert: } dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$



Example



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$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$



Example



Assume relationship between x and y is 1-1.



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$$\text{invert: } dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

The PoCverse
Variable
Transformation
12 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\overbrace{\left(\frac{y}{c} \right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

The PoCserve
Variable
Transformation
12 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



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 If $P_x(x) \rightarrow \text{non-zero constant as } x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$



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 If $P_x(x) \rightarrow \text{non-zero constant as } x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

 If $P_x(x) \rightarrow x^\beta \text{ as } x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



Example

The PoCverse
Variable
Transformation
13 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$



Example

The PoCverse
Variable
Transformation
13 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

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Exponentials arise from randomness (easy) ...



Example

Exponential distribution

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Exponentials arise from randomness (easy) ...



More later when we cover robustness.



Outline

The PoCVerse
Variable
Transformation
14 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

Variable transformation

Basics

Holtmark's Distribution

PLIPLO

References



Gravity



Select a random point in the universe $\vec{x}$.



The PoCverse
Variable
Transformation
15 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLD

References



¹Stigler's Law of Eponymy

Gravity



Select a random point in the universe $\vec{x}$.



Measure the force of gravity $F(\vec{x})$.



The PoCverse
Variable
Transformation
15 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLO

References



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Select a random point in the universe $\vec{x}$.



Measure the force of gravity $F(\vec{x})$.



Observe that $P_F(F) \sim F^{-5/2}$.



The PoCVerse
Variable
Transformation
15 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLO

References



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Distribution named after Holtsmark who was thinking about electrostatics and plasma ^[1].



The PoCVerse
Variable
Transformation
15 of 22

Variable
transformation

Basics

Holtsmark's Distribution
PLIPLO

References



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Gravity

- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtsmark who was thinking about electrostatics and plasma ^[1].
- Again, the humans naming things after humans, poorly.¹



The PoCVerse
Variable
Transformation
15 of 22

Variable
transformation

Basics

Holtsmark's Distribution
PLIPLO

References



¹Stigler's Law of Eponymy

Matter is concentrated in stars: ^[2]



F is distributed unevenly

The PoCSverse
Variable
Transformation
16 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References



Matter is concentrated in stars: ^[2]



F is distributed unevenly



Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$



Matter is concentrated in stars: ^[2]



F is distributed unevenly



Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

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Assume stars are distributed randomly in space (oops?)



Matter is concentrated in stars: ^[2]

🧱 F is distributed unevenly

🧱 Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

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🧱 Assume only one star has significant effect at $\vec{x}$.



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🧱 Law of gravity:

$$F \propto r^{-2}$$



Matter is concentrated in stars: ^[2]

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🧱 Assume only one star has significant effect at $\vec{x}$.

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$$F \propto r^{-2}$$

🧱 invert:

$$r \propto F^{-\frac{1}{2}}$$



Matter is concentrated in stars: ^[2]

 F is distributed unevenly

 Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

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 Assume stars are distributed randomly in space (oops?)

 Assume only one star has significant effect at $\vec{x}$.

 Law of gravity:

$$F \propto r^{-2}$$

 invert:

$$r \propto F^{-\frac{1}{2}}$$

 Connect differentials: $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$

The PoCverse
Variable
Transformation
17 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLO

References



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$

The PoCVerse
Variable
Transformation
17 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLO

References



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$

The PoCVerse
Variable
Transformation
17 of 22

Variable
transformation

Basics

Holtmark's Distribution
PLIPLD

References



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



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$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



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$$= F^{-1-3/2}dF$$



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



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$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



$$= F^{-1-3/2}dF$$



$$= F^{-5/2}dF.$$



Gravity:

The PoCSverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtmark's Distribution

FLIPLO

References

$$P_F(F)dF = F^{-5/2}dF$$



Gravity:

The PoCverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtmark's Distribution

FLIPLO

References



$$P_F(F)dF = F^{-5/2}dF$$

$$\gamma = 5/2$$



Gravity:

The PoCSverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtmark's Distribution

FLIPLO

References

$$P_F(F)dF = F^{-5/2}dF$$



$$\gamma = 5/2$$



Mean is finite.



Gravity:

The PoCverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtmark's Distribution
.....
PLIPLO

References

$$P_F(F)dF = F^{-5/2}dF$$



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Mean is finite.



Variance = ∞ .



Gravity:

The PoCverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtsmark's Distribution
.....
PLIPLO

References

$$P_F(F)dF = F^{-5/2}dF$$



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Mean is finite.



Variance = ∞ .



A **wild** distribution.



Gravity:

The PoCverse
Variable
Transformation
18 of 22

Variable
transformation

Basics

Holtmark's Distribution
.....
PLIPLO

References

$$P_F(F)dF = F^{-5/2}dF$$



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Variance = ∞ .



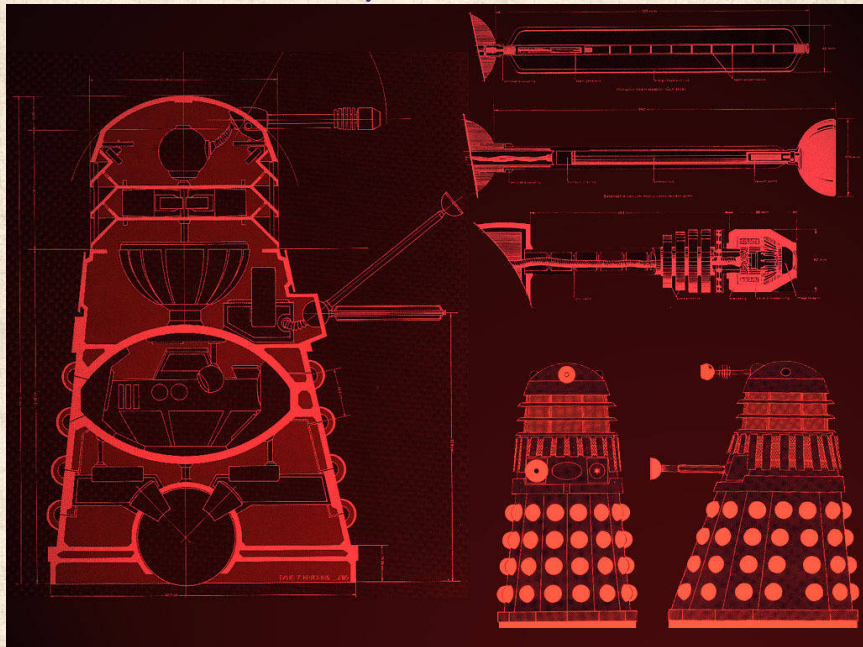
A **wild** distribution.



Upshot: Random sampling of space usually safe
but can end badly...



□ Todo: Build Dalek army.



Outline

The PoCServe
Variable
Transformation
20 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

Variable transformation

Basics

Holtmark's Distribution

PLIPLO

References



Extreme Caution!

The PoCVerse
Variable
Transformation
21 of 22

Variable
transformation

Basics

Holtsmark's Distribution

PLIPLO

References

 **PLIPLO = Power law in, power law out**



Extreme Caution!

The PoCVerse
Variable
Transformation
21 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References



PLIPLO = Power law in, power law out



Explain a power law as resulting from another unexplained power law.



Extreme Caution!

The PoCverse
Variable
Transformation
21 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References



PLIPLO = Power law in, power law out



Explain a power law as resulting from another unexplained power law.



Yet another homunculus argument ...



Extreme Caution!

The PoCVerse
Variable
Transformation
21 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

 PLIPLO = Power law in, power law out

 Explain a power law as resulting from another unexplained power law.

 Yet another homunculus argument ...

 Don't do this!!! (slap, slap)



Extreme Caution!

The PoCverse
Variable
Transformation
21 of 22

Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

 PLIPLO = Power law in, power law out

 Explain a power law as resulting from another unexplained power law.

 Yet another homunculus argument ...

 Don't do this!!! (slap, slap)

 MIWO = Mild in, Wild out is the stuff.



Extreme Caution!

 PLIPLO = Power law in, power law out

 Explain a power law as resulting from another unexplained power law.

 Yet another homunculus argument ...

 Don't do this!!! (slap, slap)

 MIWO = Mild in, Wild out is the stuff.

 In general: We need mechanisms!



References I

The PoCVerse
Variable
Transformation
22 of 22

Variable
transformation

Basics

Holtzmark's Distribution

PLIPLIO

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