

## Power-Law Mechanisms: Variable Transformation

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## Outline

## Variable transformation

Basics  
 Holtsmark's Distribution  
 PLIPLO

## References

## Variable Transformation

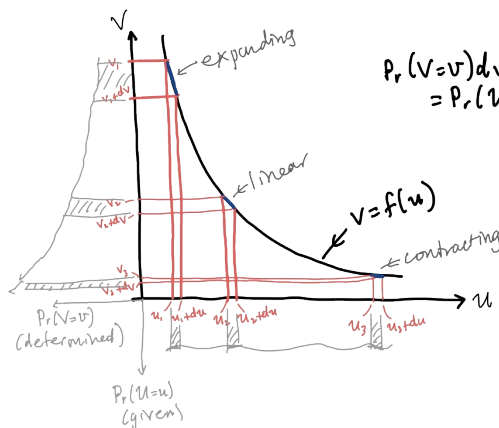
Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable  $X$  with known distribution  $P_x$
- Second random variable  $Y$  with  $y = f(x)$ .

$$\begin{aligned} P_Y(y)dy &= \sum_{x|f(x)=y} P_X(x)dx \\ &= \sum_{u|f(x)=u} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$

 Often easier to do by hand...



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### Example

- Assume relationship between  $x$  and  $y$  is 1-1.
- Power-law relationship between variables:  
 $y = cx^{-\alpha}, \alpha > 0$
- Look at  $y$  large and  $x$  small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

$$\text{invert: } dx = \frac{-1}{c\alpha} x^{\alpha+1} dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

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Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left( \overbrace{\left( \frac{y}{c} \right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

 If  $P_x(x) \rightarrow$  non-zero constant as  $x \rightarrow 0$  then

$$P_u(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

 If  $P_x(x) \rightarrow x^\beta$  as  $x \rightarrow 0$  then

$$P_u(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

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### Example

## Exponential distribution

Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

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## Gravity

- 🌀 Select a random point in the universe  $\vec{x}$ .
- 🌀 Measure the force of gravity  $F(\vec{x})$ .
- 🌀 Observe that  $P_F(F) \sim F^{-5/2}$ .
- 🌀 Distribution named after Holtsmark who was thinking about electrostatics and plasma <sup>[1]</sup>.
- 🌀 Again, the humans naming things after humans, poorly.<sup>1</sup>



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<sup>1</sup>Stigler's Law of Eponymy .

Matter is concentrated in stars: <sup>[2]</sup>

- $F$  is distributed unevenly
- Probability of being a distance  $r$  from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r)dr \propto r^2 dr$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at  $\vec{x}$ .
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

- Connect differentials:  $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$

## Transformation:

Using  $r \propto F^{-1/2}$ ,  $dr \propto F^{-3/2} dF$ , and  $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2}) F^{-3/2} dF$$



$$\propto (F^{-1/2})^2 F^{-3/2} dF$$



$$= F^{-1-3/2} dF$$



$$= F^{-5/2} dF.$$

## Gravity:

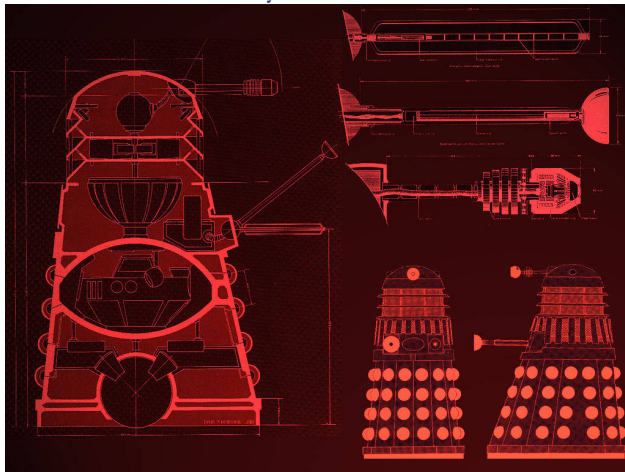


$$P_F(F)dF = F^{-5/2} dF$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance =  $\infty$ .
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

□ Todo: Build Dalek army.



## Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument [↗](#)...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

## References I

- J. Holtsmark.  
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- D. Sornette.  
[Critical Phenomena in Natural Sciences.](#)  
Springer-Verlag, Berlin, 1st edition, 2003.

