

Mechanisms for Generating Power-Law Size Distributions, Part 1

Last updated: 2026/08/19, 12:31:02 EDT

DOI:

Principles of Complex Systems,
Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

Deliverator: Calla Beauregard
Showrunner: Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
University of Vermont | Santa Fe Institute



The PoCVerse
Power-Law
Mechanisms, Pt. 1
1 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Licensed under the Creative Commons Attribution 4.0 International

These slides are brought to you by:

Sealie & Lambie Productions



The PoCVerse
Power-Law
Mechanisms, Pt. 1
2 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

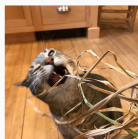
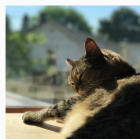
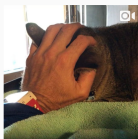
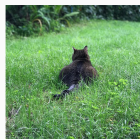
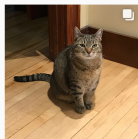
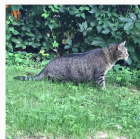
Fractional Brownian
Motion

References



These slides are also brought to you by:

Special Guest Executive Producer



The PoCverse
Power-Law
Mechanisms, Pt. 1
3 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 



Outline

Random Walks

The First Return Problem

Random River Networks

Scaling Relations

Death and Sports

Fractional Brownian Motion

References

The PoCverse
Power-Law
Mechanisms, Pt. 1
4 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Heavy-tailed distributions are **characters**.



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Heavy-tailed distributions are **characters**.



Some of these distributions have power-law tails.





Heavy-tailed distributions are **characters**.



Some of these distributions have power-law tails.



Measured exponents (γ 's and α 's) vary across systems (and measurers).



- Heavy-tailed distributions are **characters**.
- Some of these distributions have power-law tails.
- Measured exponents (γ 's and α 's) vary across systems (and measurers).
- What's their **origin story**?



Random Walks

The First Return
Problem

Random River
Networks

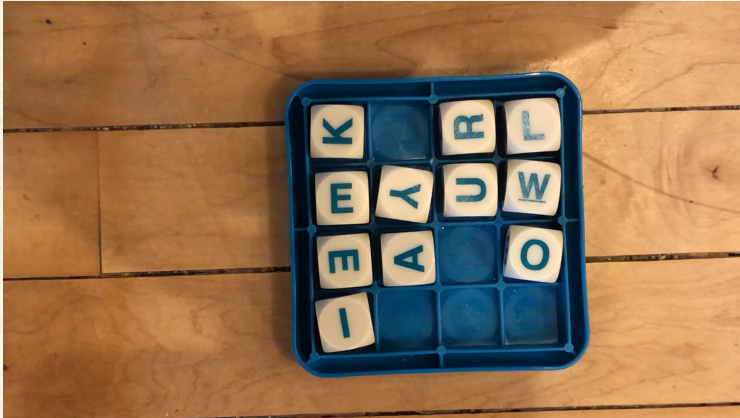
Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

The Boggoracle Speaks:



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

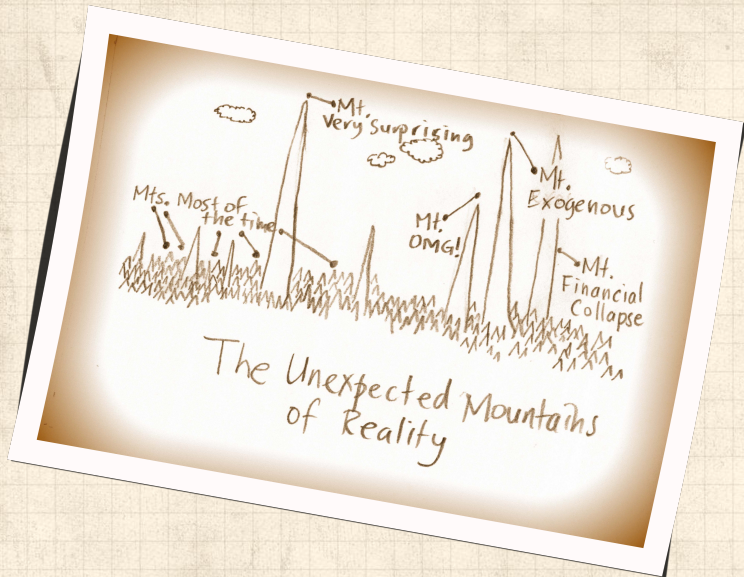
Fractional Brownian
Motion

References



The Snormals





Mechanisms:

A powerful story in the rise of complexity:

The PoCVerse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:



structure arises out of randomness.

The PoCverse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:



structure arises out of randomness.



Exhibit A: Random walks. 

The PoCverse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:

 structure arises out of randomness.

 Exhibit A: Random walks. 

The essential random walk:

 One spatial dimension.

The PoCverse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:

 structure arises out of randomness.

 Exhibit A: Random walks. 

The essential random walk:

 One spatial dimension.

 Time and space are discrete

The PoCVerse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:



structure arises out of randomness.



Exhibit A: Random walks. 

The essential random walk:



One spatial dimension.



Time and space are discrete



Random walker (e.g., a zombie texter ) starts at origin $x = 0$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Mechanisms:

A powerful story in the rise of complexity:

 structure arises out of randomness.

 Exhibit A: Random walks. 

The essential random walk:

 One spatial dimension.

 Time and space are discrete

 Random walker (e.g., a zombie texter ) starts at origin $x = 0$.

 Step at time t is ε_t :

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
10 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

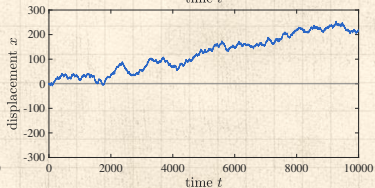
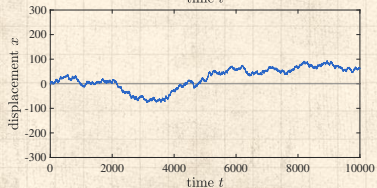
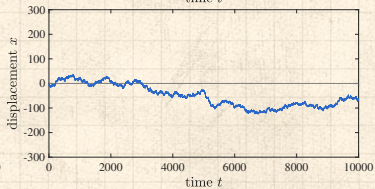
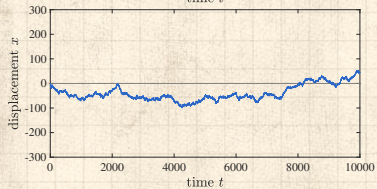
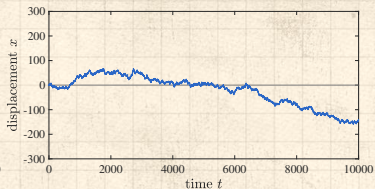
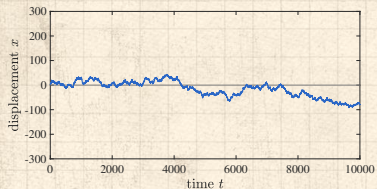
Death and Sports

Fractional Brownian
Motion

References



A few random random walks:



The PoCserve
Power-Law
Mechanisms, Pt. 1
11 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

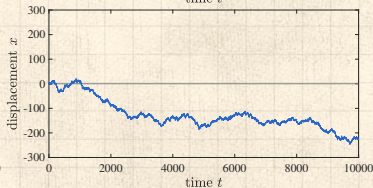
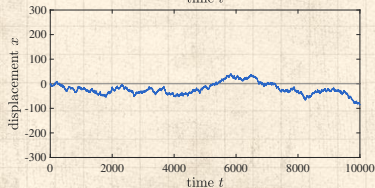
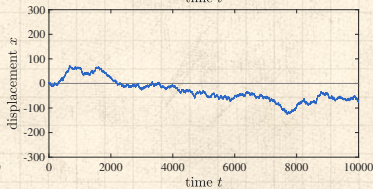
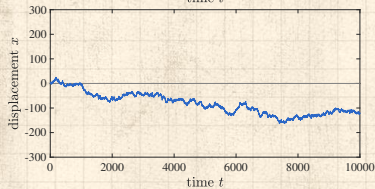
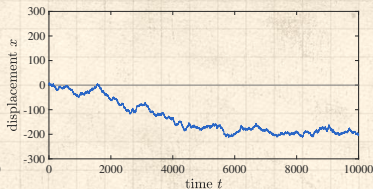
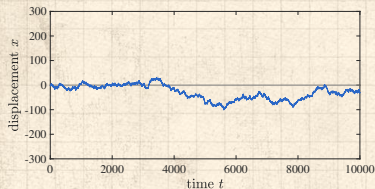
Death and Sports

Fractional Brownian
Motion

References



A few random random walks:



The PoCverse
Power-Law
Mechanisms, Pt. 1
11 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

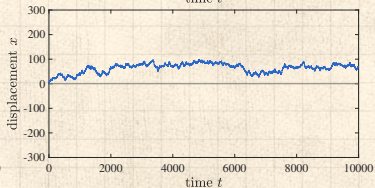
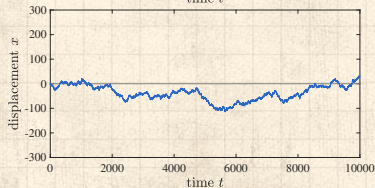
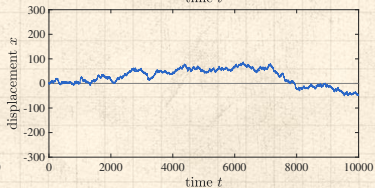
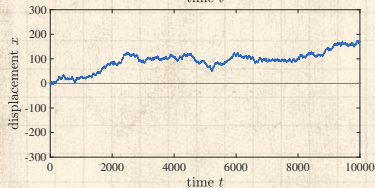
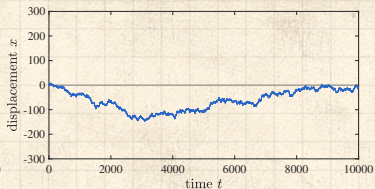
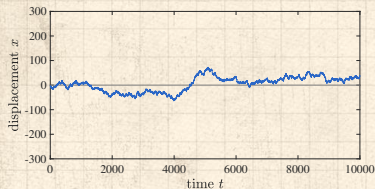
Death and Sports

Fractional Brownian
Motion

References



A few random random walks:



The PoCserve
Power-Law
Mechanisms, Pt. 1
11 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

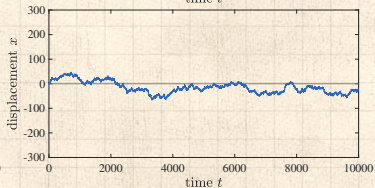
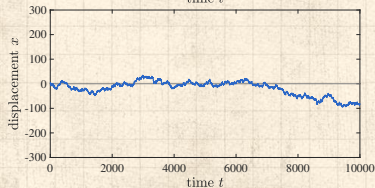
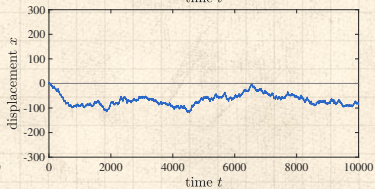
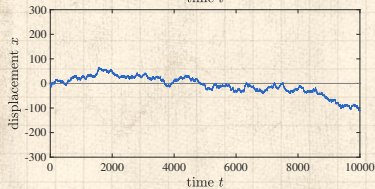
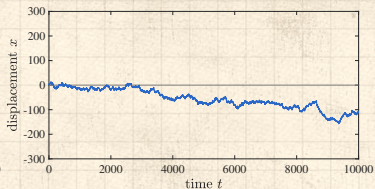
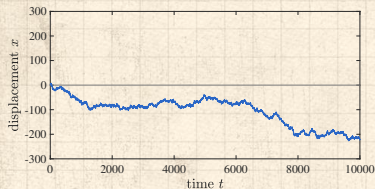
Death and Sports

Fractional Brownian
Motion

References



A few random random walks:



The PoCverse
Power-Law
Mechanisms, Pt. 1
11 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

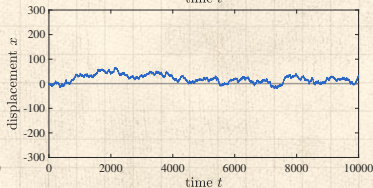
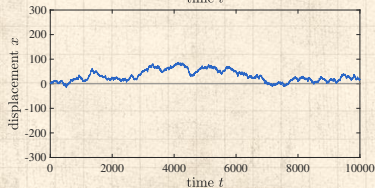
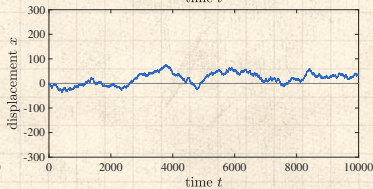
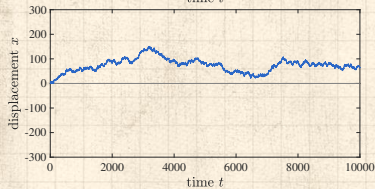
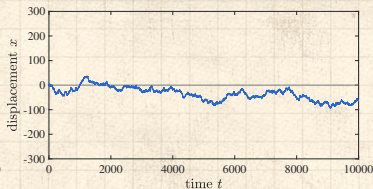
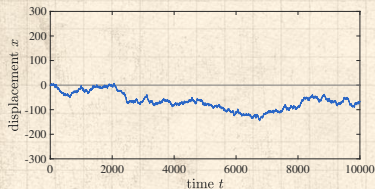
Death and Sports

Fractional Brownian
Motion

References



A few random random walks:



The PoCverse
Power-Law
Mechanisms, Pt. 1
11 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle = 0$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle = 0$$



At any time step, we ‘expect’ our zombie texter to be back at their starting place.

The PoCverse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle = 0$$



At any time step, we ‘expect’ our zombie texter to be back at their starting place.



Obviously fails for odd number of steps...



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle = 0$$

 At any time step, we ‘expect’ our zombie texter to be back at their starting place.

 Obviously fails for odd number of steps...

 But as time goes on, the chance of our texting undead friend lurching back to $x=0$ must diminish, right?

The PoCVerse
Power-Law
Mechanisms, Pt. 1
12 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\text{Var}(x_t) = \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right)$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i)\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

The PoCverse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i) = \sum_{i=1}^t 1\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

The PoCverse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i) = \sum_{i=1}^t 1 = t\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

The PoCverse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i) = \sum_{i=1}^t 1 = t\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \varepsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i) = \sum_{i=1}^t 1 = t\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$



A non-trivial scaling law arises out of
additive aggregation or accumulation.

The PoCverse
Power-Law
Mechanisms, Pt. 1
13 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Great moments in Televised Random Walks



Plinko!  from the Price is Right.

 Plinko failure .

 Also known as the bean machine , the quincunx (simulation) , and the Galton box.

The PoCverse
Power-Law
Mechanisms, Pt. 1
14 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

The PoCVerse
Power-Law
Mechanisms, Pt. 1
15 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

 Each **specific** random walk of length t appears with a chance $1/2^t$.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
15 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.

The PoCverse
Power-Law
Mechanisms, Pt. 1
15 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
15 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
15 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walk basics:

Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert assignment question 

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$



How does $P(x_t)$ behave for large t ?

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

 Take time $t = 2n$ to help ourselves.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

Take time $t = 2n$ to help ourselves.

$$x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

Take time $t = 2n$ to help ourselves.

$x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$

x_{2n} is even so set $x_{2n} = 2k$.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$



How does $P(x_t)$ behave for large t ?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert assignment question 

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert assignment question 

- The whole is different from the parts.

#nutritious

The PoCVerse
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



How does $P(x_t)$ behave for large t ?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert assignment question 

- The whole is different from the parts.
- See also: [Stable Distributions](#) 

#nutritious

The PoCSeve
Power-Law
Mechanisms, Pt. 1
16 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

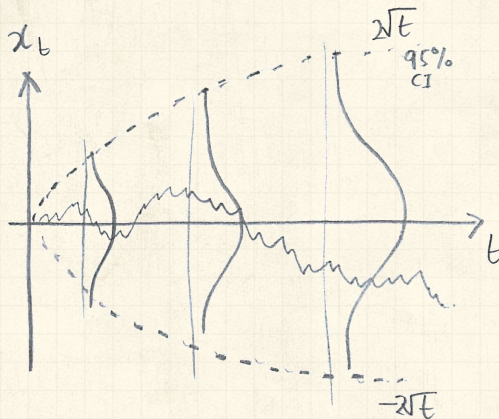
Death and Sports

Fractional Brownian
Motion

References



Universality is also not left-handed:



This is Diffusion : the most essential kind of spreading (more later).



View as Random Additive Growth Mechanism.

The PoCverse
Power-Law
Mechanisms, Pt. 1
17 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



So many things are connected:

Pascal's Triangle



Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
18 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



¹Stigler's Law of Eponymy  showing excellent form again.

So many things are connected:

Pascal's Triangle



Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.



Counting encoded in algebraic forms (and much more).

The PoC'Sverse
Power-Law
Mechanisms, Pt. 1
18 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



¹Stigler's Law of Eponymy  showing excellent form again.

So many things are connected:

Pascal's Triangle



Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.



Counting encoded in algebraic forms (and much more).



Encode heads and tails as variables h and t .

The PoC'Sverse
Power-Law
Mechanisms, Pt. 1
18 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



¹Stigler's Law of Eponymy  showing excellent form again.

So many things are connected:

Pascal's Triangle ↗



Could have been the Pyramid of Pingala ↗¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.



Counting encoded in algebraic forms (and much more).



Encode heads and tails as variables h and t .



$(h + t)^n = \sum_{k=0}^n \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

The PoC'Sverse
Power-Law
Mechanisms, Pt. 1
18 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



¹Stigler's Law of Eponymy ↗ showing excellent form again.

So many things are connected:

Pascal's Triangle ↗



Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.



Counting encoded in algebraic forms (and much more).



Encode heads and tails as variables h and t .



$(h + t)^n = \sum_{k=0}^n \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$



$(h + t)^3 = hhh + hht + hth + thh + htt + tht + tth + ttt$

¹Stigler's Law of Eponymy [↗] showing excellent form again.

The PoCverse
Power-Law
Mechanisms, Pt. 1
18 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



The Snormals



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



The Random Road
through the Forests of Forgettable Events



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

Random walks are even weirder than you might think...



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

Random walks are even weirder than you might think...



$\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.



Random walks are even weirder than you might think...

 $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.

 Think of a coin flip game with ten thousand tosses.



Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?



Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?
-  The most likely number of lead changes is...



Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?
-  The most likely number of lead changes is... 0.



Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?
-  The most likely number of lead changes is... 0.
-  In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$



Random walks are even weirder than you might think...

- ✎ $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
- ✎ Think of a coin flip game with ten thousand tosses.
- ✎ If you are behind early on, what are the chances you will make a comeback?
- ✎ The most likely number of lead changes is... 0.
- ✎ In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$
- ✎ Even crazier:
The expected time between tied scores = ∞



Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?
-  The most likely number of lead changes is... 0.
-  In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$
-  Even crazier:
The expected time between tied scores = ∞

See Feller, Intro to Probability Theory, Volume I [5]



Applied knot theory:



“Designing tie knots by random walks” ↗

Fink and Mao,

Nature, **398**, 31–32, 1999. [6]

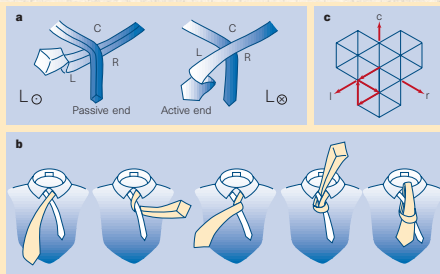


Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie.
a, The two ways of beginning a knot, $L_{\odot}$ and $L_{\otimes}$. For knots beginning with $L_{\odot}$, the tie must begin inside-out. **b**, The four-in-hand, denoted by the sequence $L_{\odot} R_{\odot} C_{\odot} T$. **c**, A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\uparrow\uparrow\downarrow$.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
22 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Hexagons are the bestagons.

Applied knot theory:

The PoCverse
Power-Law
Mechanisms, Pt. 1
23 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

Table 1 **Aesthetic tie knots**

h	γ	γ/h	$K(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		$L_0 R_0 C_0 T$
4	1	0.25	1	-1	1	Four-in-hand	$L_0 R_0 L_0 C_0 T$
5	2	0.40	2	-1	0	Pratt knot	$L_0 C_0 R_0 L_0 C_0 T$
6	2	0.33	4	0	0	Half-Windsor	$L_0 R_0 C_0 L_0 R_0 C_0 T$
7	2	0.29	6	-1	1		$L_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
7	3	0.43	4	0	1		$L_0 C_0 R_0 C_0 L_0 R_0 C_0 T$
8	2	0.25	8	0	2		$L_0 R_0 L_0 C_0 R_0 L_0 R_0 C_0 T$
8	3	0.38	12	-1	0	Windsor	$L_0 C_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
9	3	0.33	24	0	0		$L_0 R_0 C_0 L_0 R_0 C_0 L_0 R_0 C_0 T$
9	4	0.44	8	-1	2		$L_0 C_0 R_0 C_0 L_0 C_0 R_0 L_0 C_0 T$

Knots are characterized by half-winding number h , centre number γ , centre fraction γ/h , knots per class $K(h, \gamma)$, symmetry s , balance b , name and sequence.

 h = number of moves

 γ = number of center moves

 $K(h, \gamma) = 2^{\gamma-1} \binom{h-\gamma-2}{\gamma-1}$

 $s = \sum_{i=1}^h x_i$ where $x_i = -1$ for L and $x_i = +1$ for R .

 $b = \frac{1}{2} \sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.



The problem of first return:



The problem of first return:



What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?



The problem of first return:



What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?



Will our zombie texter always return to the origin?



The problem of first return:

-  What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
-  Will our zombie texer always return to the origin?
-  What about higher dimensions?



The problem of first return:

-  What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
-  Will our zombie texer always return to the origin?
-  What about higher dimensions?

Reasons for caring:



The problem of first return:

-  What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
-  Will our zombie texer always return to the origin?
-  What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an **interesting** exponent.



The problem of first return:

-  What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
-  Will our zombie texer always return to the origin?
-  What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.



The problem of first return:

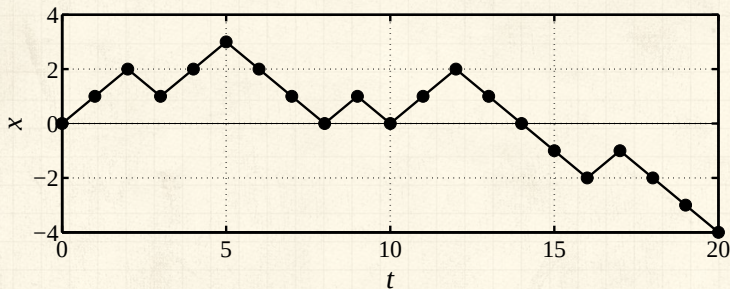
-  What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
-  Will our zombie texter always return to the origin?
-  What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an **interesting** exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.



For random walks in 1-d:



The PoCverse
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

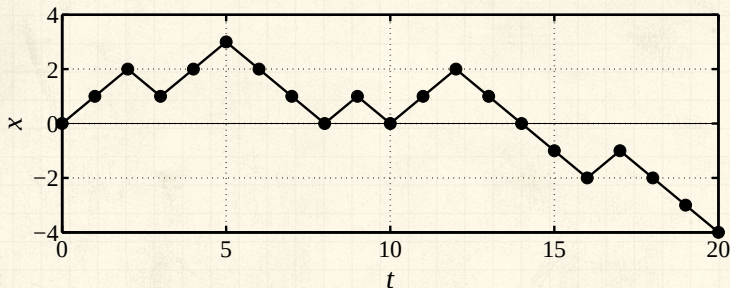
Death and Sports

Fractional Brownian
Motion

References



For random walks in 1-d:



A **return** to origin can only happen when $t = 2n$.

The PoCserve
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

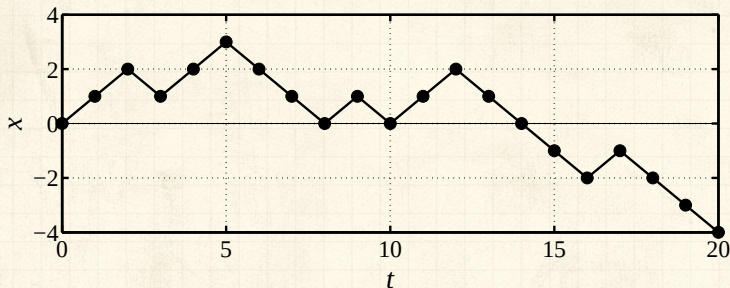
Death and Sports

Fractional Brownian
Motion

References



For random walks in 1-d:



A **return** to origin can only happen when $t = 2n$.



In example above, returns occur at $t = 8, 10,$ and 14 .

The PoCverse
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

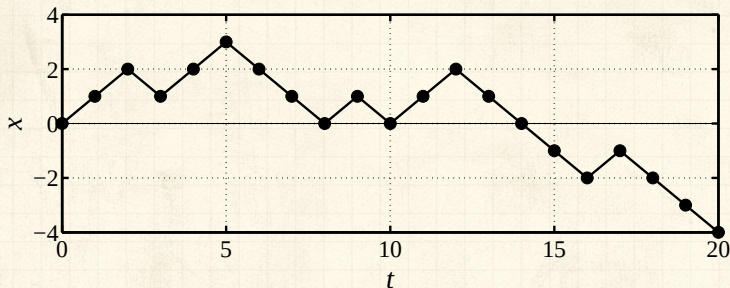
Death and Sports

Fractional Brownian
Motion

References



For random walks in 1-d:



A **return** to origin can only happen when $t = 2n$.



In example above, returns occur at $t = 8, 10$, and 14 .



Call $P_{\text{fr}}(2n)$ the probability of **first return** at $t = 2n$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

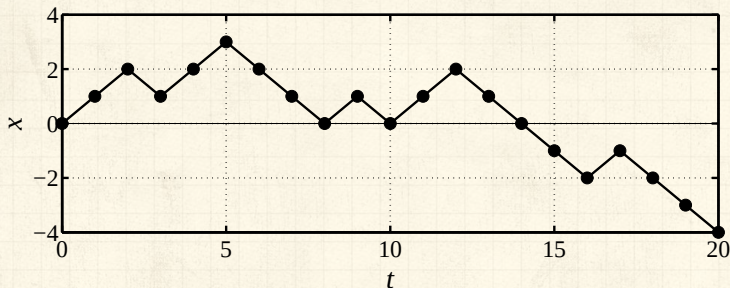
Death and Sports

Fractional Brownian
Motion

References



For random walks in 1-d:



A **return** to origin can only happen when $t = 2n$.



In example above, returns occur at $t = 8, 10,$ and 14 .



Call $P_{\text{fr}}(2n)$ the probability of **first return** at $t = 2n$.



Probability calculation $\equiv$ Counting problem
(combinatorics/statistical mechanics).

The PoCverse
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

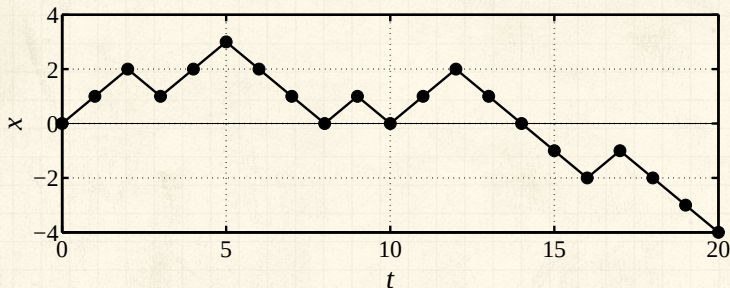
Death and Sports

Fractional Brownian
Motion

References



For random walks in 1-d:



-  A **return** to origin can only happen when $t = 2n$.
-  In example above, returns occur at $t = 8, 10$, and 14 .
-  Call $P_{\text{fr}}(2n)$ the probability of **first return** at $t = 2n$.
-  Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
-  **Idea:** Transform first return problem into an easier return problem.

The PoCverse
Power-Law
Mechanisms, Pt. 1
25 of 49

Random Walks

The First Return
Problem

Random River
Networks

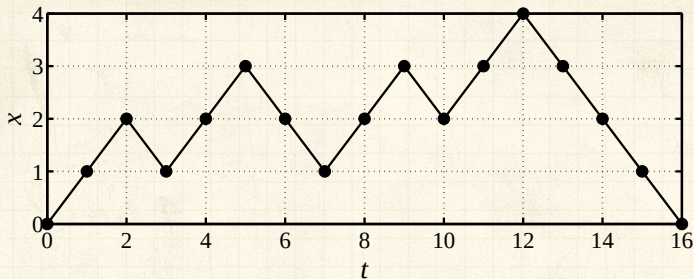
Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





Can assume zombie texter first lurches to $x = 1$.

The PoCserve
Power-Law
Mechanisms, Pt. 1
26 of 49

Random Walks

The First Return
Problem

Random River
Networks

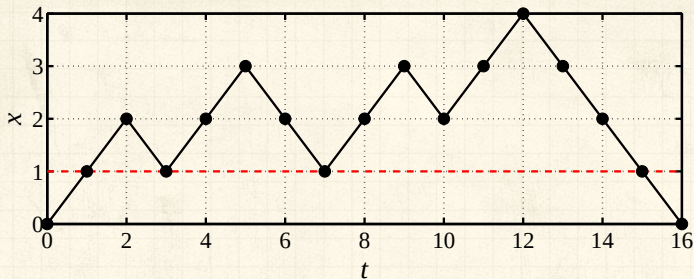
Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





Can assume zombie texter first lurches to $x = 1$.



Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).

The PoCserve
Power-Law
Mechanisms, Pt. 1
26 of 49

Random Walks

The First Return
Problem

Random River
Networks

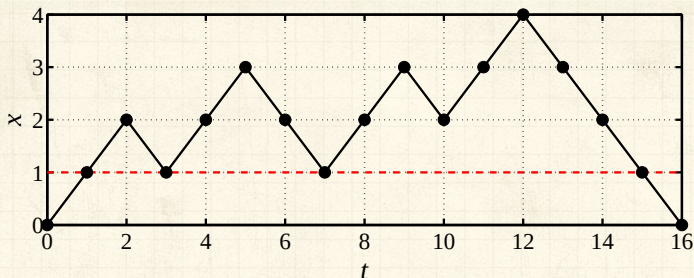
Scaling Relations

Death and Sports

Fractional Brownian
Motion

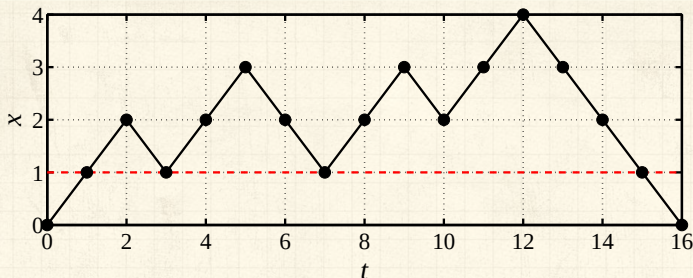
References





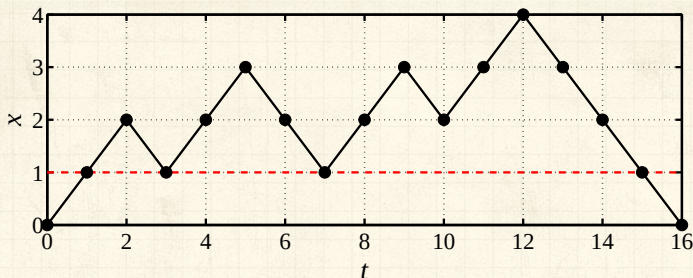
- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.





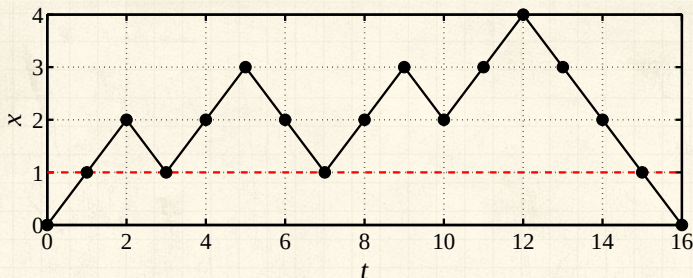
- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} \Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$





- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $$P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} \Pr(x_t \geq 1, 1 \leq t \leq 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.





- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} \Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for texters that first lurch to $x = -1$.



Counting first returns:

Approach:

The PoCVerse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

Approach:



Move to counting numbers of walks.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

Approach:

-  Move to counting numbers of walks.
-  Return to probability at end.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.

The PoCverse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea:** If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Counting first returns:

The PoCVerse
Power-Law
Mechanisms, Pt. 1
27 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea:** If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ **excluded walks**.



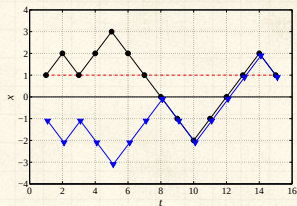
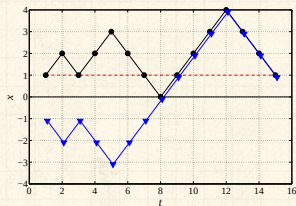
Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea:** If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ **excluded walks**.
- We'll use a method of images to identify these excluded walks.



Examples of excluded walks:



Key observation for excluded walks:



For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
28 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

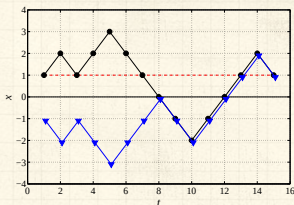
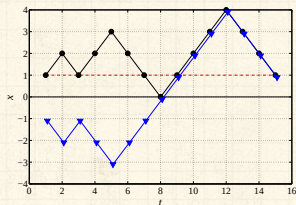
Death and Sports

Fractional Brownian
Motion

References



Examples of excluded walks:



Key observation for excluded walks:

-  For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
-  Matching path first mirrors and then tracks after first reaching $x=0$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
28 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

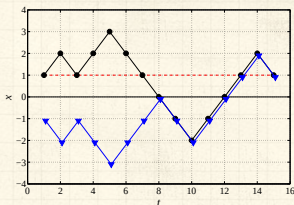
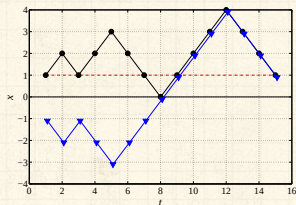
Death and Sports

Fractional Brownian
Motion

References



Examples of excluded walks:



Key observation for excluded walks:

-  For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
-  Matching path first mirrors and then tracks after first reaching $x=0$.
-  # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once

The PoC Sverse
Power-Law
Mechanisms, Pt. 1
28 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

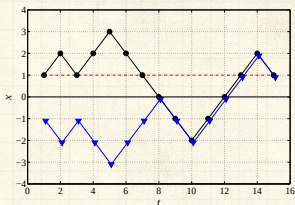
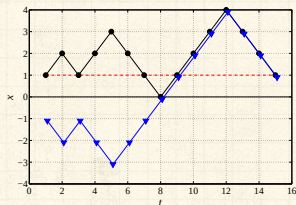
Death and Sports

Fractional Brownian
Motion

References



Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once
= # of t -step paths starting at $x=-1$ and ending at $x=+1$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
28 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

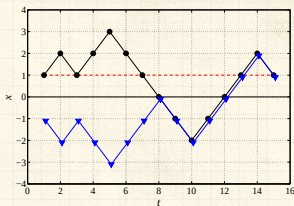
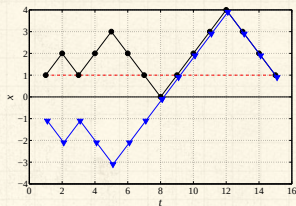
Death and Sports

Fractional Brownian
Motion

References



Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once
= # of t -step paths starting at $x=-1$ and ending at $x=+1$
= $N(-1, +1, t)$

The PoCverse
Power-Law
Mechanisms, Pt. 1
28 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
29 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



One of many related things: Catalan numbers



Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.



For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.

The PoCverse
Power-Law
Mechanisms, Pt. 1
29 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.



For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.



So $N_{\text{fr}}^+(2n) = N(+1, +1, 2n - 2) - N(-1, +1, 2n - 2)$

The PoCverse
Power-Law
Mechanisms, Pt. 1
29 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



One of many related things: Catalan numbers 



Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.



For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.



So $N_{\text{fr}}^+(2n) = N(+1, +1, 2n - 2) - N(-1, +1, 2n - 2)$



Negative side:

$$N_{\text{fr}}^-(2n) = N(-1, -1, 2n - 2) - N(+1, -1, 2n - 2)$$



 Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.

 For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.

 So $N_{\text{fr}}^+(2n) = N(+1, +1, 2n - 2) - N(-1, +1, 2n - 2)$

 Negative side:

$$N_{\text{fr}}^-(2n) = N(-1, -1, 2n - 2) - N(+1, -1, 2n - 2)$$

 Symmetry: $N_{\text{fr}}^+(2n) = N_{\text{fr}}^-(2n)$





Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.



For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.



So $N_{\text{fr}}^+(2n) = N(+1, +1, 2n - 2) - N(-1, +1, 2n - 2)$



Negative side:

$$N_{\text{fr}}^-(2n) = N(-1, -1, 2n - 2) - N(+1, -1, 2n - 2)$$



Symmetry: $N_{\text{fr}}^+(2n) = N_{\text{fr}}^-(2n)$



Both $N_{\text{fr}}(2n)$ and the one sided $N_{\text{fr}}^+(2n)$ are of mathematical and physical interest.



 Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{\text{fr}}^+(2n)$.

 For paths that first move to the negative side: $N_{\text{fr}}^-(2n)$.

 So $N_{\text{fr}}^+(2n) = N(+1, +1, 2n - 2) - N(-1, +1, 2n - 2)$

 Negative side:

$$N_{\text{fr}}^-(2n) = N(-1, -1, 2n - 2) - N(+1, -1, 2n - 2)$$

 Symmetry: $N_{\text{fr}}^+(2n) = N_{\text{fr}}^-(2n)$

 Both $N_{\text{fr}}(2n)$ and the one sided $N_{\text{fr}}^+(2n)$ are of mathematical and physical interest.

 Overall:

$$\begin{aligned} N_{\text{fr}}(2n) &= N_{\text{fr}}^+(2n) + N_{\text{fr}}^-(2n) = 2N_{\text{fr}}^+(2n) \\ &= 2N(+1, +1, 2n - 2) - 2N(-1, +1, 2n - 2). \end{aligned}$$



Probability of first return:

Insert assignment question  :

The PoCVerse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$



Normalized number of paths gives probability.

The PoCverse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$



Normalized number of paths gives probability.



Total number of possible paths = 2^{2n} .

The PoCVerse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$



Normalized number of paths gives probability.



Total number of possible paths = 2^{2n} .



$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}}.$$



Normalized number of paths gives probability.



Total number of possible paths = 2^{2n} .



$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n) \\ \simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}}.$$



Normalized number of paths gives probability.



Total number of possible paths = 2^{2n} .



$$\begin{aligned} P_{\text{fr}}(2n) &= \frac{1}{2^{2n}} N_{\text{fr}}(2n) \\ &\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}} \\ &= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \end{aligned}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Probability of first return:

Insert assignment question  :



Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}}.$$



Normalized number of paths gives probability.



Total number of possible paths = 2^{2n} .



$$\begin{aligned} P_{\text{fr}}(2n) &= \frac{1}{2^{2n}} N_{\text{fr}}(2n) \\ &\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n}^{3/2}} \\ &= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}. \end{aligned}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
30 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



The PoCVerse
Power-Law
Mechanisms, Pt. 1
31 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
31 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
31 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
31 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

 Same scaling holds for continuous space/time walks.

 $P(t)$ is normalizable.

 **Recurrence:** Random walker always returns to origin





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness



Even though walker must return, expect a long wait...





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness



Even though walker must return, expect a long wait...



One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness



Even though walker must return, expect a long wait...



One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions :





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness



Even though walker must return, expect a long wait...



One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions :



Walker in $d = 2$ dimensions must also return



 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

 Same scaling holds for continuous space/time walks.

 $P(t)$ is normalizable.

 **Recurrence:** Random walker always returns to origin

 But mean, variance, and all higher moments are infinite.
#totalmadness

 Even though walker must return, expect a long wait...

 **One moral:** Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions :

 Walker in $d = 2$ dimensions must also return

 Walker may not return in $d \geq 3$ dimensions





We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.



Same scaling holds for continuous space/time walks.



$P(t)$ is normalizable.



Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite.

#totalmadness



Even though walker must return, expect a long wait...



One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions ↗:



Walker in $d = 2$ dimensions must also return



Walker may not return in $d \geq 3$ dimensions



Associated human genius: George Pólya ↗



Random walks

On finite spaces:

The PoCverse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:



In any finite homogeneous space, a random walker will visit every site with equal probability

The PoCverse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system

The PoCverse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system
-  Non-trivial Invariant Densities arise in chaotic systems.

The PoCverse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system
-  Non-trivial Invariant Densities arise in chaotic systems.

On networks:

The PoCVerse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:

- 🧱 In any finite homogeneous space, a random walker will visit every site with equal probability
- 🧱 Call this probability the **Invariant Density** of a dynamical system
- 🧱 Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- 🧱 On networks, a random walker visits each node with frequency $\propto$ node degree

#groovy

The PoCverse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random walks

On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system
-  Non-trivial Invariant Densities arise in chaotic systems.

On networks:

-  On networks, a random walker visits each node with frequency $\propto$ node degree
-  Equal probability still present: walkers traverse **edges** with equal frequency.

#groovy

#totallygroovy

The PoCVerse
Power-Law
Mechanisms, Pt. 1
32 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Scheidegger Networks ^[17, 4]

The PoCverse
Power-Law
Mechanisms, Pt. 1
33 of 49

Random Walks

The First Return
Problem

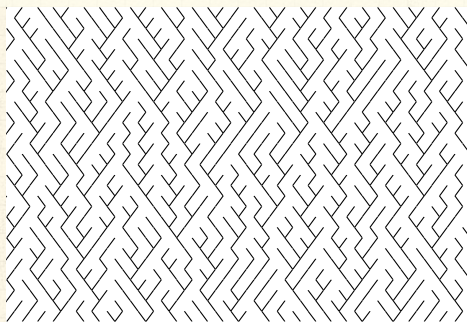
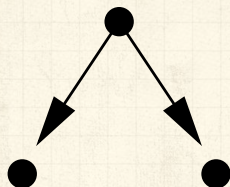
Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Random directed network on triangular lattice.



Toy model of real networks.



'Flow' is southeast or southwest with equal probability.



Scheidegger networks



Creates basins with random walk boundaries.

The PoCVerse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Scheidegger networks

-  Creates basins with random walk boundaries.
-  **Observe** that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Scheidegger networks

The PoCVerse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.



Scheidegger networks

The PoCVerse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

-  Creates basins with random walk boundaries.
-  **Observe** that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

-  Random walk with probabilistic pauses.
-  Basin termination = first return random walk problem.



Scheidegger networks

The PoCVerse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$



Scheidegger networks

The PoCverse
Power-Law
Mechanisms, Pt. 1
34 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.



Connections between exponents:



For a basin of length ℓ , width $\propto \ell^{1/2}$

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

The PoCserve
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

The PoCserve
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

The PoCserve
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
 $= a^{-4/3} da$

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

The PoCverse
Power-Law
Mechanisms, Pt. 1
35 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
 $= a^{-4/3} da$
 $= a^{-\tau} da$



Connections between exponents:

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Both basin area and length obey power law distributions

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Both basin area and length obey power law distributions



Observed for real river networks

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Both basin area and length obey power law distributions



Observed for real river networks



Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Both basin area and length obey power law distributions



Observed for real river networks



Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks
-  Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

-  Hack's law^[10]:

$$\ell \propto a^h.$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks
-  Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

-  Hack's law^[10]:

$$\ell \propto a^h.$$

-  For real, large networks^[13] $h \simeq 0.5$ (isometric scaling)

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks
-  Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

-  Hack's law^[10]:

$$\ell \propto a^h.$$

-  For real, large networks^[13] $h \simeq 0.5$ (isometric scaling)
-  Smaller basins possibly $h > 1/2$ (allometric scaling).

The PoCverse
Power-Law
Mechanisms, Pt. 1
36 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks
-  Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

-  Hack's law^[10]:

$$\ell \propto a^h.$$

-  For real, large networks^[13] $h \simeq 0.5$ (isometric scaling)
-  Smaller basins possibly $h > 1/2$ (allometric scaling).
-  Models exist with interesting values of h .



Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks
-  Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

-  Hack's law^[10]:

$$\ell \propto a^h.$$

-  For real, large networks^[13] $h \simeq 0.5$ (isometric scaling)
-  Smaller basins possibly $h > 1/2$ (allometric scaling).
-  Models exist with interesting values of h .
-  **Plan:** Redo calc with γ , τ , and h .



Connections between exponents:

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

The PoCVerse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ = \Pr(\text{basin length} = \ell)d\ell \end{aligned}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \end{aligned}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \\ &\propto (a^h)^{-\gamma}a^{h-1}da \end{aligned}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \\ &\propto (a^h)^{-\gamma}a^{h-1}da \\ &= a^{-(1+h(\gamma-1))}da \end{aligned}$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \\ &\propto (a^h)^{-\gamma}a^{h-1}da \\ &= a^{-(1+h(\gamma-1))}da \end{aligned}$$



$$\tau = 1 + h(\gamma - 1)$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
37 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:



Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$



$$d\ell \propto d(a^h) = ha^{h-1}da$$



Find τ in terms of γ and h .



$$\begin{aligned} \Pr(\text{basin area} = a)da \\ &= \Pr(\text{basin length} = \ell)d\ell \\ &\propto \ell^{-\gamma}d\ell \\ &\propto (a^h)^{-\gamma}a^{h-1}da \\ &= a^{-(1+h(\gamma-1))}da \end{aligned}$$



$$\tau = 1 + h(\gamma - 1)$$



Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.



Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
38 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$



Only one exponent is independent (take h).

The PoCverse
Power-Law
Mechanisms, Pt. 1
38 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$



Only one exponent is independent (take h).



Simplifies system description.

The PoCverse
Power-Law
Mechanisms, Pt. 1
38 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$



Only one exponent is independent (take h).



Simplifies system description.



Expect Scaling Relations where power laws are found.

The PoCverse
Power-Law
Mechanisms, Pt. 1
38 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$



Only one exponent is independent (take h).



Simplifies system description.



Expect Scaling Relations where power laws are found.



Need only characterize Universality  class with independent exponents.

The PoCverse
Power-Law
Mechanisms, Pt. 1
38 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



Death ...

Failure:

 A very simple model of failure/death

 x_t = entity's 'health' at time t

 Start with $x_0 > 0$.

 Entity fails when x hits 0.



“Explaining mortality rate plateaus” 

Weitz and Fraser,

Proc. Natl. Acad. Sci., **98**, 15383–15386, 2001. ^[18]

The PoCverse
Power-Law
Mechanisms, Pt. 1
39 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



... and the NBA:

Basketball and other sports ^[2]:

 Three arcsine laws  (Lévy ^[12]) for continuous-time random walk lasting time T :

$$\frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}.$$

The arcsine distribution  applies for:

- (1) fraction of time positive,
- (2) the last time the walk changes sign,
- and (3) the time the maximum is achieved.

 Well approximated by basketball score lines ^[8, 2].

 Australian Rules Football has some differences ^[11].

The PoCserve
Power-Law
Mechanisms, Pt. 1
40 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



More than randomness



Can generalize to Fractional Random Walks [15, 16, 14]

The PoCVerse
Power-Law
Mechanisms, Pt. 1
41 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



More than randomness



Can generalize to Fractional Random Walks ^[15, 16, 14]



Fractional Brownian Motion , Lévy flights 

The PoCverse
Power-Law
Mechanisms, Pt. 1
41 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



More than randomness

 Can generalize to Fractional Random Walks ^[15, 16, 14]

 Fractional Brownian Motion , Lévy flights 

 See Montroll and Shlesinger for example: ^[14]

“On $1/f$ noise and other distributions with long tails.”
Proc. Natl. Acad. Sci., 1982.

 In 1-d, standard deviation σ scales as

$$\sigma \sim t^\alpha$$

The PoCverse
Power-Law
Mechanisms, Pt. 1
41 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



More than randomness

 Can generalize to Fractional Random Walks ^[15, 16, 14]

 Fractional Brownian Motion , Lévy flights 

 See Montroll and Shlesinger for example: ^[14]

“On $1/f$ noise and other distributions with long tails.”
Proc. Natl. Acad. Sci., 1982.

 In 1-d, standard deviation σ scales as

$$\sigma \sim t^\alpha$$

$\alpha = 1/2$ — diffusive

$\alpha > 1/2$ — superdiffusive

$\alpha < 1/2$ — subdiffusive

The PoCVerse
Power-Law
Mechanisms, Pt. 1
41 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



More than randomness

 Can generalize to Fractional Random Walks ^[15, 16, 14]

 Fractional Brownian Motion , Lévy flights 

 See Montroll and Shlesinger for example: ^[14]

“On $1/f$ noise and other distributions with long tails.”
Proc. Natl. Acad. Sci., 1982.

 In 1-d, standard deviation σ scales as

$$\sigma \sim t^\alpha$$

$\alpha = 1/2$ — diffusive

$\alpha > 1/2$ — superdiffusive

$\alpha < 1/2$ — subdiffusive

 Extensive memory of path now matters...

The PoCVerse
Power-Law
Mechanisms, Pt. 1
41 of 49

Random Walks

The First Return
Problem

Random River
Networks

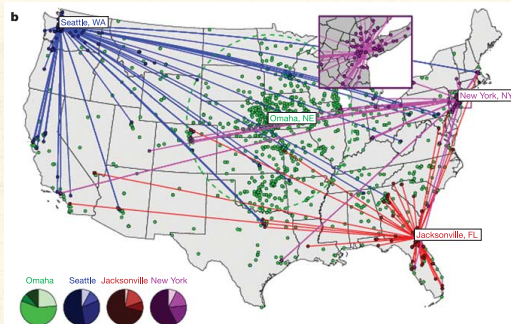
Scaling Relations

Death and Sports

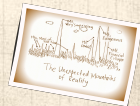
Fractional Brownian
Motion

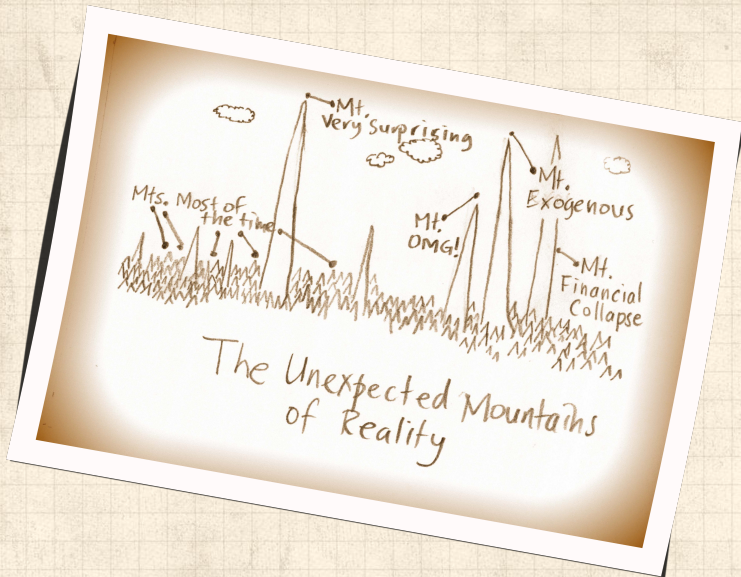
References





- First big studies of movement and interactions of people.
- Brockmann *et al.* ^[1] “Where’s George” study.
- Beyond Lévy: Superdiffusive in space but with long waiting times.
- Tracking movement via cell phones ^[9] and Twitter ^[7].





Random Walks

The First Return
Problem

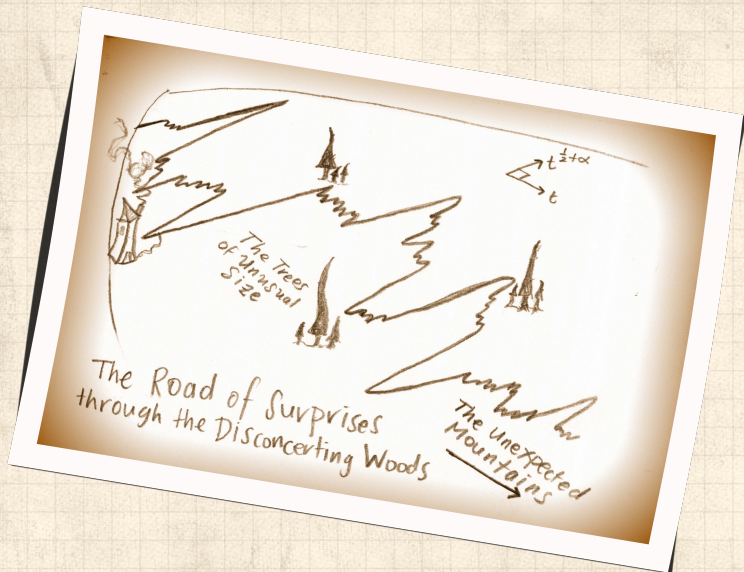
Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



References I

- [1] D. Brockmann, L. Hufnagel, and T. Geisel.
The scaling laws of human travel.
[Nature](#), pages 462–465, 2006. pdf 
- [2] A. Clauset, M. Kogan, and S. Redner.
Safe leads and lead changes in competitive team sports.
[Phys. Rev. E](#), 91:062815, 2015. pdf 
- [3] P. S. Dodds and D. H. Rothman.
Unified view of scaling laws for river networks.
[Physical Review E](#), 59(5):4865–4877, 1999. pdf 
- [4] P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
[Annu. Rev. Earth Planet. Sci.](#), 28:571–610, 2000. pdf 

The PoCVerse
Power-Law
Mechanisms, Pt. 1
45 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



References II

- [5] W. Feller.
An Introduction to Probability Theory and Its Applications,
volume I.
John Wiley & Sons, New York, third edition, 1968.
- [6] T. M. Fink and Y. Mao.
Designing tie knots by random walks.
Nature, 398:31–32, 1999. [pdf](#) 
- [7] M. R. Frank, L. Mitchell, P. S. Dodds, and C. M. Danforth.
Happiness and the patterns of life: A study of geolocated
Tweets.
Nature Scientific Reports, 3:2625, 2013. [pdf](#) 
- [8] A. Gabel and S. Redner.
Random walk picture of basketball scoring.
Journal of Quantitative Analysis in Sports, 8:1–20, 2012.

The PoCverse
Power-Law
Mechanisms, Pt. 1
46 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



References III

- [9] M. C. González, C. A. Hidalgo, and A.-L. Barabási.
Understanding individual human mobility patterns.
Nature, 453:779–782, 2008. pdf ↗
- [10] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and
Maryland.
United States Geological Survey Professional Paper,
294-B:45–97, 1957. pdf ↗
- [11] D. P. Kiley, A. J. Reagan, L. Mitchell, C. M. Danforth, and
P. S. Dodds.
The game story space of professional sports: Australian Rules
Football.
Physical Review E, 93, 2016. pdf ↗

The PoCVerse
Power-Law
Mechanisms, Pt. 1
47 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



References IV

- [12] P. Lévy and M. Loeve.
Processus stochastiques et mouvement brownien.
Gauthier-Villars Paris, 1965.
- [13] D. R. Montgomery and W. E. Dietrich.
Channel initiation and the problem of landscape scale.
Science, 255:826–30, 1992. pdf ↗
- [14] E. W. Montroll and M. F. Shlesinger.
On the wonderful world of random walks, volume XI of
Studies in statistical mechanics, chapter 1, pages 1–121.
New-Holland, New York, 1984.
- [15] E. W. Montroll and M. W. Shlesinger.
On $1/f$ noise and other distributions with long tails.
Proc. Natl. Acad. Sci., 79:3380–3383, 1982. pdf ↗

The PoCVerse
Power-Law
Mechanisms, Pt. 1
48 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References



References V

- [16] E. W. Montroll and M. W. Shlesinger.
Maximum entropy formalism, fractals, scaling phenomena,
and $1/f$ noise: a tale of tails.
[J. Stat. Phys.](#), 32:209–230, 1983.
- [17] A. E. Scheidegger.
The algebra of stream-order numbers.
[United States Geological Survey Professional Paper](#),
525-B:B187–B189, 1967. pdf ↗
- [18] J. S. Weitz and H. B. Fraser.
Explaining mortality rate plateaus.
[Proc. Natl. Acad. Sci.](#), 98:15383–15386, 2001. pdf ↗

The PoCVerse
Power-Law
Mechanisms, Pt. 1
49 of 49

Random Walks

The First Return
Problem

Random River
Networks

Scaling Relations

Death and Sports

Fractional Brownian
Motion

References

