

Mechanisms for Generating Power-Law Size Distributions, Part 1

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Principles of Complex Systems,
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Outline

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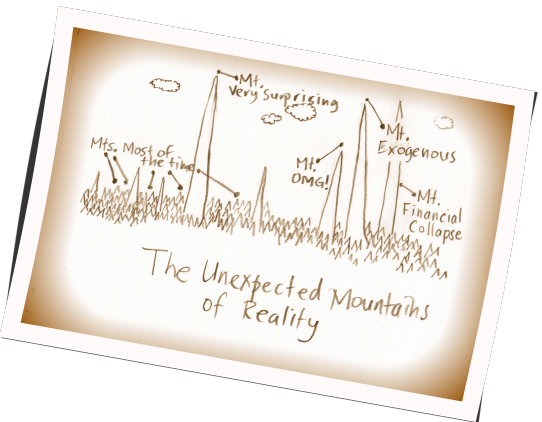
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- Heavy-tailed distributions are **characters**.
- Some of these distributions have power-law tails.
- Measured exponents (γ 's and α 's) vary across systems (and measurers).
- What's their **origin story**?

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Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

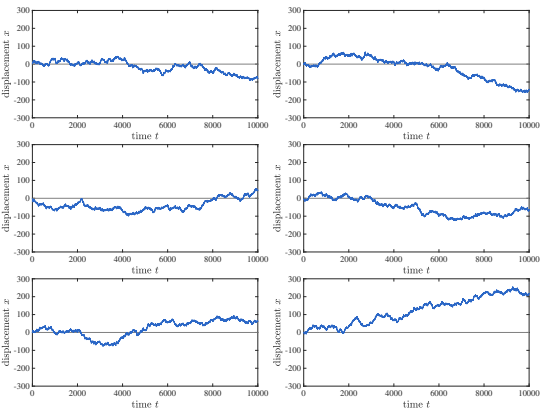
The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a zombie texter) starts at origin $x = 0$.
- Step at time t is ε_t :

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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A few random random walks:



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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \varepsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \varepsilon_i \right\rangle = \sum_{i=1}^t \langle \varepsilon_i \rangle = 0$$

- At any time step, we 'expect' our zombie texter to be back at their starting place.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our texting undead friend lurching back to $x=0$ must diminish, right?

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Variances sum:

$$\begin{aligned} \text{Var}(x_t) &= \text{Var}\left(\sum_{i=1}^t \varepsilon_i\right) \\ &= \sum_{i=1}^t \text{Var}(\varepsilon_i) = \sum_{i=1}^t 1 = t \end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.

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Random walk basics:

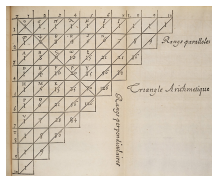
Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert assignment question 

$$N(i, j, t) = \binom{t}{(t+j-i)/2}$$

So many things are connected:

Pascal's Triangle



-  Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...
-  Binomials tend towards the Normal.
-  Counting encoded in algebraic forms (and much more).
-  Encode heads and tails as variables h and t .
-  $(h+t)^n = \sum_{k=0}^n \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
-  $(h+t)^3 = hhh + hht + hth + thh + htt + tht + tth + ttt$

¹Stigler's Law of Eponymy  showing excellent form again.

Random walks are even weirder than you might think...

-  $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
-  Think of a coin flip game with ten thousand tosses.
-  If you are behind early on, what are the chances you will make a comeback?
-  The most likely number of lead changes is... 0.
-  In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$
-  Even crazier:
The expected time between tied scores = ∞

See Feller, Intro to Probability Theory, Volume I [5]

How does $P(x_t)$ behave for large t ?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert assignment question

- 🧩 The whole is different from the parts. #nutritious
- 🧩 See also: Stable Distributions ↗



Applied knot theory:



“Designing tie knots by random walks”

Fink and Mao,
Nature, **398**, 31–32, 1999. ^[6]

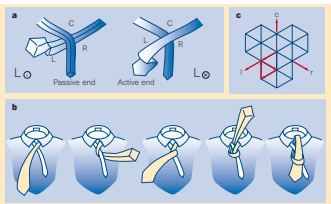
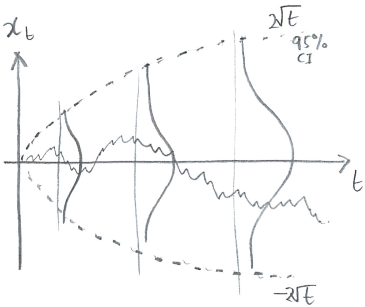


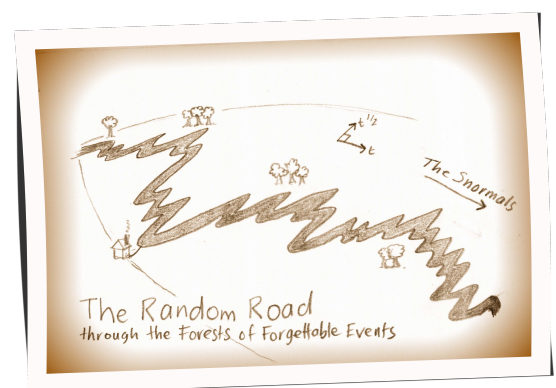
Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie.
a. The two ways of beginning a knot L_0 and L_{∞} . For knots beginning with L_0 , the tie must begin inside-out. **b.** The four-in-hand, denoted by the sequence $L_0 R_1 L_1 C_0 T$. **c.** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\tau \tau \tau \tau$.

Hexagons are the bestagons.

Universality is also not left-handed:



-  This is Diffusion : the most essential kind of spreading (more later).
-  View as Random Additive Growth Mechanism.



Applied knot theory:

h	γ	γ/h	$k(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		L,R,C,T
4	1	0.25	1	-1	1	Four-in-hand	L,R,L,C,T
5	2	0.40	2	-1	0	Pratt knot	L,C,R,L,C,T
6	2	0.33	4	0	0	Half-Windsor	L,R,C,L,R,C,T
7	2	0.29	6	-1	1		L,R,L,C,L,R,C,T
7	3	0.43	8	0	1		C,R,C,L,R,C,T
8	2	0.25	8	0	2		C,R,L,C,R,L,R,C,T
8	3	0.38	12	-1	0	Windsor	C,R,C,L,R,C,L,C,T
9	3	0.33	24	0	0		L,R,C,L,R,C,L,R,C,T
9	4	0.44	8	-1	2		L,C,R,C,L,C,R,L,C,T

Knots are characterized by half-winding number h , centre number γ , centre fraction γ/h , knots per class $k(h, \gamma)$, symmetry s , balance b , name and sequence.

- | | | | |
|---|--|---|---|
|  | h = number of moves |  | $s = \sum_{i=1}^h x_i$ where $x_i = -1$ for L and $x_i = +1$ for R . |
|  | γ = number of center moves |  | $b = \frac{1}{2} \sum_{i=2}^{h-1} \omega_i + \omega_{i-1} $ where $\omega = \pm 1$ represents winding direction. |
|  | $K(h, \gamma) = 2^{\gamma-1} \binom{h-\gamma-2}{\gamma-1}$ | | |

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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our zombie texter always return to the origin?
- What about higher dimensions?

Reasons for caring:

- We will find a power-law size distribution with an interesting exponent.
- Some physical structures may result from random walks.
- We'll start to see how different scalings relate to each other.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Probability of first return:

Insert assignment question

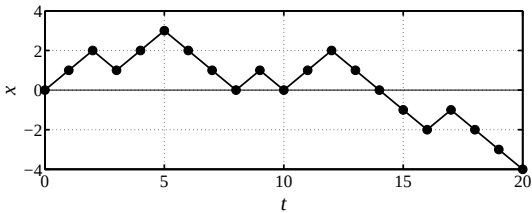
Find

$$N_{fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n} .

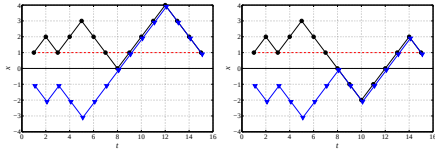
$$P_{fr}(2n) = \frac{1}{2^{2n}} N_{fr}(2n)$$
$$\approx \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}$$
$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}.$$

For random walks in 1-d:



- A **return** to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8, 10$, and 14 .
- Call $P_{fr}(2n)$ the probability of **first return** at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

Examples of excluded walks:



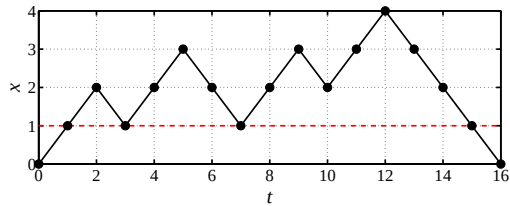
Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once
= # of t -step paths starting at $x=-1$ and ending at $x=+1$
= $N(-1, +1, t)$

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. **#totalmadness**
- Even though walker must return, expect a long wait...
- One moral:** Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- Associated human **genius**: George Pólya



- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for texters that first lurch to $x = -1$.

- Call the number of paths that return after $t = 2n$ time steps after first moving to the positive side $N_{fr}^+(2n)$.
- For paths that first move to the negative side: $N_{fr}^-(2n)$.
- So $N_{fr}^+(2n) = N(+1, +1, 2n-2) - N(-1, +1, 2n-2)$
- Negative side:
 $N_{fr}^-(2n) = N(-1, -1, 2n-2) - N(+1, -1, 2n-2)$
- Symmetry: $N_{fr}^+(2n) = N_{fr}^-(2n)$
- Both $N_{fr}(2n)$ and the one sided $N_{fr}^+(2n)$ are of mathematical and physical interest.
- Overall:

$$N_{fr}(2n) = N_{fr}^+(2n) + N_{fr}^-(2n) = 2N_{fr}^+(2n)$$
$$= 2N(+1, +1, 2n-2) - 2N(-1, +1, 2n-2).$$

One of many related things: Catalan numbers

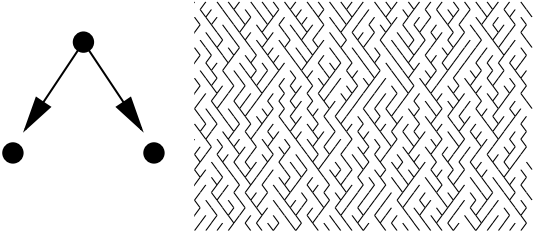
Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the **Invariant Density** of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree **#groovy**
- Equal probability still present: walkers traverse **edges** with equal frequency. **#totallygroovy**



- ☿ Random directed network on triangular lattice.
- ☿ Toy model of real networks.
- ☿ ‘Flow’ is southeast or southwest with equal probability.

Connections between exponents:

- ☿ Both basin area and length obey power law distributions
- ☿ Observed for real river networks
- ☿ Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- ☿ Hack’s law ^[10]:
 $\ell \propto a^h$.
- ☿ For real, large networks ^[13] $h \simeq 0.5$ (isometric scaling)
- ☿ Smaller basins possibly $h > 1/2$ (allometric scaling).
- ☿ Models exist with interesting values of h .
- ☿ Plan: Redo calc with γ , τ , and h .

Death ...

- Failure:
- ☿ A very simple model of failure/death
 - ☿ x_t = entity’s ‘health’ at time t
 - ☿ Start with $x_0 > 0$.
 - ☿ Entity fails when x hits 0.



“Explaining mortality rate plateaus”
Weitz and Fraser,
Proc. Natl. Acad. Sci., **98**, 15383–15386, 2001. ^[18]

Scheidegger networks

- ☿ Creates basins with random walk boundaries.
- ☿ Observe that subtracting one random walk from another gives random walk with increments:

$$\varepsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- ☿ Random walk with probabilistic pauses.
- ☿ Basin termination = first return random walk problem.
- ☿ Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- ☿ For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Connections between exponents:

- ☿ For a basin of length ℓ , width $\propto \ell^{1/2}$
- ☿ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- ☿ Invert: $\ell \propto a^{2/3}$
- ☿ $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$
- ☿ $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
 $= a^{-4/3} da$
 $= a^{-\tau} da$

Connections between exponents:

- ☿ Given
 $\ell \propto a^h$, $P(a) \propto a^{-\tau}$, and $P(\ell) \propto \ell^{-\gamma}$

- ☿ $d\ell \propto d(a^h) = h a^{h-1} da$
- ☿ Find τ in terms of γ and h .
- ☿ $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-\gamma} d\ell$
 $\propto (a^h)^{-\gamma} a^{h-1} da$
 $= a^{-(1+h(\gamma-1))} da$

☿ $\tau = 1 + h(\gamma - 1)$

- ☿ Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.

Connections between exponents:

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies to: ^[3]

$\tau = 2 - h$

and

$\gamma = 1/h$

- ☿ Only one exponent is independent (take h).
- ☿ Simplifies system description.
- ☿ Expect Scaling Relations where power laws are found.
- ☿ Need only characterize Universality class with independent exponents.

... and the NBA:

- Basketball and other sports ^[2]:
- ☿ Three arcsine laws (Lévy ^[12]) for continuous-time random walk lasting time T :

$$\frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}.$$

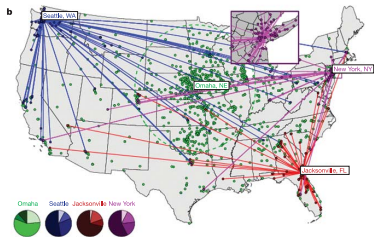
- The arcsine distribution applies to:
(1) fraction of time positive,
(2) the last time the walk changes sign,
and (3) the time the maximum is achieved.
- ☿ Well approximated by basketball score lines ^[8, 2].
- ☿ Australian Rules Football has some differences ^[11].

More than randomness

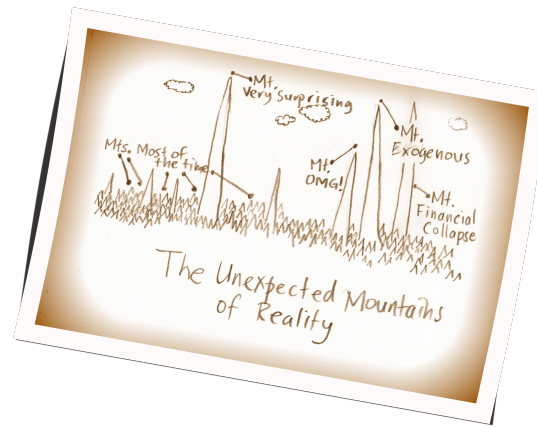
- ☿ Can generalize to Fractional Random Walks ^[15, 16, 14]
- ☿ Fractional Brownian Motion, Lévy flights
- ☿ See Montroll and Shlesinger for example: ^[14]
“On $1/f$ noise and other distributions with long tails.”
Proc. Natl. Acad. Sci., 1982.
- ☿ In 1-d, standard deviation σ scales as

$\sigma \sim t^\alpha$

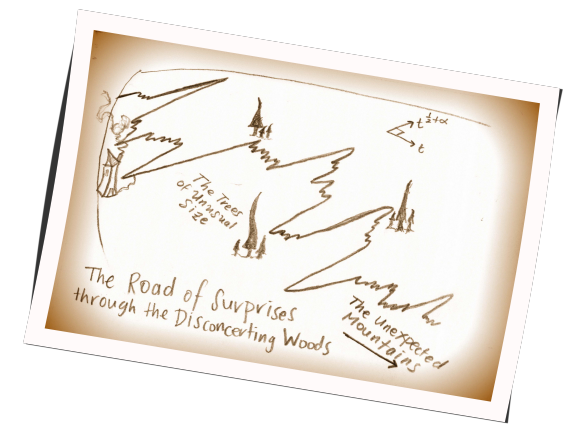
- $\alpha = 1/2$ — diffusive
- $\alpha > 1/2$ — superdiffusive
- $\alpha < 1/2$ — subdiffusive
- ☿ Extensive memory of path now matters...



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- First big studies of movement and interactions of people.
- Brockmann *et al.* ^[1] “Where’s George” study.
- Beyond Lévy: Superdiffusive in space but with long waiting times.
- Tracking movement via cell phones ^[9] and Twitter ^[7].

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