

Lognormals and friends

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2025–2026

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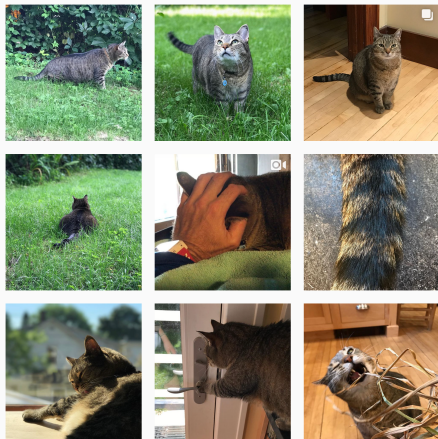
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There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

There are other ‘heavy-tailed’ distributions:

1. The Log-normal distribution ↗

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2. Weibull distributions ↗

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^\mu} dx$$

CCDF = stretched exponential ↗.

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CCDF = stretched exponential ↗.

3. Also: Gamma distribution ↗, Erlang distribution ↗, and
more.

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$



$\ln x$ is distributed according to a normal distribution with mean μ and variance σ .



Appears in economics and biology where growth increments are distributed normally.



Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

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For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \quad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \quad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$



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All moments of lognormals are **finite**.

Derivation from a normal distribution

Take Y as distributed normally:

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Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

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Derivation from a normal distribution

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Set $Y = \ln X$:

Derivation from a normal distribution

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Transform according to $P(x)dx = P(y)dy$:

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$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$

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Confusion between lognormals and pure power laws

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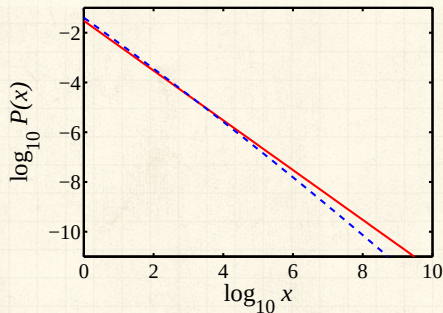
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Near agreement
over four orders of
magnitude!



For lognormal (blue), $\mu = 0$ and $\sigma = 10$.



For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

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What's happening:

$$\begin{aligned}\ln P(x) &= \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\} \\ &= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}\end{aligned}$$

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$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

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If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right) \ln x + \text{const.}} \Rightarrow \boxed{\gamma = 1 - \frac{\mu}{\sigma^2}}$$



If $\mu < 0, \gamma > 1$ which is totally cool.

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 If $\sigma^2 \gg 1$ and μ ,

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 Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

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 Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

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 $\Rightarrow$ If you find a -1 exponent,
you may have a lognormal distribution...

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Generating lognormals:

Random multiplicative growth:



$$x_{n+1} = rx_n$$

where $r > 0$ is a random growth variable

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In log space, growth is by addition:

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$\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?



Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).

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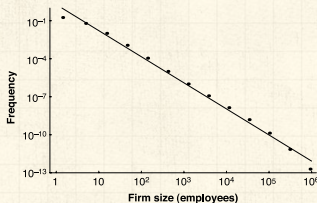
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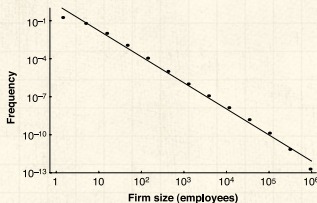
$$\text{Freq} \propto (\text{size})^{-\gamma}$$
$$\gamma \simeq 2$$

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$$\text{Freq} \propto (\text{size})^{-\gamma}$$
$$\gamma \simeq 2$$

🧱 One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. ^[1].

An explanation

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Axtel cites Malcai et al.'s (1999) argument ^[5] for why power laws appear with exponent $\gamma \simeq 2$

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 The set up: N entities with size $x_i(t)$

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 Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

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 The set up: N entities with size $x_i(t)$

 Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

 Same as for lognormal but one extra piece.

 Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

Some math later...

Insert assignment question 

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Some math later...

Insert assignment question 



Find $P(x) \sim x^{-\gamma}$

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Find $P(x) \sim x^{-\gamma}$



where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

N = total number of firms.

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$$\text{Now, if } c/N \ll 1 \text{ and } \gamma > 2 \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$

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$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

Some math later...

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Groovy... c small $\Rightarrow \gamma \simeq 2$

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The second tweak

Ages of firms/people/... may not be the same

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Allow the number of updates for each size x_i to vary

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The second tweak

Ages of firms/people/... may not be the same



Allow the number of updates for each size x_i to vary



Example: $P(t)dt = ae^{-at}dt$ where t = age.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- Back to no bottom limit: each x_i follows a lognormal

The second tweak

Ages of firms/people/... may not be the same

⌘ Allow the number of updates for each size x_i to vary

⌘ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.

⌘ Back to no bottom limit: each x_i follows a lognormal

⌘ Sizes are distributed as ^[6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

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Now averaging different lognormal distributions.

Averaging lognormals



$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x \sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) dt$$

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Insert fabulous calculation (team is spared).

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Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$

The second tweak



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Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

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Double-Pareto distribution 

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Double-Pareto distribution 



First noticed by Montroll and Shlesinger ^[7, 8]

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Later: Huberman and Adamic ^[3, 4]: Number of pages per website

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Lognormals and power laws can be awfully similar

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Random Multiplicative Growth leads to lognormal distributions

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Enforcing a minimum size leads to a power law tail

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-  Take-home message: Be careful out there...

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