

Branching Networks II

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Principles of Complex Systems,
Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
University of Vermont | Santa Fe Institute



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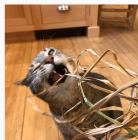
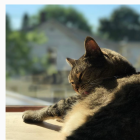
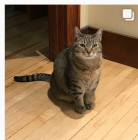
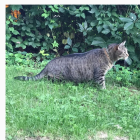
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 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

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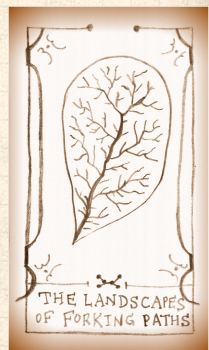
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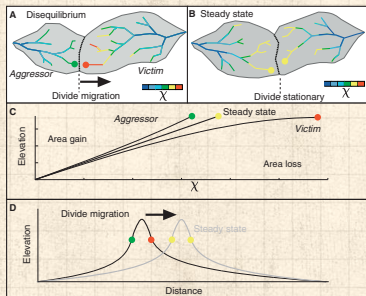




Piracy on the high χ 's:



“Dynamic Reorganization of River Basins” [↗](#)
 Willett et al.,
 Science, **343**, 1248765, 2014. ^[21]

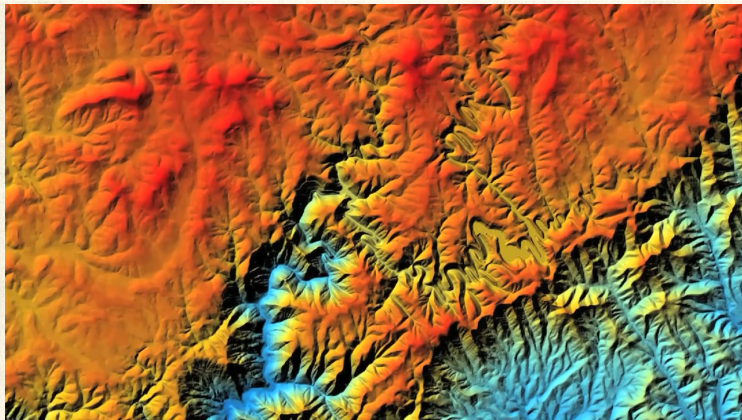


$$\frac{\partial z(x, t)}{\partial t} = U - K A^m \left| \frac{\partial z(x, t)}{\partial x} \right|^n$$

$$z(x) = z_b + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_b}^x \left(\frac{A_0}{A(x')} \right)^{m/n} dx'$$

Piracy on the high χ 's:



Story: How river networks move across a landscape  (Science Daily)

Source: https://www.youtube.com/watch?v=FnroL1_-12c 





THE LAWS OF
BRANCHING

Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

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Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:



In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.

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Horton and Tokunaga seem different:

-  In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
-  Oddly, Horton's laws have **four** parameters and Tokunaga has **two** parameters.



Can Horton and Tokunaga be happy?

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-  R_n, R_a, R_ℓ , and R_s **versus** T_1 and R_T . One simple redundancy: $R_\ell = R_s$.
Insert assignment question 



Can Horton and Tokunaga be happy?

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-  To make a connection, clearest approach is to start with Tokunaga's law ...



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Insert assignment question 
-  To make a connection, clearest approach is to start with Tokunaga's law ...
-  Known result: Tokunaga $\rightarrow$ Horton ^[18, 19, 20, 9, 2]



Let us make them happy

We need one more ingredient:

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Let us make them happy

We need one more ingredient:

Space-fillingness

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Let us make them happy

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Space-fillingness



A network is **space-filling** if the average distance between adjacent streams is roughly constant.

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Reasonable for river and cardiovascular networks

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Let us make them happy

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-  A network is **space-filling** if the average distance between adjacent streams is roughly constant.
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-  For river networks:
Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.



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-  In terms of basin characteristics:

$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}}$$



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$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_{\omega} \bar{s}_{\omega}}{a_{\Omega}}$$



More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

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Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$



Start looking for Horton's stream number law:

$$n_\omega / n_{\omega+1} = R_n.$$



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Estimate n_ω , the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.



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Observe that each stream of order ω terminates by either:



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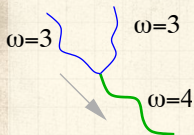
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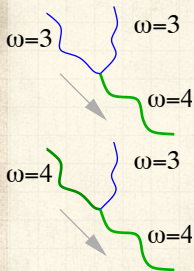
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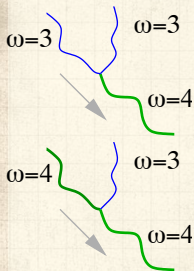
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▶ $2n_{\omega+1}$ streams of order ω do this
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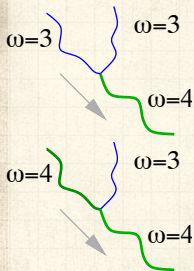
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 Observe that each stream of order ω terminates by either:



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▶ $2n_{\omega+1}$ streams of order ω do this
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...
▶ $n_{\omega'} T_{\omega'-\omega}$ streams of order ω do this



More with the happy-making thing

Putting things together:



$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} +$$

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More with the happy-making thing

Putting things together:



$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$

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Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain R_n .



Insert assignment question 



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Insert assignment question 



Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



Finding other Horton ratios

Connect Tokunaga to R_s

 Now use uniform drainage density ρ_{dd} .

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Finding other Horton ratios

Connect Tokunaga to R_s



Now use uniform drainage density ρ_{dd} .



Assume side streams are roughly separated by distance $1/\rho_{dd}$.

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- For an order ω **stream segment**, expected length is

$$\bar{s}_{\omega} \simeq \rho_{\text{dd}}^{-1} \left(1 + \sum_{k=1}^{\omega-1} T_k \right)$$



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$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right)$$



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$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^\omega$$



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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T$$

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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



Recall $R_\ell = R_s$ so

$$R_\ell = R_s = R_T$$



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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



Recall $R_\ell = R_s$ so

$$R_\ell = R_s = R_T$$



And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$



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Some observations:

 R_n and R_ℓ depend on T_1 and R_T .



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Some observations:

 R_n and R_ℓ depend on T_1 and R_T .

 Seems that R_a must as well ...



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Some observations:

- 🧱 R_n and R_ℓ depend on T_1 and R_T .
- 🧱 Seems that R_a must as well ...
- 🧱 Suggests Horton's laws must contain some redundancy



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Some observations:

-  R_n and R_ℓ depend on T_1 and R_T .
-  Seems that R_a must as well ...
-  Suggests Horton's laws must contain some redundancy
-  We'll in fact see that $R_a = R_n$.



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-  R_n and R_ℓ depend on T_1 and R_T .
-  Seems that R_a must as well ...
-  Suggests Horton's laws must contain some redundancy
-  We'll in fact see that $R_a = R_n$.
-  Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]



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The other way round



Note: We can invert the expressions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.



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 Note: We can invert the expressions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.



$$R_T = R_\ell,$$



$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$



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$$R_T = R_\ell,$$



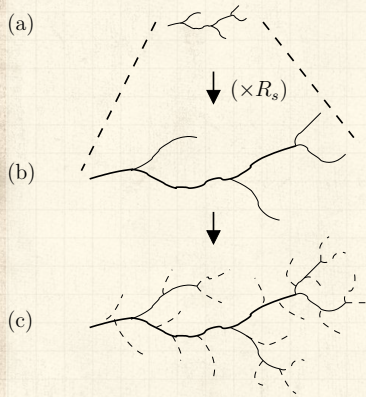
$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$

 Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...



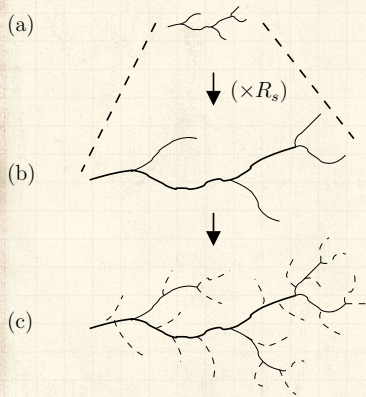
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From Horton to Tokunaga [2]



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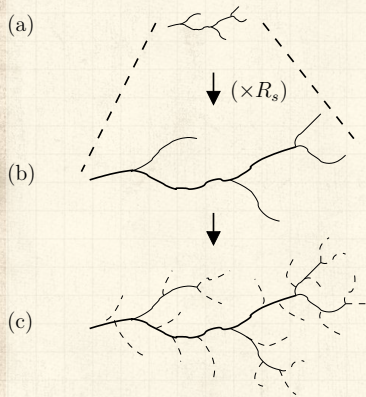


Assume Horton's laws hold
for number and length



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From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length

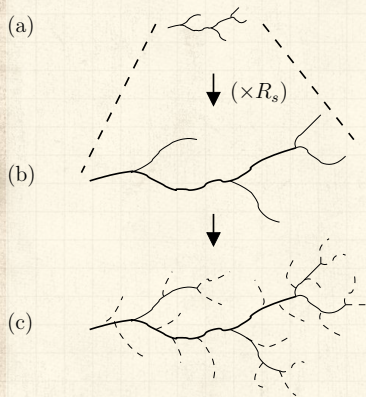


Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.



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From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length



Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.

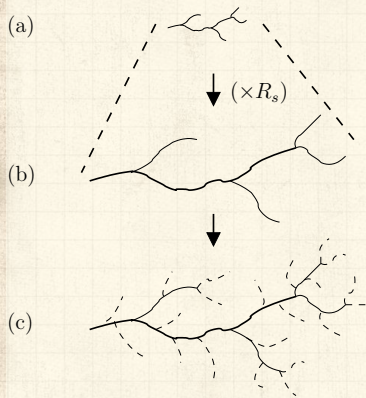


Scale up by a factor of R_ℓ , orders increment to $\omega + 1$ and ω .



Horton and Tokunaga are friends

From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length



Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.



Scale up by a factor of R_ℓ , orders increment to $\omega + 1$ and ω .



Maintain drainage density by adding new order $\omega - 1$ streams



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...and in detail:



Must retain same drainage density.

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Horton and Tokunaga are friends

...and in detail:

-  Must retain same drainage density.
-  Add an extra $(R_\ell - 1)$ first order streams for each original tributary.

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...and in detail:

-  Must retain same drainage density.
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-  Since by definition, an order $\omega + 1$ stream segment has T_ω order 1 side streams, we have:



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$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right).$$



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-  For large ω , Tokunaga's law is the solution—let's check ...



Horton and Tokunaga are friends

Just checking:



Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right)$$

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Horton and Tokunaga are friends

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$$\begin{aligned} T_k &= (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_1 R_\ell^{i-1} \right) \\ &= (R_\ell - 1) \left(1 + T_1 \frac{R_\ell^{k-1} - 1}{R_\ell - 1} \right) \end{aligned}$$

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$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{k-1}}{R_\ell - 1}$$



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Horton's laws of area and number:

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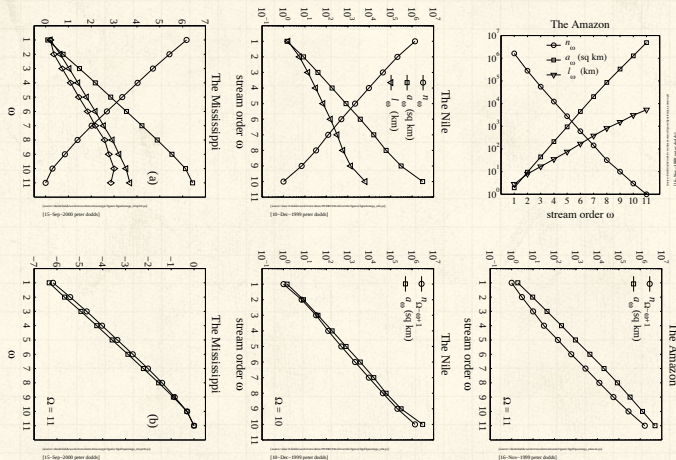
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In bottom plots, stream number graph has been flipped vertically.



Highly suggestive that $R_w \equiv R_{w-1}$

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How robust are our estimates of ratios?



Measuring Horton ratios is tricky:

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How robust are our estimates of ratios?



Rule of thumb: discard data for two smallest and two largest orders.



Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3, 8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7, 8]	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019



Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

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Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

 $a_\Omega \propto$ sum of all stream segment lengths in a order Ω basin
(assuming uniform drainage density)

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$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega}}_{n_\omega} \cdot \overset{n_\Omega}{\hat{1}}$$



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 So:

$$\begin{aligned} a_\Omega &\simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\text{dd}} \\ &\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \hat{1}}_{n_\omega} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_\omega} \\ &= \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^\omega \end{aligned}$$



Reducing Horton's laws:

Continued ...



$$a_{\Omega} \propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega}$$

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Reducing Horton's laws:

Continued ...



$$\begin{aligned} a_{\Omega} &\propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega} \\ &= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)} \end{aligned}$$

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Reducing Horton's laws:

Continued ...



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Reducing Horton's laws:

Continued ...



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So, a_{Ω} is growing like R_n^{Ω} and therefore:

$$R_n \equiv R_a$$



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Not quite:



...But this only a rough argument as Horton's laws do not imply a strict hierarchy



Reducing Horton's laws:

Not quite:

- ...But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.



Reducing Horton's laws:

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Not quite:

 ...But this only a rough argument as Horton's laws do not imply a strict hierarchy

 Need to account for sidebranching.

 Insert assignment question 



Equipartitioning:

Intriguing division of area:

 Observe: Combined area of basins of order ω independent of ω .

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Equipartitioning:

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-  Observe: Combined area of basins of order ω independent of ω .
-  Not obvious: basins of low orders not necessarily contained in basin on higher orders.



Equipartitioning:

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- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$



Equipartitioning:

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$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

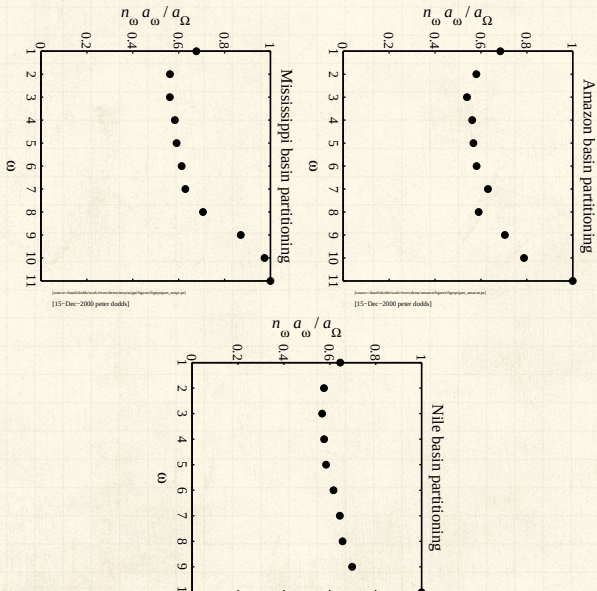
 Reason:

$$n_\omega \propto (R_n)^{-\omega}$$
$$\bar{a}_\omega \propto (R_a)^\omega \propto n_\omega^{-1}$$



Equipartitioning:

Some examples:



Neural Reboot: Fwoompf

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<https://www.youtube.com/watch?v=5mUs70SqD4o?rel=0> 



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The story so far:



Natural branching networks are **hierarchical, self-similar** structures

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The story so far:



Natural branching networks are **hierarchical, self-similar** structures



Hierarchy is **mixed**



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The story so far:

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 Tokunaga's law describes detailed architecture:
$$T_k = T_1 R_T^{k-1}.$$



Scaling laws

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 Only two Horton laws are independent ($R_n = R_a$)



Scaling laws

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 Only two Horton laws are independent ($R_n = R_a$)

 Only **two** parameters are **independent**:
$$(T_1, R_T) \Leftrightarrow (R_n, R_s)$$



Scaling laws

A little further ...

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A little further ...



Ignore stream ordering for the moment

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A little further ...



Ignore stream ordering for the moment



Pick a random location on a branching network p .



Scaling laws

A little further ...

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-  Each point p is associated with a basin and a longest stream length



Scaling laws

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 $P(a) \propto a^{-\tau}$ for large a
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Scaling laws

A little further ...

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 $P(a) \propto a^{-\tau}$ for large a
-  Q: What is probability that the longest stream from p has length ℓ ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
-  Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$



Scaling laws

Probability distributions with power-law decays

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Probability distributions with power-law decays



We see them everywhere:



Earthquake magnitudes (Gutenberg-Richter law)



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Probability distributions with power-law decays



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Earthquake magnitudes (Gutenberg-Richter law)



City sizes (Zipf's law)



Scaling laws

Probability distributions with power-law decays



We see them everywhere:

-  Earthquake magnitudes (Gutenberg-Richter law)
-  City sizes (Zipf's law)
-  Word frequency (Zipf's law) ^[22]



Probability distributions with power-law decays



We see them everywhere:

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Probability distributions with power-law decays



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Probability distributions with power-law decays



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A big part of the story of complex systems



Probability distributions with power-law decays



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A big part of the story of complex systems



Arise from **mechanisms**: growth, randomness, optimization,

...



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A big part of the story of complex systems



Arise from **mechanisms**: growth, randomness, optimization,

...



Our task is always to illuminate the mechanism ...



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Connecting exponents



We have the detailed picture of branching networks
(Tokunaga and Horton)

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Connecting exponents



We have the detailed picture of branching networks (Tokunaga and Horton)



Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story^[17, 1, 2]



Connecting exponents



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Connecting exponents



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Our first fudge: assume Horton's laws hold throughout a basin of order Ω .



Scaling laws

Connecting exponents

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Scaling laws

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- ✉ Next: place stick between teeth.



Scaling laws

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Scaling laws

Finding γ :

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Scaling laws

Finding γ :

 Often useful to work with **cumulative distributions**, especially when dealing with power-law distributions.

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Scaling laws

Finding γ :

- Often useful to work with **cumulative distributions**, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell) d\ell$$



Scaling laws

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$$P_{>}(\ell_*) = 1 - P(\ell < \ell_*)$$



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Also known as the exceedance probability.



Scaling laws

Finding γ :



The connection between $P(x)$ and $P_{>}(x)$ when $P(x)$ has a power law tail is simple:

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Scaling laws

Finding γ :



The connection between $P(x)$ and $P_{>}(x)$ when $P(x)$ has a power law tail is simple:



Given $P(\ell) \sim \ell^{-\gamma}$ large ℓ then for large enough ℓ_*

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$$= \left. \frac{\ell^{-(\gamma-1)}}{-(\gamma-1)} \right|_{\ell=\ell_*}^{\ell_{\max}}$$



Scaling laws

Finding γ :

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$$\begin{aligned} P_{>}(\ell_*) &= \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell) d\ell \\ &\sim \int_{\ell=\ell_*}^{\ell_{\max}} \ell^{-\gamma} d\ell \\ &= \frac{\ell^{-(\gamma-1)}}{-(\gamma-1)} \bigg|_{\ell=\ell_*}^{\ell_{\max}} \\ &\propto \ell_*^{-(\gamma-1)} \quad \text{for } \ell_{\max} \gg \ell_* \end{aligned}$$



Scaling laws

Finding γ :

 **Aim:** determine probability of randomly choosing a point on a network with main stream length $> \ell_*$

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Scaling laws

Finding γ :

-  **Aim:** determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
-  Assume some spatial sampling resolution Δ



Scaling laws

Finding γ :

-  **Aim:** determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
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-  Landscape is broken up into grid of $\Delta \times \Delta$ sites



Scaling laws

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-  **Aim:** determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
-  Assume some spatial sampling resolution Δ
-  Landscape is broken up into grid of $\Delta \times \Delta$ sites
-  Approximate $P_>(\ell_*)$ as

$$P_>(\ell_*) = \frac{N_>(\ell_*; \Delta)}{N_>(0; \Delta)}.$$

where $N_>(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.



Scaling laws

Finding γ :

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where $N_>(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.

-  Use Horton's law of stream segments: $\bar{s}_\omega / \bar{s}_{\omega-1} = R_s \dots$



Scaling laws

Finding γ :

 Set $\ell_* = \bar{\ell}_\omega$ for some $1 \ll \omega \ll \Omega$.

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$$P_{>}(\bar{\ell}_\omega) = \frac{N_{>}(\bar{\ell}_\omega; \Delta)}{N_{>}(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}$$



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Finding γ :

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 Δ 's cancel



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 Denominator is $a_\Omega \rho_{\text{dd}}$, a constant.



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 So ...

$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'}$$



Scaling laws

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$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega}$$



Scaling laws

Finding γ :

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$$P_{>}(\bar{\ell}_\omega) = \frac{N_{>}(\bar{\ell}_\omega; \Delta)}{N_{>}(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \cancel{\Delta}}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \cancel{\Delta}}$$

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 So ...using Horton's laws ...

$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})$$



Scaling laws

Finding γ :

 Set $\ell_* = \bar{\ell}_\omega$ for some $1 \ll \omega \ll \Omega$.



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$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$



Scaling laws

Finding γ :



We are here:

$$P_{>}(\bar{\ell}_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$

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Finding γ :



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Cleaning up irrelevant constants:

$$P_{>}(\bar{\ell}_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega'}$$



Scaling laws

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Change summation order by substituting $\omega'' = \Omega - \omega'$.



Scaling laws

Finding γ :



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Scaling laws

Finding γ :



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Change summation order by substituting $\omega'' = \Omega - \omega'$.



Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)



Scaling laws

Finding γ :



$$P_{>}(\bar{\ell}_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n} \right)^{\Omega-\omega''}$$

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Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,



Scaling laws

Finding γ :



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again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$



Scaling laws

Finding γ :



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again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$



Scaling laws

Finding γ :



Nearly there:

$$P_{>}(\bar{\ell}_{\omega}) \propto \left(\frac{R_n}{R_s} \right)^{-\omega}$$

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Scaling laws

Finding γ :



Nearly there:

$$P_{>}(\bar{\ell}_{\omega}) \propto \left(\frac{R_n}{R_s} \right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$



Scaling laws

Finding γ :



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Need to express right hand side in terms of $\bar{\ell}_{\omega}$.



Scaling laws

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 Recall that $\bar{\ell}_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$.



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Scaling laws

Finding γ :



Therefore:

$$P_{>}(\bar{\ell}_{\omega}) \propto e^{-\omega \ln(R_n/R_s)}$$

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Scaling laws

Finding γ :



Therefore:

$$P_{>}(\bar{\ell}_{\omega}) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}\right)^{-\ln(R_n/R_s)/\ln(R_s)}$$

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$$= \bar{\ell}_{\omega}^{-\ln R_n / \ln R_s + 1}$$

$$= \bar{\ell}_{\omega}^{-\gamma + 1}$$



Scaling laws

Finding γ :



And so we have:

$$\gamma = \ln R_n / \ln R_s$$

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Scaling laws

Finding γ :



And so we have:

$$\gamma = \ln R_n / \ln R_s$$



Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert assignment question



Scaling laws

Finding γ :



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Insert assignment question 



Such connections between exponents are called **scaling relations**



Scaling laws

Finding γ :



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Insert assignment question 



Such connections between exponents are called **scaling relations**



Let's connect to one last relationship: Hack's law



Scaling laws

Hack's law: ^[6]



$$\ell \propto a^h$$

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Scaling laws

Hack's law: ^[6]



$$\ell \propto a^h$$



Typically observed that $0.5 \lesssim h \lesssim 0.7$.



Scaling laws

Hack's law: ^[6]



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Use Horton laws to connect h to Horton ratios:

$$\bar{\ell}_\omega \propto R_s^\omega \text{ and } \bar{a}_\omega \propto R_n^\omega$$



Scaling laws

Hack's law: ^[6]



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Observe:

$$\bar{\ell}_\omega \propto e^{\omega \ln R_s}$$



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$$\bar{\ell}_\omega \propto e^{\omega \ln R_s} \propto (e^{\omega \ln R_n})^{\ln R_s / \ln R_n}$$



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$$\bar{\ell}_\omega \propto e^{\omega \ln R_s} \propto (e^{\omega \ln R_n})^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n}$$



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Hack's law: ^[6]



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$$\bar{\ell}_\omega \propto e^{\omega \ln R_s} \propto (e^{\omega \ln R_n})^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n} \propto \bar{a}_\omega^{\ln R_s / \ln R_n}$$



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Hack's law: ^[6]



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Use Horton laws to connect h to Horton ratios:

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Observe:

$$\bar{\ell}_\omega \propto e^{\omega \ln R_s} \propto (e^{\omega \ln R_n})^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n} \propto \bar{a}_\omega^{\ln R_s / \ln R_n} \Rightarrow \boxed{h = \ln R_s / \ln R_n}$$



We mentioned there were a good number of ‘laws’: [2]

Relation:	Name or description:
$T_k = T_1(R_T)^{k-1}$	Tokunaga’s law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega/n_{\omega+1} = R_n$	Horton’s law of stream numbers
$\ell_{\omega+1}/\ell_\omega = R_\ell$	Horton’s law of main stream lengths
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	Horton’s law of basin areas
$\bar{s}_{\omega+1}/\bar{s}_\omega = R_s$	Horton’s law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack’s law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein’s law
$\lambda \sim L^\varphi$	variation of Langbein’s law



Connecting exponents

Only 3 parameters are independent:

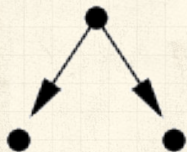
e.g., take d , R_n , and R_s

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$ $R_T = R_s$
$n_\omega/n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



Scheidegger's model

Directed random networks ^[11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$



Functional form of all scaling laws exhibited but exponents differ from real world ^[15, 16, 14]



Useful and interesting test case



A toy model—Scheidegger's model

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Reducing Horton

Scaling relations

Fluctuations

Models

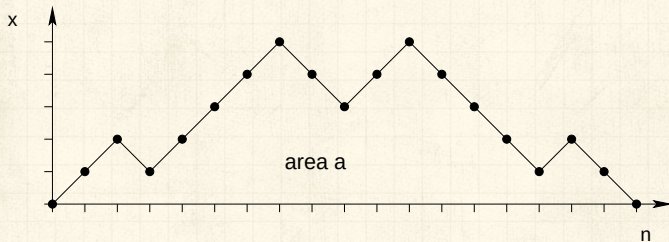
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Random walk basins:



Boundaries of basins are random walks



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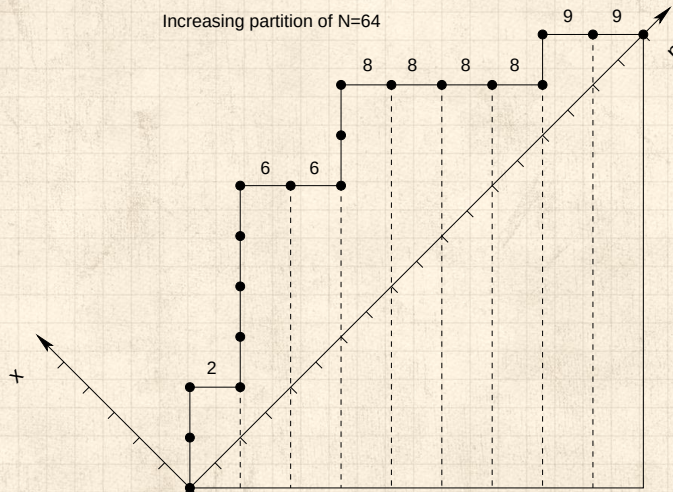
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Scheidegger's model

Prob for first return of a random walk in $(1+1)$ dimensions
(from CSYS/MATH 300):

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Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions
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$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so $P(\ell) \propto \ell^{-3/2}$.

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Typical area for a walk of length n is $\propto n^{3/2}$:

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Find $\tau = 4/3, h = 2/3, \gamma = 3/2, d = 1$.



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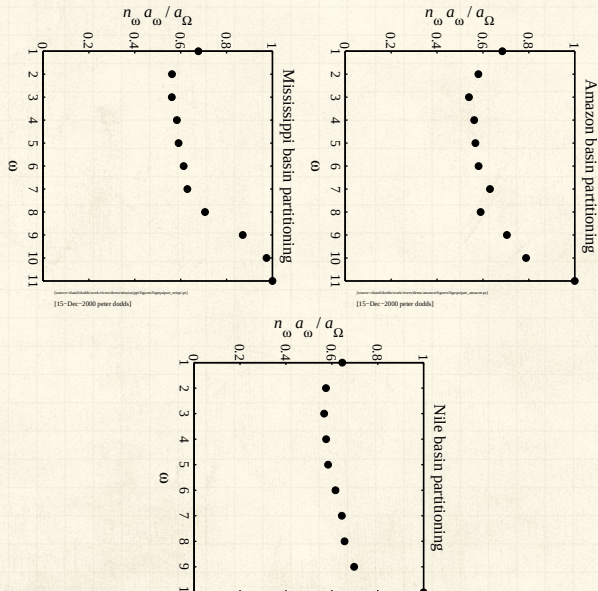


R_n and R_ℓ have not been derived analytically.



Equipartitioning reexamined:

Recall this story:



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What about

$$P(a) \sim a^{-\tau} \quad ?$$



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Since $\tau > 1$, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$



Equipartitioning

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$P(a)$ overcounts basins within basins ...



Equipartitioning



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Since $\tau > 1$, suggests no equipartitioning:

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$P(a)$ overcounts basins within basins ...



while stream ordering separates basins ...



Fluctuations

Moving beyond the mean:

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Moving beyond the mean:

 Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$



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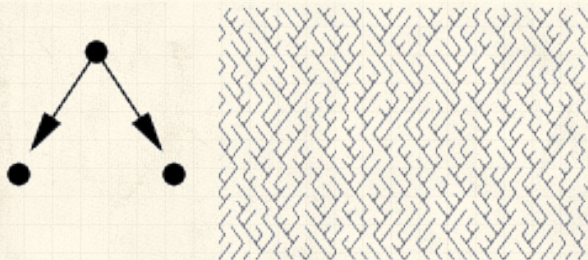
$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$

- Natural generalization to consider relationships between **probability distributions**
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



A toy model—Scheidegger's model

Directed random networks ^[11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$



Flow is directed downwards

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Generalizing Horton's laws



$$\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$$

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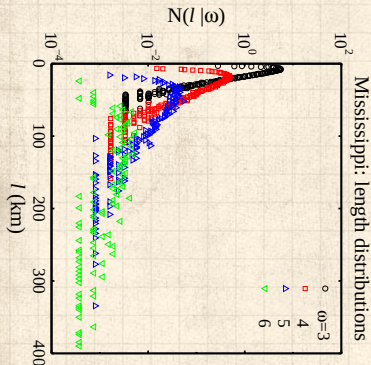

$$\bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$$



Generalizing Horton's laws

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[source: <https://www.researchgate.net/publication/221111111>]

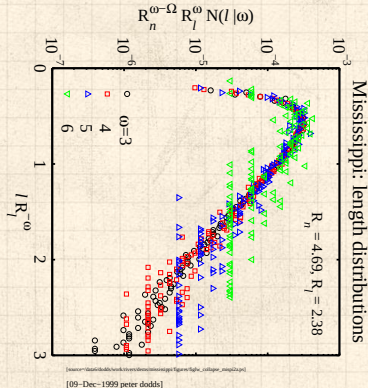
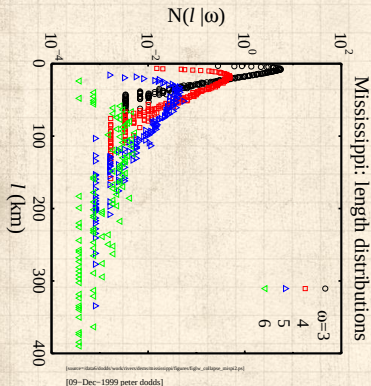
[09-Dec-1999 peter dodds]



Generalizing Horton's laws

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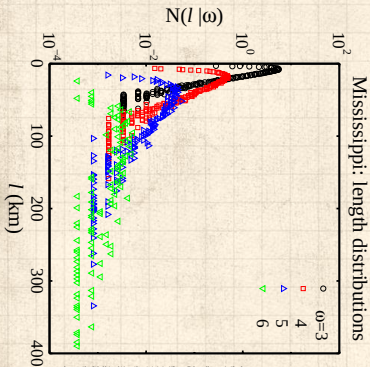
 Scaling collapse works well for intermediate orders



Generalizing Horton's laws

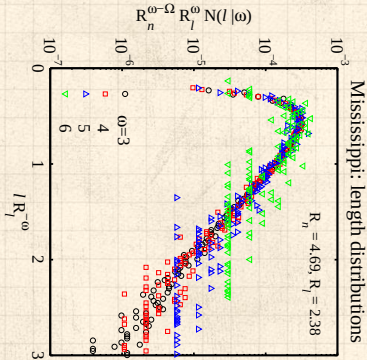
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[source="data.dodds@work/courses/courses/indiana/figures/figure_misg2.pdf"]

[09-Dec-1999 peter dodds]



[source="data.dodds@work/courses/courses/indiana/figures/figure_misg2.pdf"]

[09-Dec-1999 peter dodds]

 Scaling collapse works well for intermediate orders

 All **moments** grow exponentially with order



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Reducing Horton

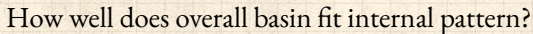
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[10-Dec-1999 peter dodds]



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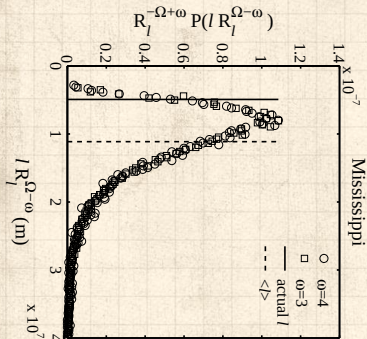
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How well does overall basin fit internal pattern?



Actual length = 4920 km (at 1 km res)



[source: "data of the Mississippi river system" by Peter Dodds, 10-Dec-1999]

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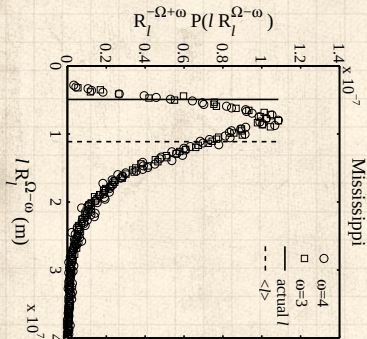
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How well does overall basin fit internal pattern?



[source: "data@addbbs.wisc.edu/river/streams/mississippi/figure_30new.jpg"]

[10-Dec-1999 peter dadds]

Actual length = 4920 km (at 1 km res)

Predicted Mean length = 11100 km



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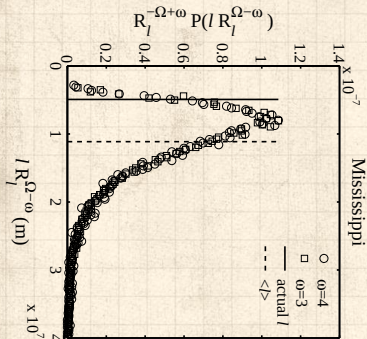
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[source: "data of the Mississippi River" by Peter Dodds, 1999]
[10-Dec-1999 peter dodds]



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Predicted Mean length = 11100 km



Predicted Std dev = 5600 km



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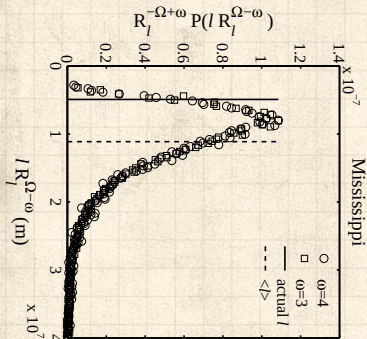
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 How well does overall basin fit internal pattern?



[source: "data of the Mississippi River network" by Peter Dodds, 1999]
[10-Dec-1999 peter dodds]

 Actual length = **4920 km** (at 1 km res)

 Predicted Mean length = **11100 km**

 Predicted Std dev = **5600 km**

 Actual length/Mean length = **44 %**



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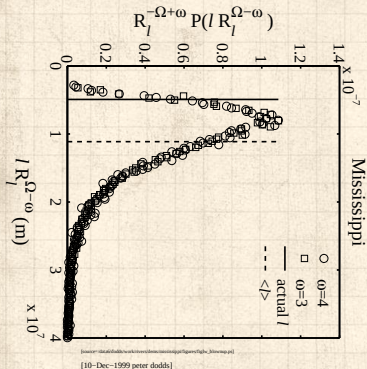
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How well does overall basin fit internal pattern?



Actual length = **4920 km** (at 1 km res)



Predicted Mean length = **11100 km**



Predicted Std dev = **5600 km**



Actual length/Mean length = **44 %**



Okay.



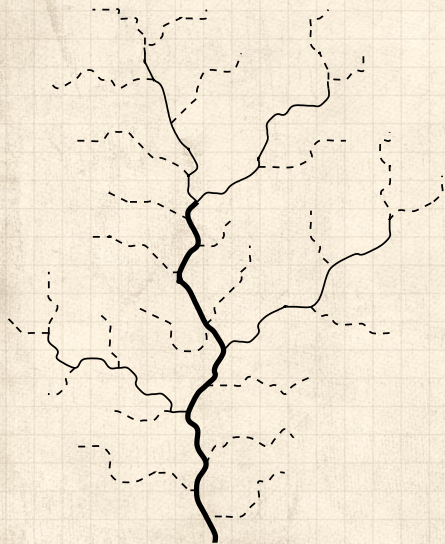
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



Combining stream segments distributions:



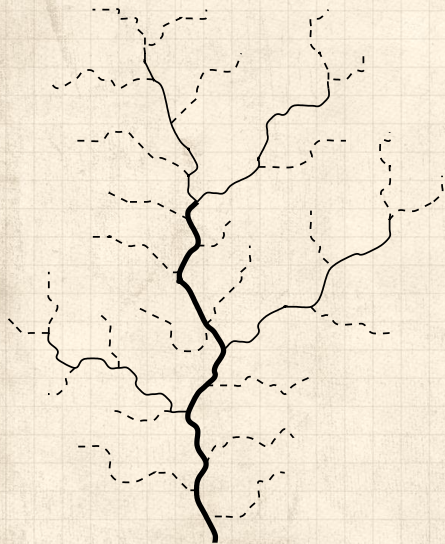
Stream segments sum
to give main stream
lengths



$$\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$



Combining stream segments distributions:



Stream segments sum
to give main stream
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$$\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$



$P(\ell_{\omega})$ is a
convolution of
distributions for the
 s_{ω}



Generalizing Horton's laws



Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \dots * N(s|\omega)$$

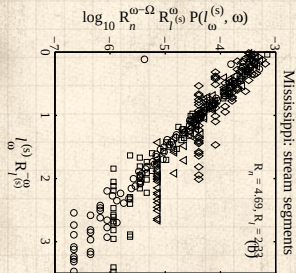


Generalizing Horton's laws



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[107-Dec-2000 peters.docx]

$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F(s/R_\ell^\omega)$$

$$F(x) = e^{-x/\xi}$$

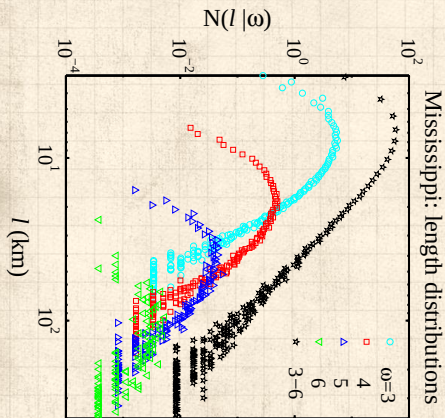
Mississippi: $\xi \simeq 900$ m.



Generalizing Horton's laws



Next level up: Main stream length distributions must combine to give overall distribution for stream length



[source: [data/dodds/works/river/dens/intro/figs/figs_figure_powerlawmain_mispt.ps](#)]

[22-Mar-2000 peter dodds]



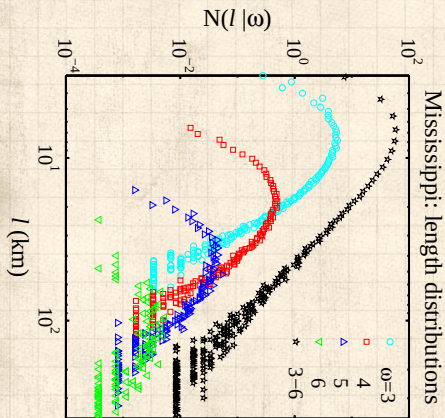
$$P(\ell) \sim \ell^{-\gamma}$$



Generalizing Horton's laws



Next level up: Main stream length distributions must combine to give overall distribution for stream length



[source: data/dodds/worksheets/dodds/mississippi_figures/figure_powerlaw_misippi.pdf]

[22-Mar-2000 peter dodds]



$$P(\ell) \sim \ell^{-\gamma}$$



Another round of convolutions [3]



Interesting ...



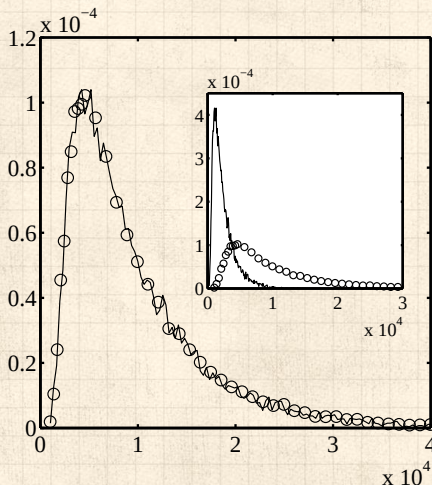
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Number and area distributions for the Scheidegger model [3]

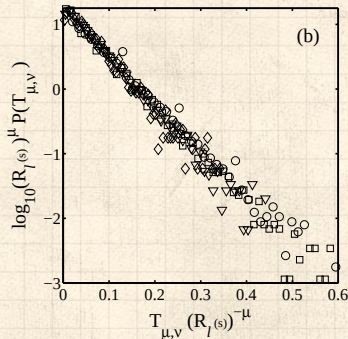
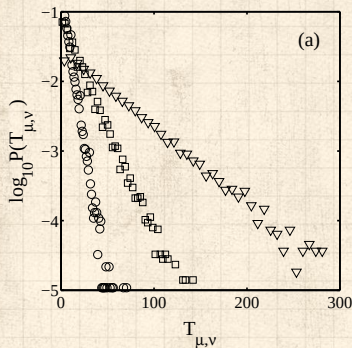


$P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.



Generalizing Tokunaga's law

Scheidegger:



Observe exponential distributions for $T_{\mu, \nu}$

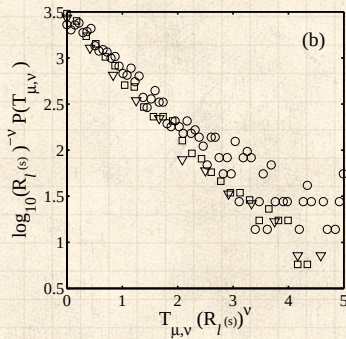
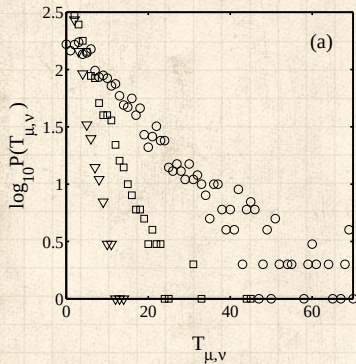


Scaling collapse works using R_s



Generalizing Tokunaga's law

Mississippi:



Same data collapse for Mississippi ...



Generalizing Tokunaga's law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t [T_{\mu,\nu}/(R_s)^{\mu-\nu-1}]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})$$



Exponentials arise from randomness.



Look at joint probability $P(s_\mu, T_{\mu,\nu})$.



Generalizing Tokunaga's law

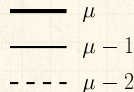
Network architecture:



Inter-tributary lengths exponentially distributed



Leads to random spatial distribution of stream segments



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Follow streams segments down stream from their beginning



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 Follow streams segments down stream from their beginning

 Probability (or rate) of an order μ stream segment terminating is **constant**:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$



Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is **constant**:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1} \xi_s$$

- Probability decays exponentially with stream order



Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
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Generalizing Tokunaga's law

 Follow streams segments down stream from their beginning

 Probability (or rate) of an order μ stream segment terminating is **constant**:

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 Probability decays exponentially with stream order

 Inter-tributary lengths exponentially distributed

 $\Rightarrow$ random spatial distribution of stream segments



Generalizing Tokunaga's law



Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

where



p_ν = probability of absorbing an order ν side stream



Generalizing Tokunaga's law



Joint distribution for generalized version of Tokunaga's law:

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where



p_ν = probability of absorbing an order ν side stream



$\tilde{p}_\mu$ = probability of an order μ stream terminating



Generalizing Tokunaga's law

 Joint distribution for generalized version of Tokunaga's law:

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where

 p_ν = probability of absorbing an order ν side stream

 $\tilde{p}_\mu$ = probability of an order μ stream terminating

 Approximation: depends on distance units of s_μ

 In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



Generalizing Tokunaga's law



Now deal with this thing:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

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Generalizing Tokunaga's law



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Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.



Generalizing Tokunaga's law

 Now deal with this thing:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

 Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.

 Obtain

$$P(x, y) = Nx^{-1/2} [F(y/x)]^x$$

where

$$F(v) = \left(\frac{1-v}{q} \right)^{-(1-v)} \left(\frac{v}{p} \right)^{-v}.$$



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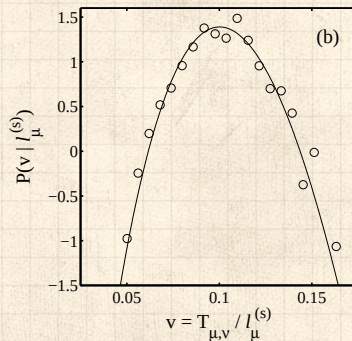
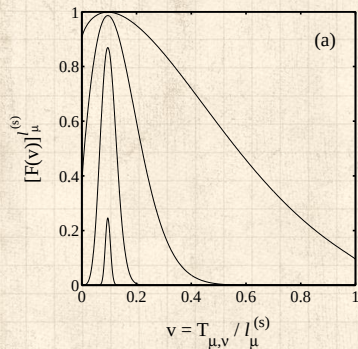
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Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:



Generalizing Tokunaga's law

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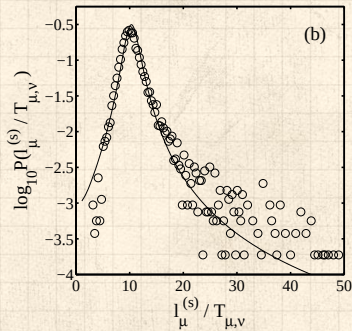
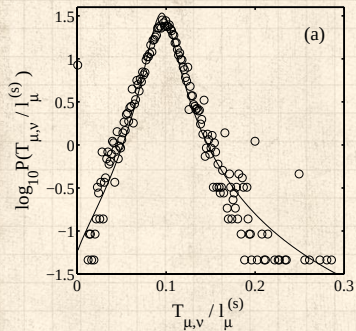
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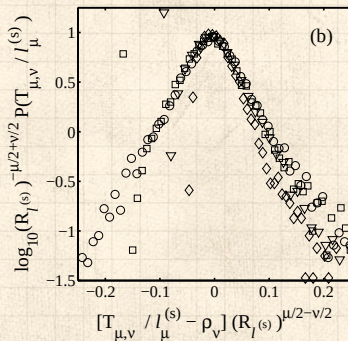
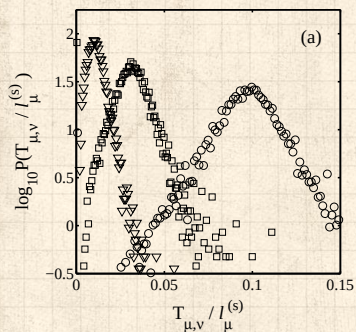
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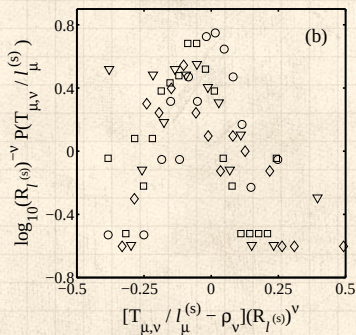
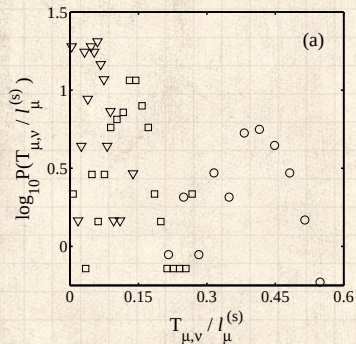
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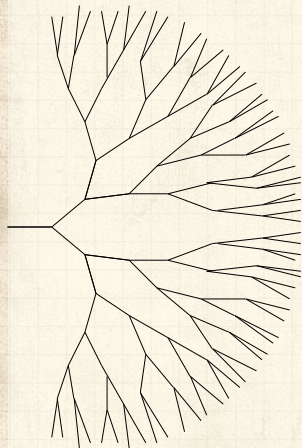


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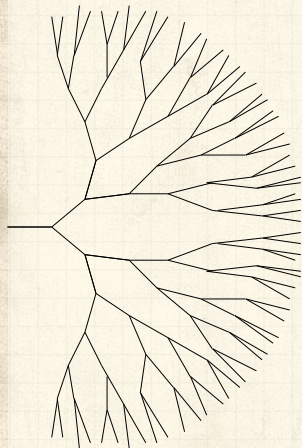
Mississippi:



Random subnetworks on a Bethe lattice ^[13]



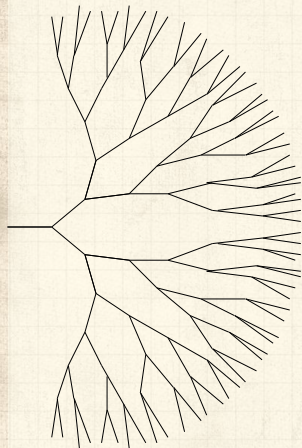
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Dominant theoretical concept for several decades.



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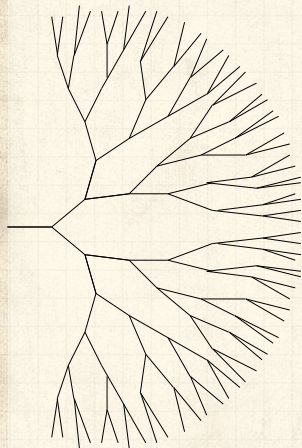
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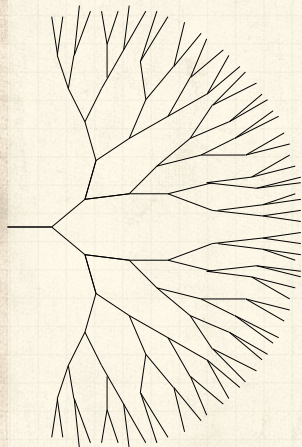
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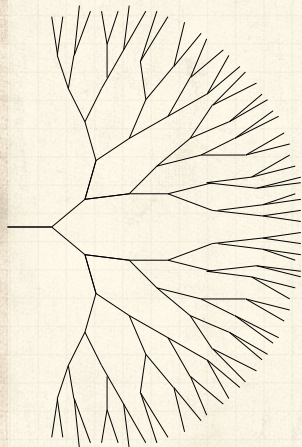
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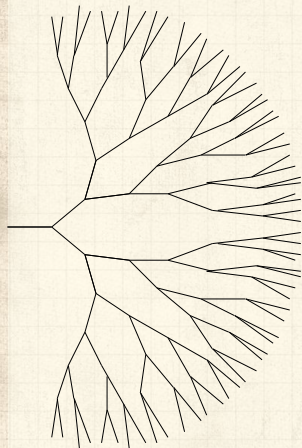
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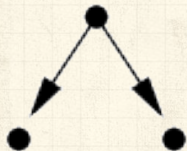


So let's move on ...



Scheidegger's model

Directed random networks ^[11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$



Functional form of all scaling laws exhibited but exponents differ from real world ^[15, 16, 14]



Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. ^[10]

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 Landscapes obtained numerically give exponents near that of real networks.

 **But:** numerical method used matters.

 **And:** Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network ^[8]



Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$$h \Rightarrow \ell \propto a^h \text{ (Hack's law).}$$

$$d \Rightarrow \ell \propto L_{\parallel}^d \text{ (stream self-affinity).}$$



Branching networks II Key Points:



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-  For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
-  Laws can be extended nicely to laws of distributions.
-  Numerous models of branching network evolution exist: nothing rock solid yet ...?



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