

Branching Networks I

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Principles of Complex Systems, Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

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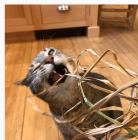
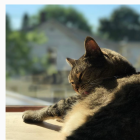
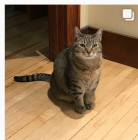
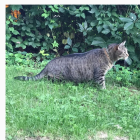
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 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

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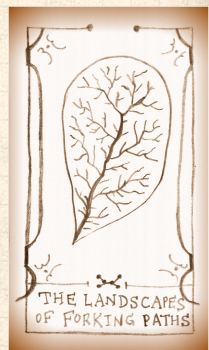
Horton's Laws

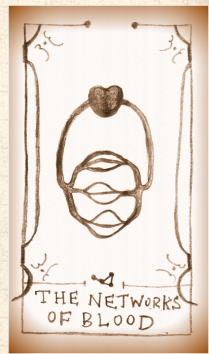
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Branching networks are useful things:



Fundamental to material **supply and collection**

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Branching networks are useful things:



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Supply: From one source to many sinks in 2- or 3-d.



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Branching networks are useful things:



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Supply: From one source to many sinks in 2- or 3-d.



Collection: From many sources to one sink in 2- or 3-d.



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Branching networks are useful things:

-  Fundamental to material **supply and collection**
-  **Supply:** From one source to many sinks in 2- or 3-d.
-  **Collection:** From many sources to one sink in 2- or 3-d.
-  Typically observe hierarchical, recursive self-similar structure



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Examples:



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Examples:

- 🧱 River networks (our focus)



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Examples:

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-  Plants
-  Evolutionary trees



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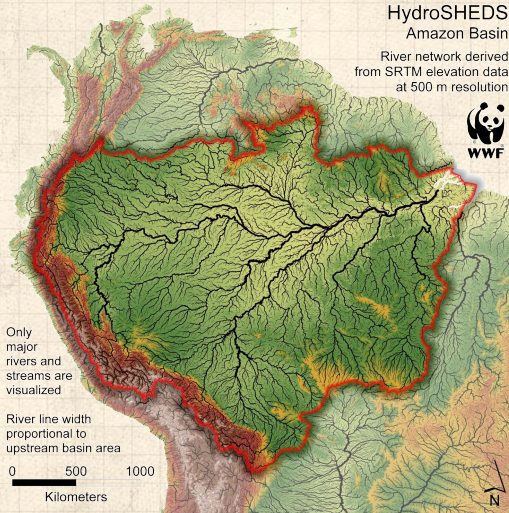
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-  Typically observe hierarchical, recursive self-similar structure

Examples:

-  River networks (our focus)
-  Cardiovascular networks
-  Plants
-  Evolutionary trees
-  Organizations (only in theory ...)



Branching networks are everywhere ...



<http://hydrosheds.cr.usgs.gov/>

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Branching networks are everywhere ...

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<http://en.wikipedia.org/wiki/Image:Applebox.JPG> 



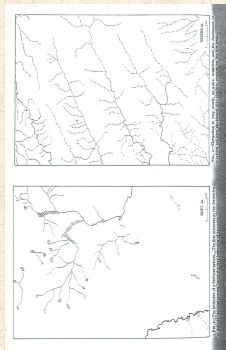
An early thought piece: Extension and Integration



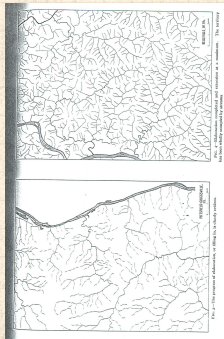
“The Development of Drainage Systems: A Synoptic View” ↗

Waldo S. Glock,

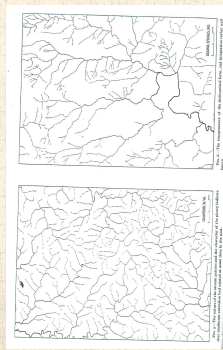
The Geographical Review, **21**, 475–482, 1931. [2]



Initiation,
Elongation



Elaboration, Piracy.



Abstraction,
Absorption.

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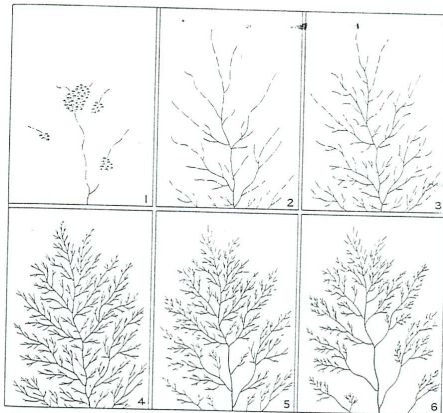


FIG. 8—An ideal diagrammatic summary of the development of a drainage system given for purposes of comparison only. The first four parts show extension, thus: 1, initiation; 2, elongation; 3, elaboration; and 4, maximum extension. Parts 5 and 6 represent steps during integration.

The sequential stages recognized in the evolution of a drainage system are “extension” and “integration”; the first, a stage of increasing complexity; the second, of simplification.



Shaw and Magnasco's beautiful erosion simulations



Unpublished.



Though to be destroyed and lost.



The VHS.

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Drainage basin for a point p is the complete region of land from which overland flow drains through p .



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 Definition most sensible for a point in a stream.



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 **Recursive structure:** Basins contain basins and so on.



Geomorphological networks

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 In principle, a drainage basin is defined at every point on a landscape.



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 On flat hillslopes, drainage basins are effectively linear.



Definitions

-  **Drainage basin** for a point p is the complete region of land from which overland flow drains through p .
-  Definition most sensible for a point in a stream.
-  **Recursive structure:** Basins contain basins and so on.
-  In principle, a drainage basin is defined at every point on a landscape.
-  On flat hillslopes, drainage basins are effectively linear.
-  We treat subsurface and surface flow as following the gradient of the surface.

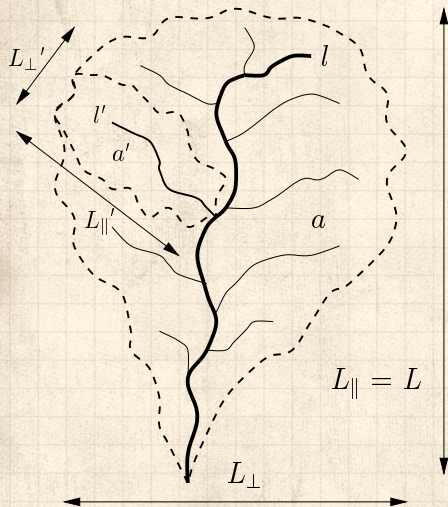


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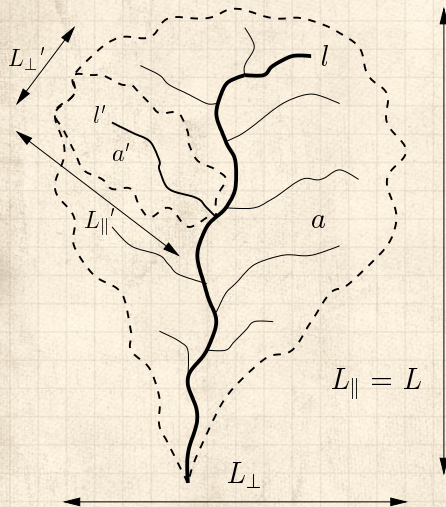
-  **Drainage basin** for a point p is the complete region of land from which overland flow drains through p .
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-  On flat hillslopes, drainage basins are effectively linear.
-  We treat subsurface and surface flow as following the gradient of the surface.
-  Okay for large-scale networks ...



Basic basin quantities: $a, l, L_{\parallel}, L_{\perp}$:



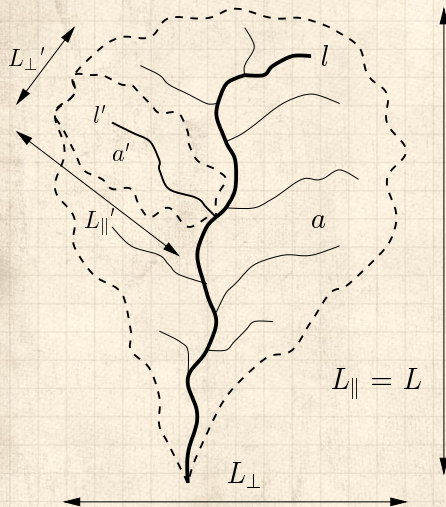
Basic basin quantities: $a, l, L_{\parallel}, L_{\perp}$:



a = drainage basin
area



Basic basin quantities: a , l , $L_{\parallel}$, $L_{\perp}$:



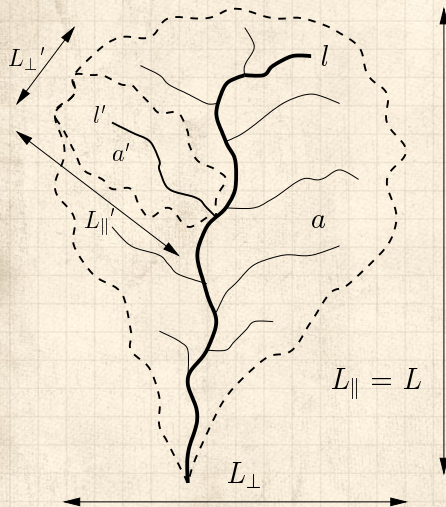
a = drainage basin area



l = length of longest (main) stream (which may be fractal)



Basic basin quantities: $a, l, L_{\parallel}, L_{\perp}$:



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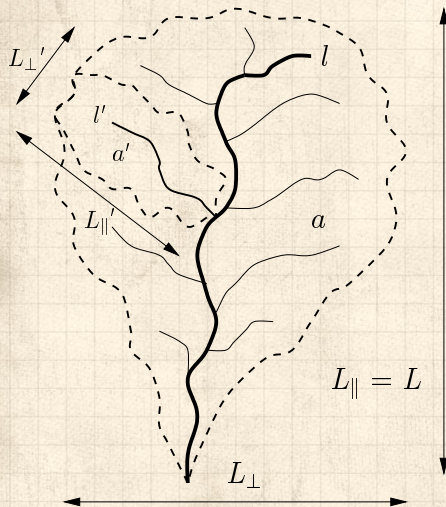
l = length of longest (main) stream (which may be fractal)



$L = L_{\parallel}$ = longitudinal length of basin



Basic basin quantities: $a, l, L_{\parallel}, L_{\perp}$:



-  a = drainage basin area
-  l = length of longest (main) stream (which may be fractal)
-  $L = L_{\parallel}$ = longitudinal length of basin
-  $L = L_{\perp}$ = width of basin



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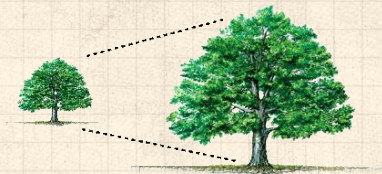
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Isometry:

dimensions scale
linearly with each
other.

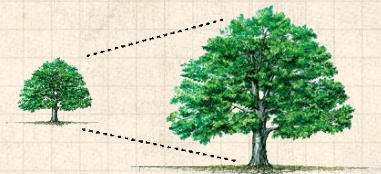


Allometry



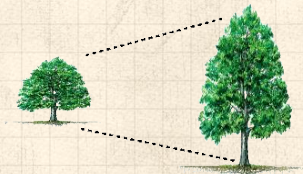
Isometry:

dimensions scale
linearly with each
other.



Allometry:

dimensions scale
nonlinearly.



Basin allometry

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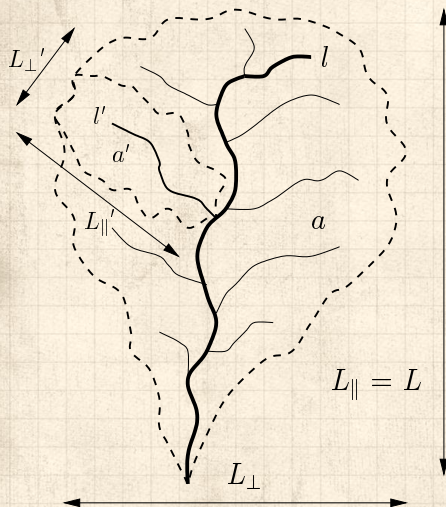
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relationships:



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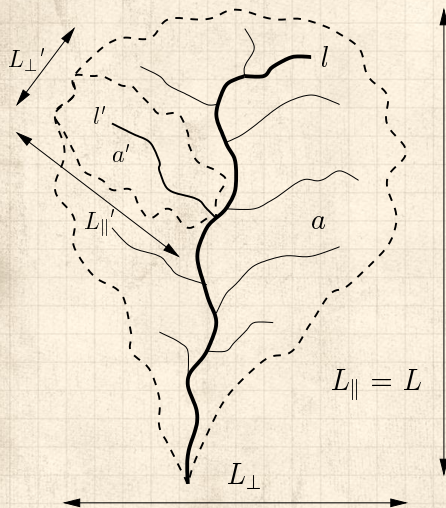
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Allometric
relationships:



$$\ell \propto a^h$$



Basin allometry

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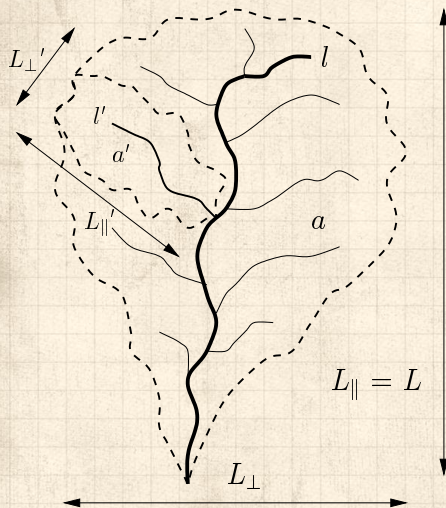
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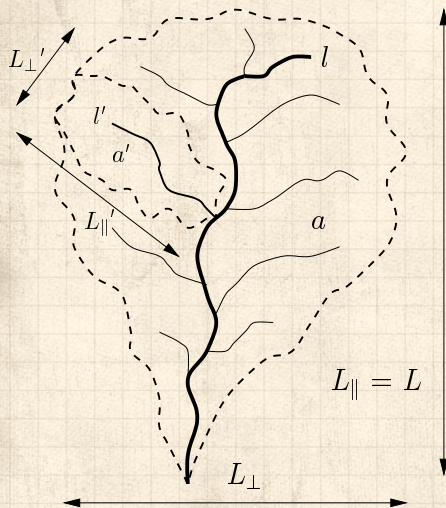
$$\ell \propto a^h$$



$$\ell \propto L^d$$



Basin allometry



Allometric relationships:



$$\ell \propto a^h$$



$$\ell \propto L^d$$



Combine above:

$$a \propto L^{d/h} \equiv L^D$$



‘Laws’



Hack's law (1957) ^[3]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$

‘Laws’



Hack's law (1957) ^[3]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$



Scaling of main stream length with basin size:

$$\ell \propto L_{||}^d$$

reportedly $1.0 < d < 1.1$

‘Laws’



Hack's law (1957) ^[3]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$



Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^d$$

reportedly $1.0 < d < 1.1$



Basin allometry:

$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

$D < 2 \rightarrow$ basins elongate.

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There are a few more ‘laws’: ^[1]

Relation:	Name or description:
$T_k = T_1(R_T)^{k-1}$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega/n_{\omega+1} = R_n$	Horton's law of stream numbers
$\ell_{\omega+1}/\ell_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1}/\bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law



Reported parameter values: ^[1]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05



Kind of a mess ...

Order of business:

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Kind of a mess ...

Order of business:

1. Find out how these relationships are connected.



Kind of a mess ...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.



Kind of a mess ...

Order of business:

1. Find out how these relationships are connected.
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3. Explain origins of these parameter values



Kind of a mess ...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): **Many attempts: not yet sorted out ...**





Stream Ordering:

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Method for describing network architecture:



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Method for describing network architecture:



Introduced by Horton (1945) ^[4]



Stream Ordering:

Method for describing network architecture:

 Introduced by Horton (1945) ^[4]

 Modified by Strahler (1957) ^[7]



Stream Ordering:

Method for describing network architecture:

- Introduced by Horton (1945) ^[4]
- Modified by Strahler (1957) ^[7]
- Term: Horton-Strahler Stream Ordering ^[5]



Stream Ordering:

Method for describing network architecture:

-  Introduced by Horton (1945) ^[4]
-  Modified by Strahler (1957) ^[7]
-  Term: Horton-Strahler Stream Ordering ^[5]
-  Can be seen as **iterative trimming** of a network.



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Some definitions:



A **channel head** is a point in landscape where flow becomes focused enough to form a stream.



Stream Ordering:

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 A **channel head** is a point in landscape where flow becomes focused enough to form a stream.

 A **source stream** is defined as the stream that reaches from a channel head to a junction with another stream.



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 Roughly analogous to capillary vessels.



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 Roughly analogous to capillary vessels.

 Use symbol $\omega = 1, 2, 3, \dots$ for stream order.



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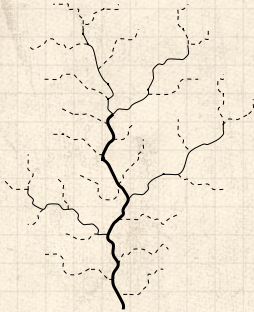
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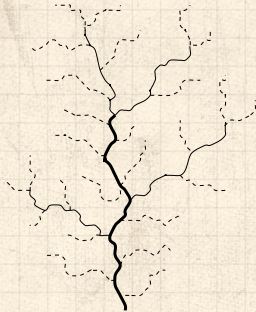
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1. Label all **source streams** as **order $\omega = 1$** and remove.



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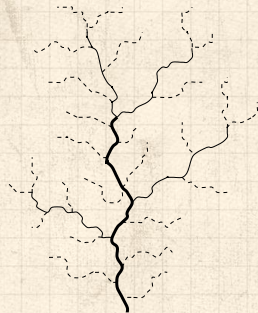
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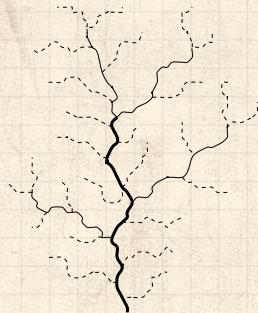
Stream Ordering:



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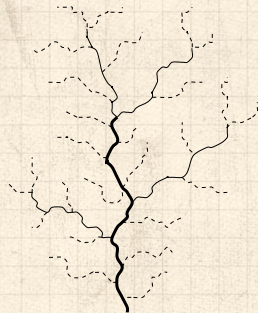
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Stream Ordering:



1. Label all **source streams** as **order $\omega = 1$** and remove.
2. Label all **new** source streams as **order $\omega = 2$** and remove.
3. Repeat until one stream is left (order = Ω)



Stream Ordering:



1. Label all **source streams** as **order $\omega = 1$** and remove.
2. Label all **new** source streams as **order $\omega = 2$** and remove.
3. Repeat until one stream is left (order = Ω)
4. Basin is said to be of the order of the last stream removed.



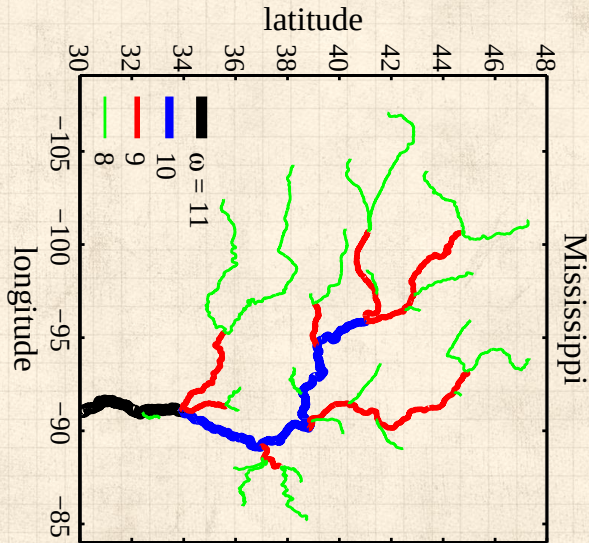
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5. Example above is a basin of order $\Omega = 3$.



Stream Ordering—A large example:



[source=/data6/dodds/work/rivers/dems/mississippi/figures/figorder_paths_mispi10.ps]

[21-Mar-2000 peter dodds]



Stream Ordering:

Another way to define ordering:

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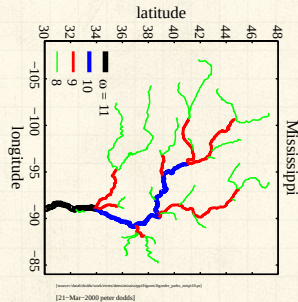
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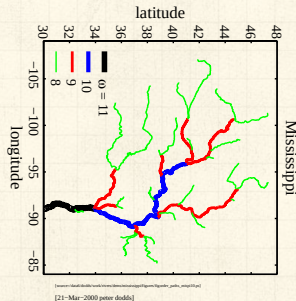


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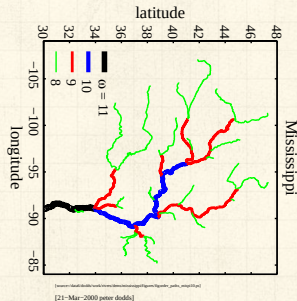


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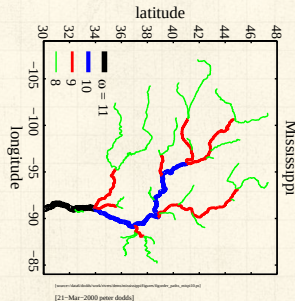
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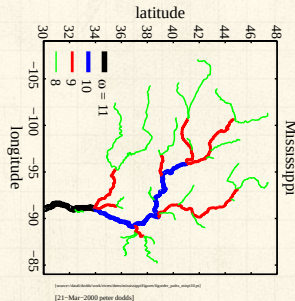


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- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.



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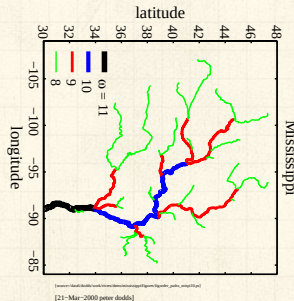
- As before, label all **source streams** as **order $\omega = 1$** .
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- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.

- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.



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One problem:



Resolution of data messes with ordering



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One problem:

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- Micro-description changes (e.g., order of a basin may increase)



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- Stream ordering helpfully discretizes a network.



Stream Ordering:

One problem:

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Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand **network architecture**



Basic algorithm for extracting networks from Digital Elevation Models (DEMs):

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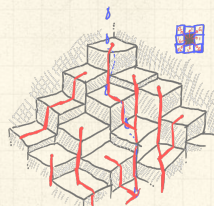
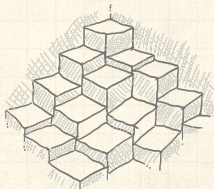
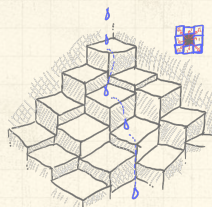
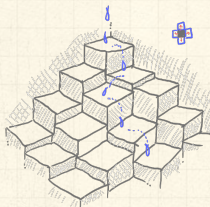
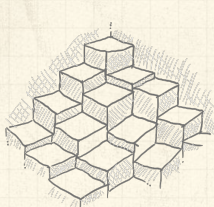
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Also:

`/Users/dodds/work/rivers/1998dems/kevinlakewaster.c`



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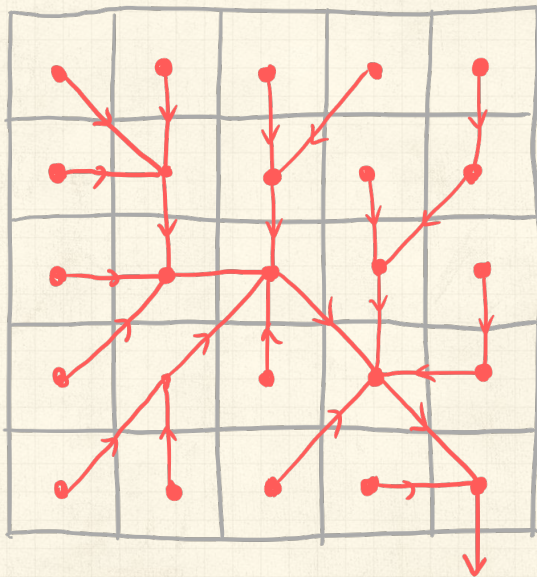
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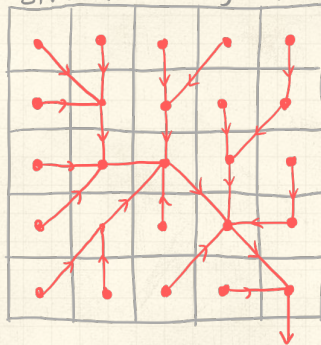
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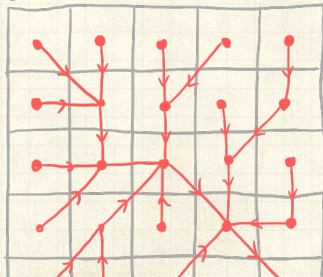
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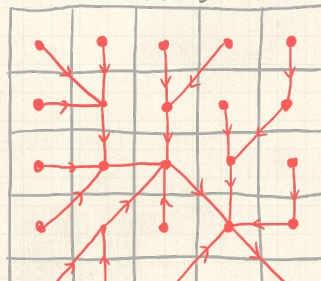
stream ordering w :



basin area a :



main stream length l :



Stream Ordering:

Resultant definitions:



A basin of order Ω has n_ω streams (or sub-basins) of order ω .



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1. an order ω stream segment is only that part of the stream which is actually of order ω
2. an order ω stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams





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BRANCHING

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Self-similarity of river networks

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Horton's laws

Self-similarity of river networks



First quantified by Horton (1945)^[4], expanded by Schumm (1956)^[6]



Horton's laws

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Three laws:



Horton's laws

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Three laws:

 Horton's law of stream numbers:

$$n_{\omega}/n_{\omega+1} = R_n > 1$$



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 Horton's law of basin areas:

$$\bar{a}_{\omega+1}/\bar{a}_{\omega} = R_a > 1$$



Horton's laws

Horton's Ratios:



So ...laws are defined by three ratios:

$$R_n, R_\ell, \text{ and } R_a.$$



Horton's Ratios:

 So ...laws are defined by three ratios:

$$R_n, R_\ell, \text{ and } R_a.$$

 Horton's laws describe **exponential decay or growth**:

$$\begin{aligned}n_\omega &= n_{\omega-1}/R_n \\&= n_{\omega-2}/R_n^2 \\&\vdots \\&= n_1/R_n^{\omega-1} \\&= n_1 e^{-(\omega-1)\ln R_n}\end{aligned}$$



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Similar story for area and length:

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Horton's laws

Similar story for area and length:



$$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$



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Horton's laws

Similar story for area and length:



$$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$



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As stream order increases, **number drops** and **area and length increase**.



Horton's laws

A few more things:

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A few more things:



Horton's laws are laws of averages.



Horton's laws

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-  Horton's ratios go a long way to defining a branching network ...



Horton's laws

A few more things:

-  Horton's laws are laws of averages.
-  Averaging for number is **across** basins.
-  Averaging for stream lengths and areas is **within** basins.
-  Horton's ratios go a long way to defining a branching network ...
-  But we need one other piece of information ...



Horton's laws

A bonus law:



Horton's law of stream segment lengths:

$$\bar{s}_{\omega+1} / \bar{s}_{\omega} = R_s > 1$$



Horton's laws

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 Can show that $R_s = R_{\ell}$.



Horton's laws

A bonus law:

 Horton's law of stream segment lengths:

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 Can show that $R_s = R_{\ell}$.

 Insert assignment question 



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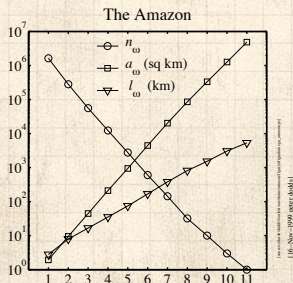
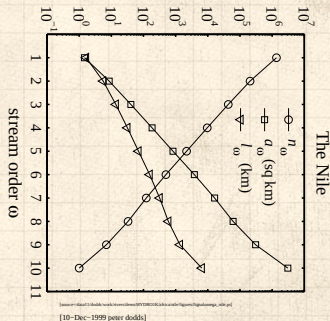
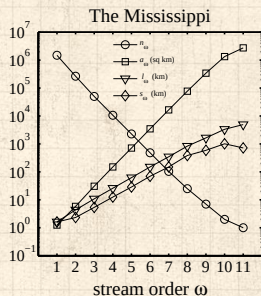
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Horton's laws-at-large

Blood networks:

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Blood networks:



Horton's laws hold for sections of cardiovascular networks

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Blood networks:

-  Horton's laws hold for sections of cardiovascular networks
-  Measuring such networks is tricky and messy ...



Horton's laws-at-large

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Blood networks:

-  Horton's laws hold for sections of cardiovascular networks
-  Measuring such networks is tricky and messy ...
-  Vessel diameters obey an analogous Horton's law.



Data from real blood networks

Network	R_n	R_r	R_ℓ	$-\frac{\ln R_r}{\ln R_n}$	$-\frac{\ln R_\ell}{\ln R_n}$	α
West <i>et al.</i>	–	–	–	1/2	1/3	3/4
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73
cat (PAT) ^[11]	3.67	1.71	1.78	0.41	0.44	0.79
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90
pig (LCX)	3.57	1.89	2.20	0.50	0.62	0.62
pig (RCA)	3.50	1.81	2.12	0.47	0.60	0.65
pig (LAD)	3.51	1.84	2.02	0.49	0.56	0.65
human (PAT)	3.03	1.60	1.49	0.42	0.36	0.83
human (PAT)	3.36	1.56	1.49	0.37	0.33	0.94



Horton's laws

Observations:



Horton's ratios vary:

$$R_n \quad 3.0-5.0$$

$$R_a \quad 3.0-6.0$$

$$R_\ell \quad 1.5-3.0$$



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 No accepted explanation for these values.



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Horton's laws tell us how quantities vary from level to level ...



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-------	---------

R_ℓ	1.5–3.0
----------	---------

 No accepted explanation for these values.

 Horton's laws tell us how quantities vary from level to level ...

 ...but they don't explain how networks are structured.



Tokunaga's law

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Delving deeper into network architecture:



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Delving deeper into network architecture:



Tokunaga (1968) identified a clearer picture of network structure [8, 9, 10]



Tokunaga's law

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 As per Horton-Strahler, use **stream ordering**.



Tokunaga's law

Delving deeper into network architecture:



Tokunaga (1968) identified a clearer picture of network structure [8, 9, 10]



As per Horton-Strahler, use **stream ordering**.



Focus: describe how streams of different orders connect to each other.



Tokunaga's law

Delving deeper into network architecture:

-  Tokunaga (1968) identified a clearer picture of network structure [8, 9, 10]
-  As per Horton-Strahler, use **stream ordering**.
-  **Focus:** describe how streams of different orders connect to each other.
-  Tokunaga's law is also a law of averages.



Definition:

 $T_{\mu,\nu}$ = the average number of **side streams** of **order ν** that enter as tributaries to streams of **order μ**



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 Recall each stream segment of order μ is 'generated' by two streams of order $\mu - 1$



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 $\mu, \nu = 1, 2, 3, \dots$

 $\mu \geq \nu + 1$

 Recall each stream segment of order μ is 'generated' by two streams of order $\mu - 1$

 These generating streams are not considered side streams.



Network Architecture

Tokunaga's law ^[8, 9, 10]

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Property 1: Scale independence—depends only on difference between orders:

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Tokunaga's law ^[8, 9, 10]



Property 1: Scale independence—depends only on difference between orders:

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Network Architecture

Tokunaga's law ^[8, 9, 10]

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Network Architecture

Tokunaga's law ^[8, 9, 10]

 Property 1: Scale independence—depends only on difference between orders:

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Network Architecture

Tokunaga's law ^[8, 9, 10]

 Property 1: Scale independence—depends only on difference between orders:

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 Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1(R_T)^{\mu-\nu-1}$$

 We usually write Tokunaga's law as:

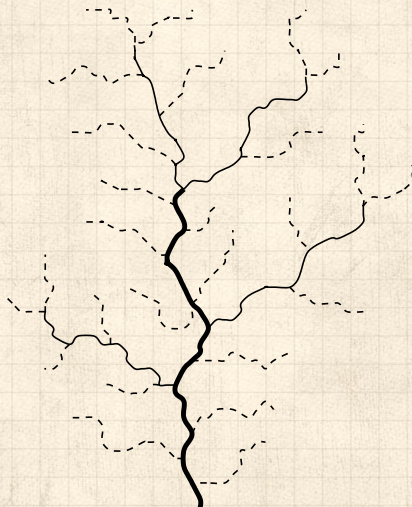
$$T_k = T_1(R_T)^{k-1} \text{ where } R_T \simeq 2$$



Tokunaga's law—an example:

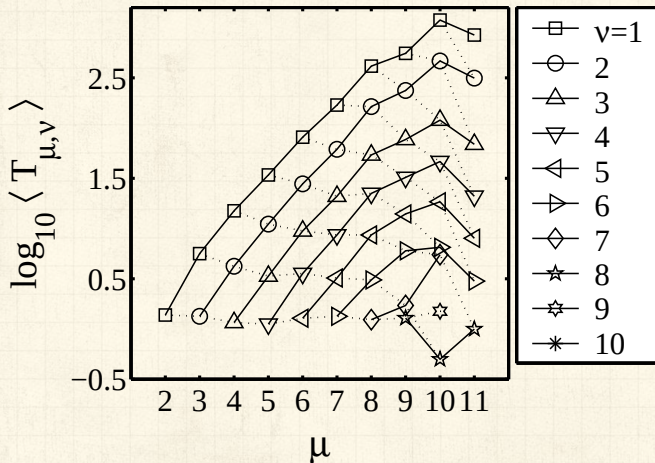
$$T_1 \simeq 2$$

$$R_T \simeq 4$$



The Mississippi

A Tokunaga graph:



Nutshell:



Branching networks show remarkable **self-similarity** over many scales.



Nutshell:



Branching networks show remarkable **self-similarity** over many scales.



There are many interrelated scaling laws.



Nutshell:



Branching networks show remarkable **self-similarity** over many scales.



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Horton-Strahler **Stream ordering** gives one useful way of getting at the architecture of branching networks.



Nutshell:



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Horton's laws reveal self-similarity.



Nutshell:



Branching networks show remarkable **self-similarity** over many scales.



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Horton's laws can be misinterpreted as suggesting a pure hierarchy.



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Branching networks show remarkable **self-similarity** over many scales.



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Tokunaga's laws neatly describe network architecture.



Nutshell:

-  Branching networks show remarkable **self-similarity** over many scales.
-  There are many interrelated scaling laws.
-  Horton-Strahler **Stream ordering** gives one useful way of getting at the architecture of branching networks.
-  **Horton's laws** reveal self-similarity.
-  Horton's laws can be misinterpreted as suggesting a pure hierarchy.
-  **Tokunaga's laws** neatly describe network architecture.
-  Branching networks exhibit a mixed hierarchical structure.



Nutshell:

- Branching networks show remarkable **self-similarity** over many scales.
- There are many interrelated scaling laws.
- Horton-Strahler **Stream ordering** gives one useful way of getting at the architecture of branching networks.
- Horton's laws** reveal self-similarity.
- Horton's laws can be misinterpreted as suggesting a pure hierarchy.
- Tokunaga's laws** neatly describe network architecture.
- Branching networks exhibit a mixed hierarchical structure.
- Horton and Tokunaga can be connected analytically.



Nutshell:

- Branching networks show remarkable **self-similarity** over many scales.
- There are many interrelated scaling laws.
- Horton-Strahler **Stream ordering** gives one useful way of getting at the architecture of branching networks.
- Horton's laws** reveal self-similarity.
- Horton's laws can be misinterpreted as suggesting a pure hierarchy.
- Tokunaga's laws** neatly describe network architecture.
- Branching networks exhibit a mixed hierarchical structure.
- Horton and Tokunaga can be connected analytically.
- Surprisingly:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$



Crafting landscapes—Far Lands or Bust



The screenshot shows the homepage of the 'Far Lands or Bust' website, which is set against a background of a Minecraft landscape. The website has a blue header with the title 'FAR LANDS OR BUST!' in large, orange, pixelated letters. Below the title is a navigation bar with links for 'home', 'about', 'world map', 'contact', and 'shirts & gifts'. To the right of the navigation bar are social media icons for YouTube, YouTube, Pinterest, Twitter, Twitch, and a generic social icon. A 'Helloooo!' message from Kurt, the creator, is displayed, mentioning his Let's Play series on YouTube and his Guinness World Record. Below this is a section titled 'The Latest Far Lands or Bust Episode!' featuring a video player for 'Minecraft Far Lands or Bust - #570 - You Can Do It'. The video player has a large '670' in the bottom left corner. Below the video player is a section titled '\$407,300 Raised for Child's Play Charity since 2011!' with a paragraph of text and a link to 'Learn more about the series...'. At the bottom of the page is a 'Child's Play' logo with the tagline 'Where Gamers Give Back' and the website 'www.childsplaycharity.org'. A small disclaimer is at the very bottom.

FAR LANDS OR BUST!

home about world map contact shirts & gifts

Helloooo! My name is Kurt and I have a Let's Play series on YouTube where, since March 2011, I have been traveling on an expedition to reach the fabled Far Lands of Minecraft Beta 1.7.3, documenting every step of the way. Now featured in the [Guinness World Records 2016 Gamer's Edition!](#)

The Latest Far Lands or Bust Episode!

Minecraft Far Lands or Bust - #570 - You Can Do It

FAR LANDS OR BUST!

670

\$407,300 Raised for Child's Play Charity since 2011!

Since starting the Far Lands or Bust fundraiser in June, 2011, generous Farlanders from around the world have raised over **\$400,000** for charity. [Learn more about the series...](#)

 Where Gamers Give Back

Child's Play
www.childsplaycharity.org

Mumbo Jumbo

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