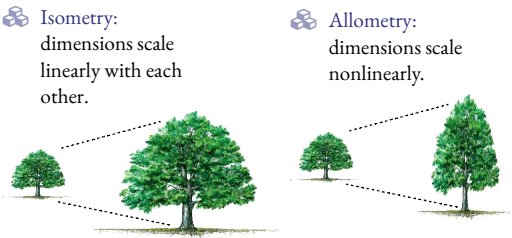
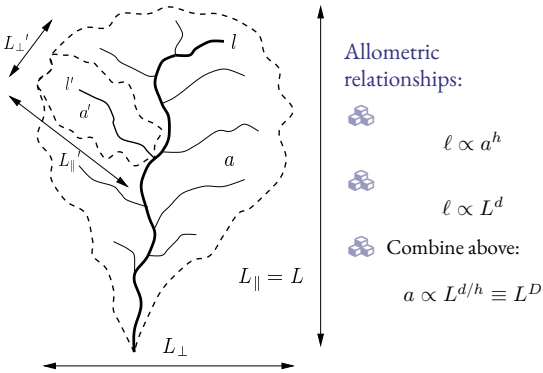


Allometry



Basin allometry



Allometric relationships:

$\ell \propto a^h$

$\ell \propto L^d$

Combine above:

$a \propto L^{d/h} \equiv L^D$

‘Laws’

Hack’s law (1957) [3]:

$\ell \propto a^h$

reportedly $0.5 < h < 0.7$

Scaling of main stream length with basin size:

$\ell \propto L_{||}^d$

reportedly $1.0 < d < 1.1$

Basin allometry:

$L_{||} \propto a^{h/d} \equiv a^{1/D}$

$D < 2 \rightarrow$ basins elongate.

There are a few more ‘laws’: [1]

Relation:	Name or description:
$T_k = T_1 (R_T)^{k-1}$	Tokunaga’s law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton’s law of stream numbers
$\ell_{\omega+1} / \ell_\omega = R_\ell$	Horton’s law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton’s law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton’s law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack’s law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein’s law
$\lambda \sim L^\varphi$	variation of Langbein’s law

Reported parameter values: [1]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05

Kind of a mess ...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out ...

Stream Ordering:

Method for describing network architecture:

- Introduced by Horton (1945) [4]
- Modified by Strahler (1957) [7]
- Term: Horton-Strahler Stream Ordering [5]
- Can be seen as iterative trimming of a network.

Stream Ordering:

Some definitions:

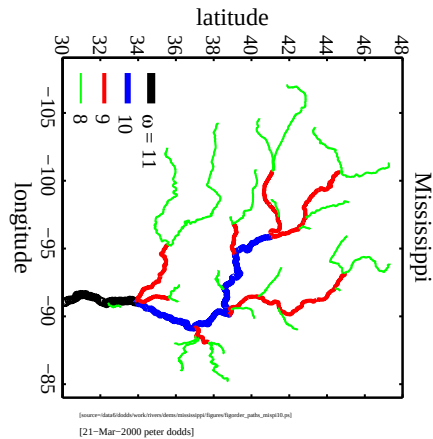
- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \dots$ for stream order.

Stream Ordering:



1. Label all source streams as order $\omega = 1$ and remove.
2. Label all new source streams as order $\omega = 2$ and remove.
3. Repeat until one stream is left (order = Ω)
4. Basin is said to be of the order of the last stream removed.
5. Example above is a basin of order $\Omega = 3$.

Stream Ordering—A large example:



Stream Ordering:

Another way to define ordering:

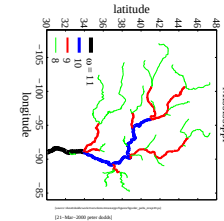
- As before, label all **source streams** as **order $\omega = 1$** .
- Follow all labelled streams downstream
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.

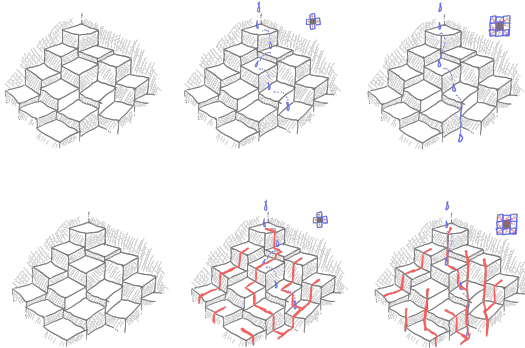
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.



Basic algorithm for extracting networks from Digital Elevation Models (DEMs):



Also:
`/Users/dodds/work/rivers/1998dems/kevinlakewaster.c`

Stream Ordering:

Resultant definitions:

- A basin of order Ω has n_ω streams (or sub-basins) of order ω .
 - $n_\omega > n_{\omega+1}$
- An order ω basin has **area a_ω** .
- An order ω basin has a **main stream length ℓ_ω** .
- An order ω basin has a **stream segment length s_ω** .
 - an order ω stream segment is only that part of the stream which is actually of order ω
 - an order ω stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams

Horton's laws

Self-similarity of river networks

- First quantified by Horton (1945)^[4], expanded by Schumm (1956)^[6]

Three laws:

- Horton's law of stream numbers:

$$n_\omega / n_{\omega+1} = R_n > 1$$

- Horton's law of stream lengths:

$$\ell_{\omega+1} / \ell_\omega = R_\ell > 1$$

- Horton's law of basin areas:

$$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a > 1$$

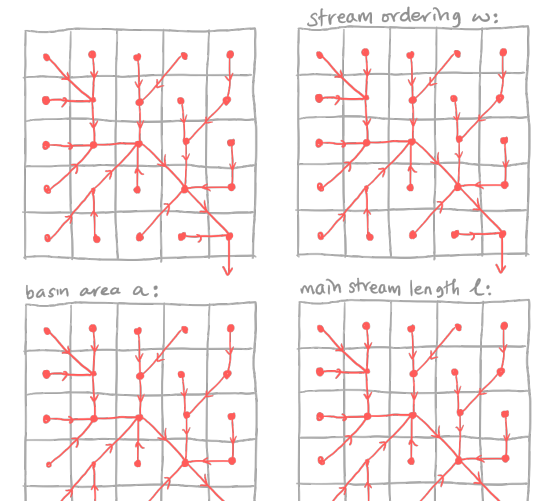
Stream Ordering:

One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ...but relationships based on ordering appear to be robust to resolution changes.

Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand **network architecture**



Horton's laws

Horton's Ratios:

- So ...laws are defined by three ratios:

$$R_n, R_\ell, \text{ and } R_a.$$

- Horton's laws describe **exponential decay or growth**:

$$\begin{aligned} n_\omega &= n_{\omega-1} / R_n \\ &= n_{\omega-2} / R_n^2 \\ &\vdots \\ &= n_1 / R_n^{\omega-1} \\ &= n_1 e^{-(\omega-1) \ln R_n} \end{aligned}$$

Horton’s laws

Similar story for area and length:



$$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$

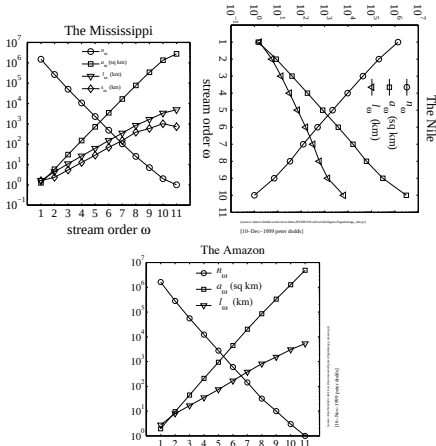


$$\bar{\ell}_\omega = \bar{\ell}_1 e^{(\omega-1)\ln R_\ell}$$



As stream order increases, **number drops** and **area and length increase**.

Horton’s laws in the real world:



Horton’s laws

Observations:



Horton’s ratios vary:

R_n	3.0–5.0
R_a	3.0–6.0
R_ℓ	1.5–3.0



No accepted explanation for these values.



Horton’s laws tell us how quantities vary from level to level ...



...but they don’t explain how networks are structured.

Horton’s laws

A few more things:



Horton’s laws are laws of averages.



Averaging for number is **across** basins.



Averaging for stream lengths and areas is **within** basins.



Horton’s ratios go a long way to defining a branching network ...



But we need one other piece of information ...

Horton’s laws-at-large

Blood networks:



Horton’s laws hold for sections of cardiovascular networks



Measuring such networks is tricky and messy ...



Vessel diameters obey an analogous Horton’s law.

Tokunaga’s law

Delving deeper into network architecture:



Tokunaga (1968) identified a clearer picture of network structure [8, 9, 10]



As per Horton-Strahler, use **stream ordering**.



Focus: describe how streams of different orders connect to each other.



Tokunaga’s law is also a law of averages.

Horton’s laws

A bonus law:



Horton’s law of stream segment lengths:

$$\bar{s}_{\omega+1}/\bar{s}_\omega = R_s > 1$$



Can show that $R_s = R_\ell$.



Insert assignment question ↗

Data from real blood networks

Network	R_n	R_r	R_ℓ	$-\frac{\ln R_r}{\ln R_n}$	$-\frac{\ln R_\ell}{\ln R_n}$	α
West <i>et al.</i>	–	–	–	1/2	1/3	3/4
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73
cat (PAT) [11]	3.67	1.71	1.78	0.41	0.44	0.79
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90
pig (LCX)	3.57	1.89	2.20	0.50	0.62	0.62
pig (RCA)	3.50	1.81	2.12	0.47	0.60	0.65
pig (LAD)	3.51	1.84	2.02	0.49	0.56	0.65
human (PAT)	3.03	1.60	1.49	0.42	0.36	0.83
human (PAT)	3.36	1.56	1.49	0.37	0.33	0.94

Network Architecture

Definition:



$T_{\mu,\nu}$ = the average number of **side streams** of **order ν** that enter as tributaries to streams of **order μ**



$\mu, \nu = 1, 2, 3, \dots$



$\mu \geq \nu + 1$



Recall each stream segment of order μ is ‘generated’ by two streams of order $\mu - 1$



These generating streams are not considered side streams.

Network Architecture

Tokunaga’s law [8, 9, 10]

- Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

- Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1(R_T)^{\mu-\nu-1}$$

- We usually write Tokunaga’s law as:

$T_k = T_1(R_T)^{k-1}$

 where $R_T \simeq 2$

Nutshell:

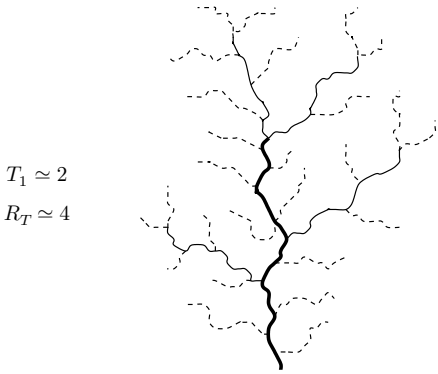
- Branching networks show remarkable **self-similarity** over many scales.
- There are many interrelated scaling laws.
- Horton-Strahler **Stream ordering** gives one useful way of getting at the architecture of branching networks.
- Horton’s laws** reveal self-similarity.
- Horton’s laws can be misinterpreted as suggesting a pure hierarchy.
- Tokunaga’s laws** neatly describe network architecture.
- Branching networks exhibit a mixed hierarchical structure.
- Horton and Tokunaga can be connected analytically.
- Surprisingly:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

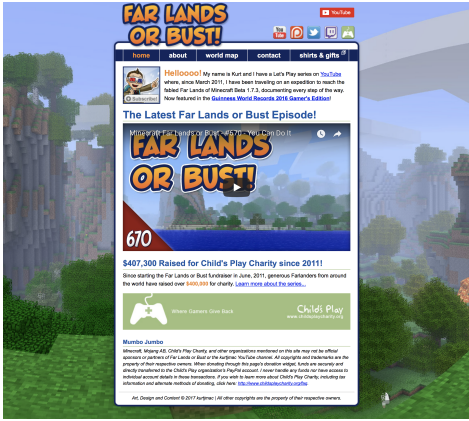
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Tokunaga’s law—an example:

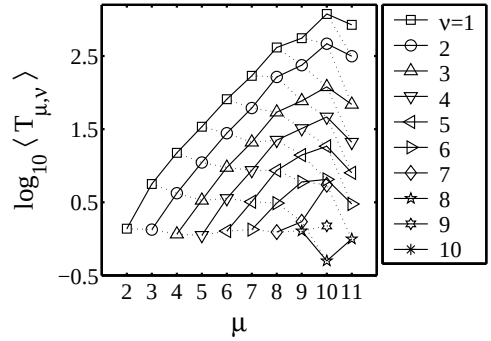


Crafting landscapes—Far Lands or Bust



The Mississippi

A Tokunaga graph:



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