# Optimal Supply Networks III: Redistribution

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## Outline





## Many sources, many sinks

### How do we distribute sources?

- Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.





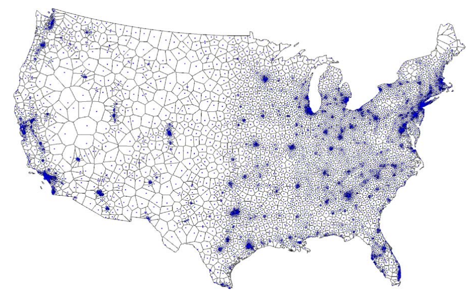

## Optimal source allocation: Size-density law

### Solidifying the basic problem

- Given a region with some population distribution $\rho$, most likely uneven.
- Given resources to build and maintain $N$ facilities.
- Q: How do we locate these $N$ facilities so as to minimize the average distance between an individual's residence and the nearest facility?



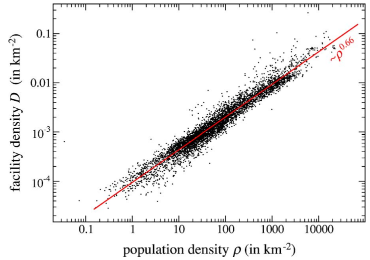
"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.



## Optimal source allocation




- Optimal facility density $\rho_{\mathrm{fac}}$ vs. population density $\rho_{\mathrm{pop}}$.
- Fit is $\rho_{\mathrm{fac}} \propto \rho_{\mathrm{pop}}^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power ...



## Optimal source allocation

### Size-density law:

- $$\rho_{\mathrm{fac}} \propto \rho_{\mathrm{pop}}^{2/3}$$
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.



## Optimal source allocation

"Territorial division: The least-time constraint behind the formation of subnational boundaries"
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer Simpson principle.

## Optimal source allocation



- Consider a region of area $A$ and population $P$ with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center as $\langle d \rangle$ and assume average speed of travel is $\langle v \rangle$.
- Assume isometry: average travel distance $\langle d \rangle$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\langle d \rangle / \langle v \rangle = cA^{1/2} / \langle v \rangle$$

where $c$ is an unimportant shape factor.

## Optimal source allocation



- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity $\tau$.
- If burden of mainenance is shared then average cost per person is $\tau / P$ where $P$ = population.
- Replace $P$ by $\rho_{\text{pop}} A$ where $\rho_{\text{pop}}$ is density.
- Important assumption: uniform density.
- Total average time cost per person:

$$T = \langle d \rangle / \langle v \rangle + \tau / (\rho_{\text{pop}} A) = cA^{1/2} / \langle v \rangle + \tau / (\rho_{\text{pop}} A).$$

- Now Minimize with respect to $A$ ...

## Optimal source allocation



- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A} \left( cA^{1/2} / \langle v \rangle + \tau / (\rho_{\text{pop}} A) \right)$$

$$= \frac{c}{2 \langle v \rangle A^{1/2}} - \frac{\tau}{\rho_{\text{pop}} A^2} = 0$$

- Rearrange:

$$A = \left( \frac{2 \langle v \rangle \tau}{c \rho_{\text{pop}}} \right)^{2/3} \propto \rho_{\text{pop}}^{-2/3}$$

- # facilities per unit area $\rho_{\text{fac}}$:

$$\rho_{\text{fac}} \propto A^{-1} \propto \rho_{\text{pop}}^{2/3}$$

- Groovy ...

## Optimal source allocation



### An issue:

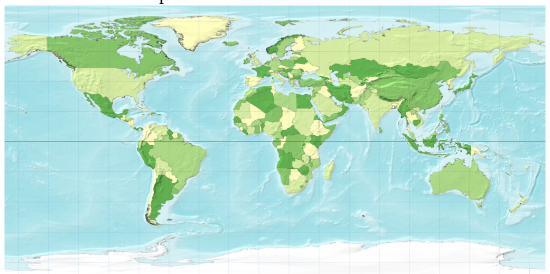
- Maintenance ($\tau$) is assumed to be independent of population and area ($P$ and $A$)

- Stephan's online book "The Division of Territory in Society" is here.
- (It used to be here.)
- The Readme is well worth reading (1995).

## Cartograms



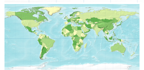
Standard world map:



## Cartograms



Cartogram of countries 'rescaled' by population:

## Cartograms



### Diffusion-based cartograms:

- Idea of cartograms is to distort areas to more accurately represent some local density $\rho_{\text{pop}}$ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [1] is based on standard diffusion:

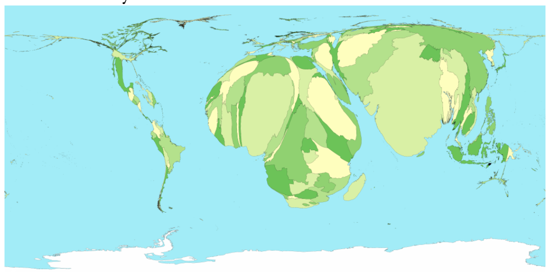
$$\nabla^2 \rho_{\text{pop}} - \frac{\partial \rho_{\text{pop}}}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\langle \rho \rangle_{\text{pop}}$.

## Cartograms



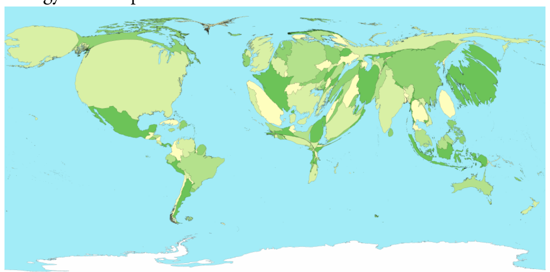
Child mortality:



## Cartograms



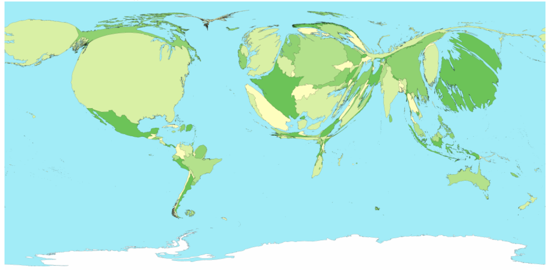
Energy consumption:



# Cartograms



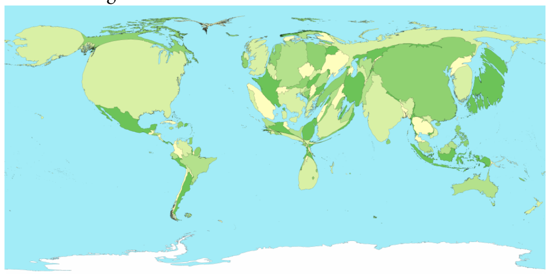
Gross domestic product:



# Cartograms



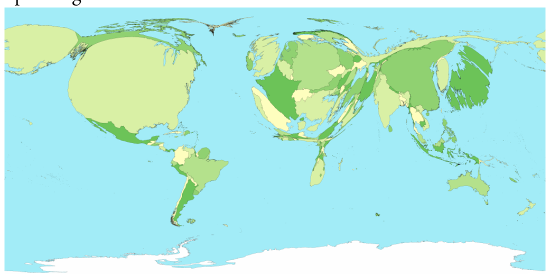
Greenhouse gas emissions:



# Cartograms



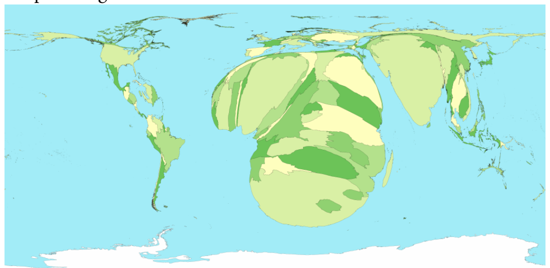
Spending on healthcare:



# Cartograms



People living with HIV:



# Cartograms

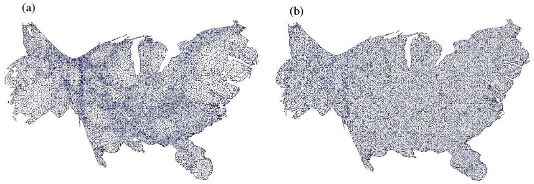


- The preceding sampling of Gastner & Newman's cartograms lives here.
- A larger collection can be found at worldmapper.org.

WORLDMAPPER The world as you've never seen it before

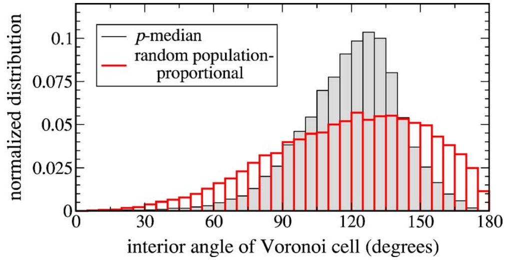


"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]




- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.






From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.



## Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility. [2]
- Assume given a fixed population density $\rho_{\text{pop}}$ defined on a spatial region $\Omega$.
- Formally, we want to find the locations of $n$ sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_\Omega \rho_{\text{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \,.$$

- Also known as the p-median problem, and connected to cluster analysis.
- Not easy ...in fact this one is an NP-hard problem. [2]
- Approximate solution originally due to Gusein-Zade [3].

# Size-density law



## Approximations:

- For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region $\Omega$ is divided up into Voronoi cells, one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say $i$) as

$$c_i A(\vec{x})^{1/2}$$

where $c_i$ is a shape factor for the $i$th Voronoi cell.
- Approximate $c_i$ as a constant $c$.

## Size-density law

### Carrying on:

- The cost function is now

$$F = c\int_{\Omega} \rho_{\text{pop}}(\vec{x})A(\vec{x})^{1/2}\mathrm{d}\vec{x}\,.$$

- We also have that the constraint that Voronoi cells divide up the overall area of $\Omega$: $\sum_{i=1}^{n} A(\vec{x}_i) = A_\Omega$.
- Sneakily turn this into an integral constraint:

$$\int_\Omega \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$

- Within each cell, $A(\vec{x})$ is constant.
- So ...integral over each of the $n$ cells equals 1.



### Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

$$G(A) = c\int_\Omega \rho_{\text{pop}}(\vec{x})A(\vec{x})^{1/2}\mathrm{d}\vec{x} - \lambda\left(n - \int_\Omega [A(\vec{x})]^{-1}\,\mathrm{d}\vec{x}\right)$$

- I Can Haz Calculus of Variations?
- Compute $\delta G/\delta A$, the functional derivative of the functional $G(A)$.
- This gives

$$\int_\Omega \left[\frac{c}{2}\rho_{\text{pop}}(\vec{x})A(\vec{x})^{-1/2} - \lambda\,[A(\vec{x})]^{-2}\right]\mathrm{d}\vec{x} = 0.$$

- Setting the integrand to be zilch, we have:

$$\rho_{\text{pop}}(\vec{x}) = 2\lambda c^{-1}A(\vec{x})^{-3/2}.$$



## Size-density law

### Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3}\rho_{\text{pop}}^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $\rho_{\text{fac}}(\vec{x})$, an approximation of the local source density.
- Substituting $\rho_{\text{fac}} = 1/A$, we have

$$\rho_{\text{fac}}(\vec{x}) = \left(\frac{c}{2\lambda}\rho_{\text{pop}}\right)^{2/3}.$$

- Normalizing (or solving for $\lambda$):

$$\rho_{\text{fac}}(\vec{x}) = n\frac{[\rho_{\text{pop}}(\vec{x})]^{2/3}}{\int_\Omega [\rho_{\text{pop}}(\vec{x})]^{2/3}\mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/3}.$$



## Global redistribution networks

### One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gastner and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\text{maint}} + \gamma C_{\text{travel}}.$$

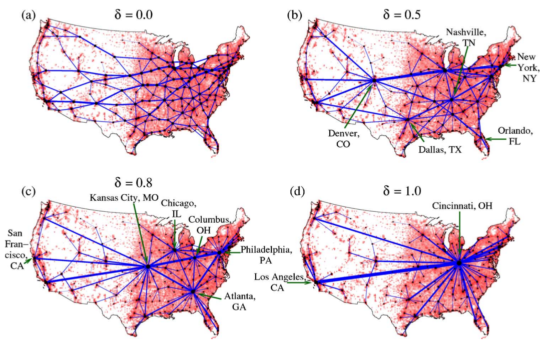
- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance $\ell_{ij}$ and number of legs to journey:

$$(1-\delta)\ell_{ij} + \delta(\#\text{hops}).$$

- When $\delta = 1$, only number of hops matters.

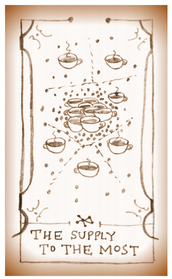


## Global redistribution networks




From Gastner and Newman (2006) [2]








## Public versus private facilities

### Beyond minimizing distances:

- "Scaling laws between population and facility densities" by Um *et al.*, Proc. Natl. Acad. Sci., 2009. [6]
- Um *et al.* find empirically and argue theoretically that the connection between facility and population density

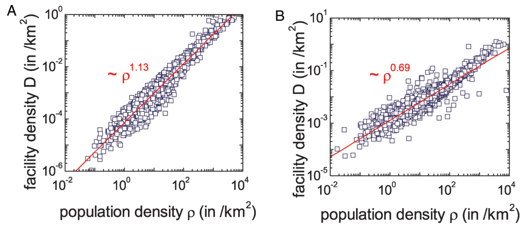
$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

does not universally hold with $\alpha = 2/3$.

- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.



## Public versus private facilities: evidence




- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\text{pop}} \simeq 100$.



## Public versus private facilities: evidence

| US facility | α (SE) | $R^2$ |
|---|---|---|
| Ambulatory hospital | 1.13(1) | 0.93 |
| Beauty care | 1.08(1) | 0.86 |
| Laundry | 1.05(1) | 0.90 |
| Automotive repair | 0.99(1) | 0.92 |
| Private school | 0.95(1) | 0.82 |
| Restaurant | 0.93(1) | 0.89 |
| Accommodation | 0.89(1) | 0.70 |
| Bank | 0.88(1) | 0.89 |
| Gas station | 0.86(1) | 0.94 |
| Death care | 0.79(1) | 0.80 |
| * Fire station | 0.78(3) | 0.93 |
| * Police station | 0.71(6) | 0.75 |
| Public school | 0.69(1) | 0.87 |

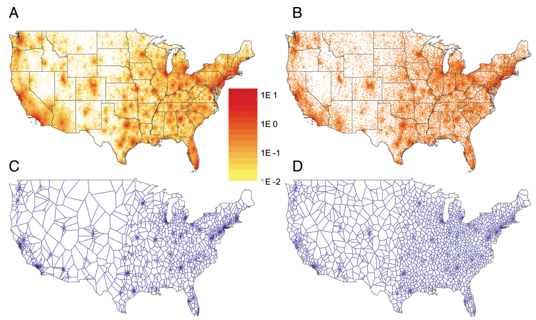
| SK facility | α (SE) | $R^2$ |
|---|---|---|
| Bank | 1.18(2) | 0.96 |
| Parking place | 1.13(2) | 0.91 |
| * Primary clinic | 1.09(2) | 1.00 |
| * Hospital | 0.96(5) | 0.97 |
| * University/college | 0.93(9) | 0.89 |
| Market place | 0.87(2) | 0.90 |
| * Secondary school | 0.77(3) | 0.98 |
| * Primary school | 0.77(3) | 0.97 |
| Social welfare org. | 0.75(2) | 0.84 |
| * Police station | 0.71(5) | 0.94 |
| Government office | 0.70(1) | 0.93 |
| * Fire station | 0.60(4) | 0.93 |
| * Public health center | 0.09(5) | 0.19 |

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

# Public versus private facilities: evidence






A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

# Public versus private facilities: the story



## So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defns: For the $i$th facility and its Voronoi cell $V_i$, define
  - $n_i$ = population of the $i$th cell;
  - $\langle r_i \rangle$ = the average travel distance to the $i$th facility.
  - $A_i$ = area of $i$th cell ($s_i$ in Um *et al.* [6])
- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$

- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.

# Public versus private facilities: the story



- Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)} \mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}.$$

- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.



| System type: | Dominant cost/benefit scaling: | Dominant constraint scaling: | Scaling of number of events per partition: | Density scaling: | Quantity equalized across partitions: |
|---|---|---|---|---|---|
| General form | $\rho_{\text{event}} V^{\alpha}$ $0 < \alpha \leq 1$ | $V^{-\beta}$ $1 - \alpha \leq \beta \leq 1$ | $N \propto V^{1-\alpha-\beta}$ | $\rho_{\text{partition}} \propto \rho_{\text{event}}^{1/(\alpha+\beta)}$ | $NV^{\alpha+\beta-1}$ |
| **I.** Event rate equalizing with partition number constrained (for-profit) | $\sim \rho_{\text{event}} \ln V$ | $V^{-1}$ | $N \propto V^{0}$ | $\rho_{\text{partition}} \propto \rho_{\text{event}}^{1}$ | $N$ |
| **II.** Minimizing average event access time with partition number constrained ($p$-median problem, pro-social) | $\rho_{\text{event}} V^{1/d}$ | $V^{-1}$ | $N \propto V^{-1/d}$ | $\rho_{\text{partition}} \propto \rho_{\text{event}}^{d/(d+1)}$ | $NV^{1/d}$ |
| **III.** System under stochastic threat with partition boundary constrained (HOT model) | $\rho_{\text{event}} V^{1}$ | $V^{-1/d}$ | $N \propto V^{-1/d}$ | $\rho_{\text{partition}} \propto \rho_{\text{event}}^{d/(d+1)}$ | $NV^{1/d}$ |
| **IV.** System under stochastic threat with partition number constrained | $\rho_{\text{event}} V^{1}$ | $V^{-1}$ | $N \propto V^{-1}$ | $\rho_{\text{partition}} \propto \rho_{\text{event}}^{1/2}$ | $NV$ |

# References I



[1] M. T. Gastner and M. E. J. Newman.
Diffusion-based method for producing density-equalizing maps.
Proc. Natl. Acad. Sci., 101:7499–7504, 2004. pdf

[2] M. T. Gastner and M. E. J. Newman.
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# References II



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