Random Networks Nutshell

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2024–2025

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Computational Story Lab | Vermont Complex Systems Center
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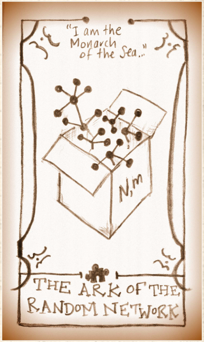
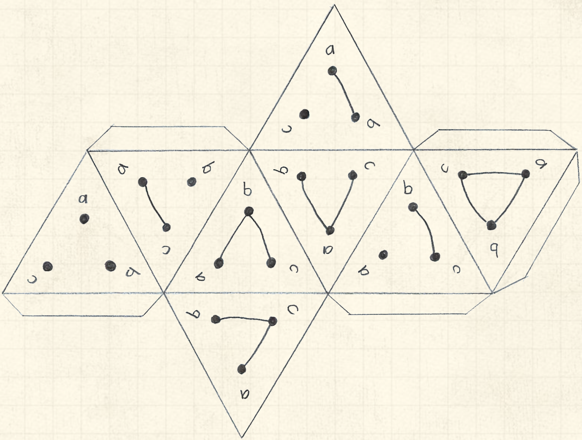
Outline

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions
Generalized Random Networks
Configuration model
How to build in practice
Motifs
Strange friends
Largest component
References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS


"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK

Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

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- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

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- Real world: links are usually costly so real networks are almost always sparse.

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   - Best for adding relatively small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

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- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

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Next slides:

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Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples for N=500

m = 100
$\langle k \rangle$ = 0.4

m = 200
$\langle k \rangle$ = 0.8

m = 230
$\langle k \rangle$ = 0.92

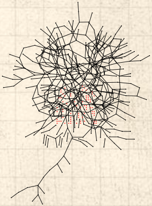
m = 240
$\langle k \rangle$ = 0.96

m = 250
$\langle k \rangle$ = 1

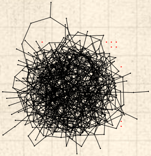
m = 260
$\langle k \rangle$ = 1.04

m = 280
$\langle k \rangle$ = 1.12

m = 300
$\langle k \rangle$ = 1.2

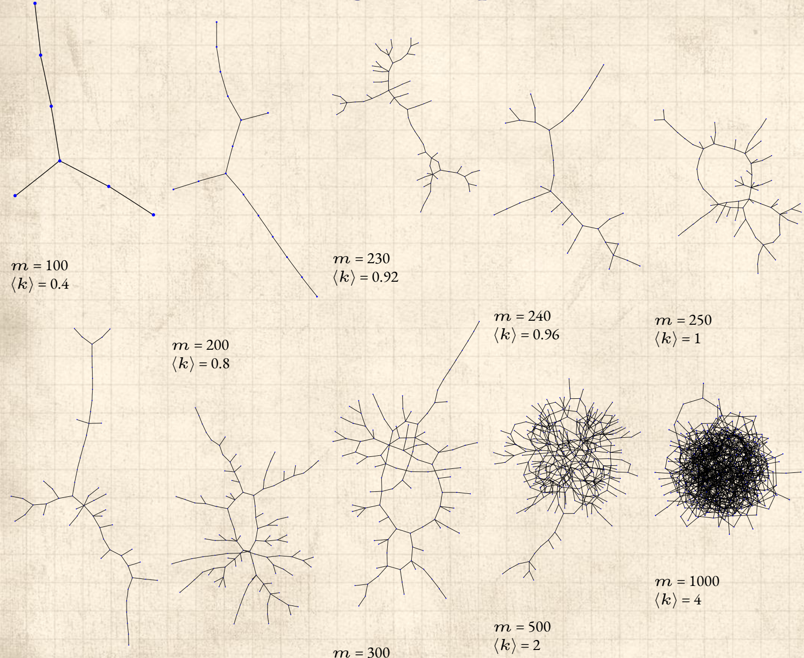


m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4

Random networks: largest components



Random networks: examples for N=500

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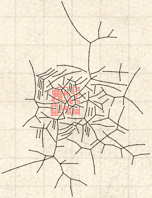

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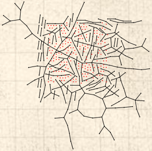

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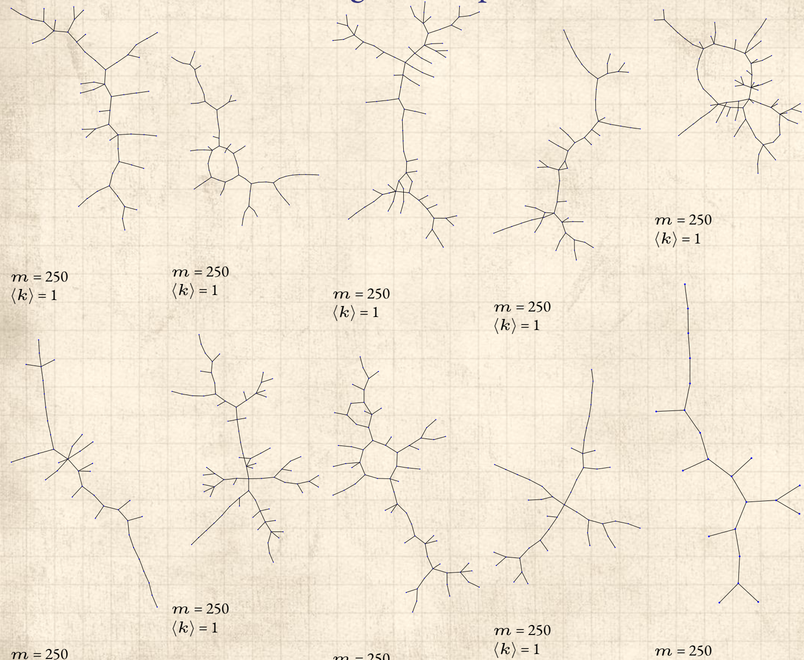
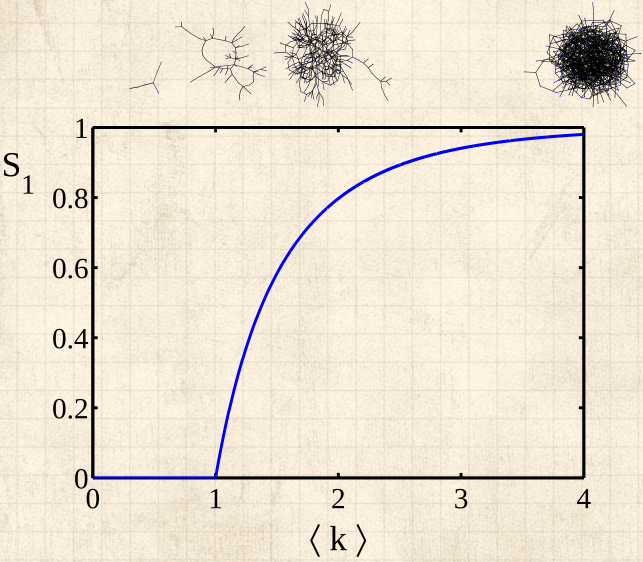


Giant component



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Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?

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$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

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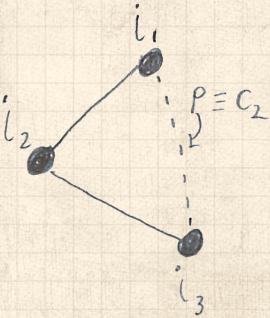

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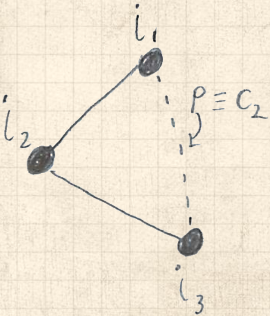

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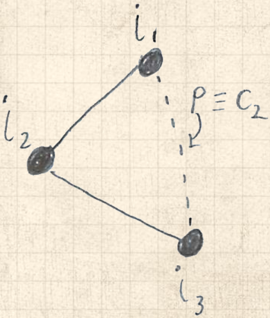

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- For standard random networks, we have simply that

$$C_2 = p.$$

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- No small loops.

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- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \to \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

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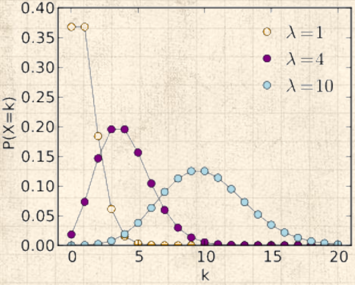
- This is a Poisson distribution with mean $\langle k \rangle$.

Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.:
  phone calls/minute,
  horse-kick deaths.
- 'Law of small numbers'

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- Note: This is a special property of Poisson distribution and can trip us up...

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Random networks: examples

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Example realizations of random networks with power law degree distributions:

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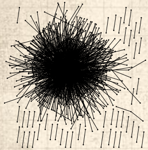
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

Random networks: examples for N=1000



γ = 2.1
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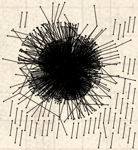

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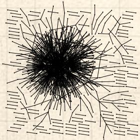

γ = 2.28
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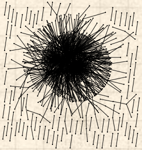

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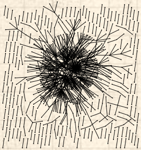

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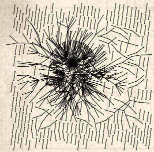

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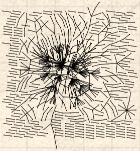

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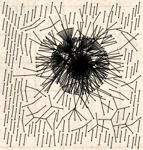

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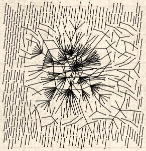

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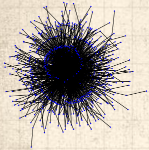
Random networks: largest components



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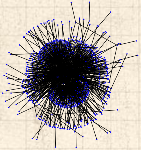

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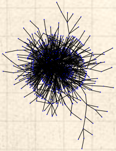

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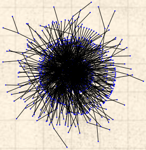

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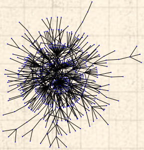

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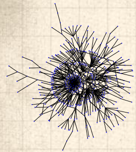

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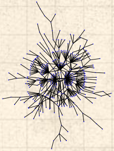

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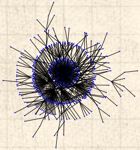

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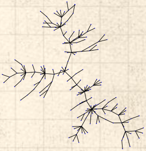

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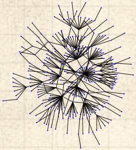

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Building random networks: Stubs

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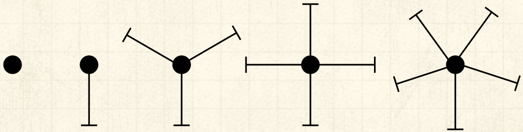
- **Idea:** start with a soup of unconnected nodes with stubs (half-edges):



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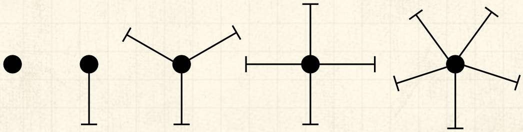

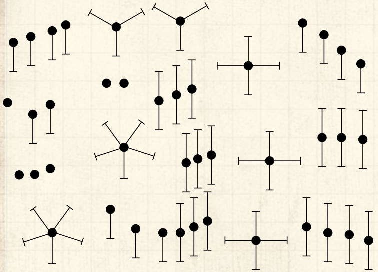

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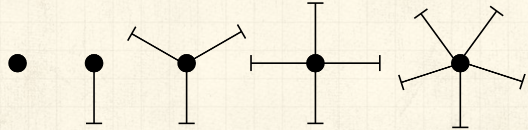

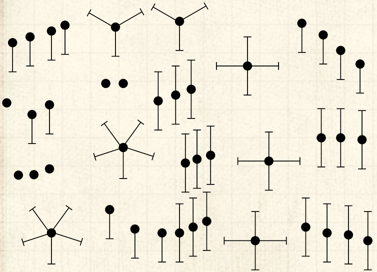

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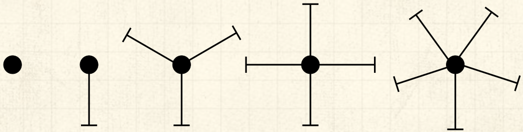

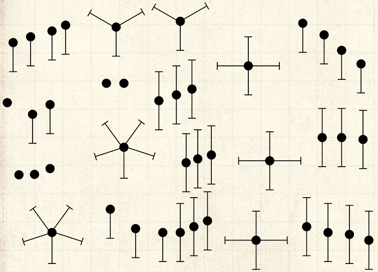

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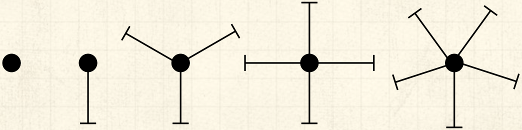

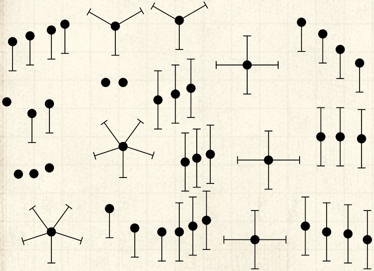

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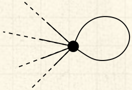
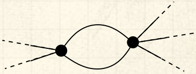
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.

(A) 

(B) 

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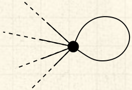
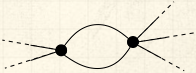
(A)  (B) 

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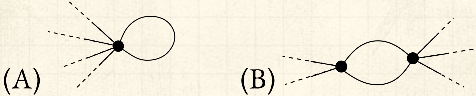
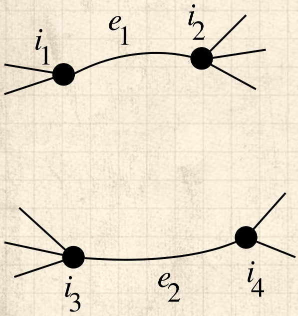


- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

General random rewiring algorithm



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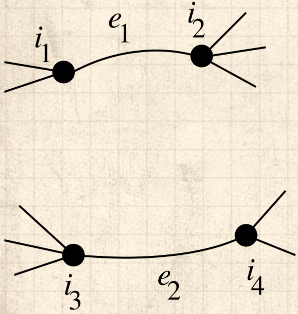

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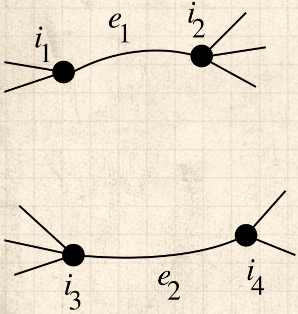

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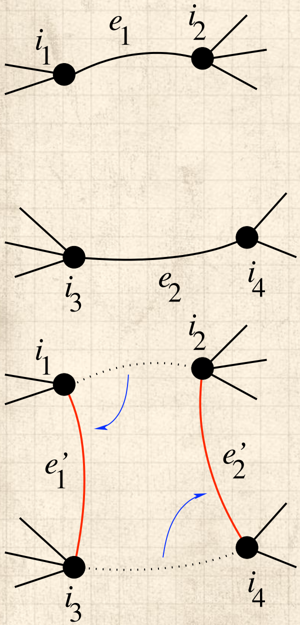

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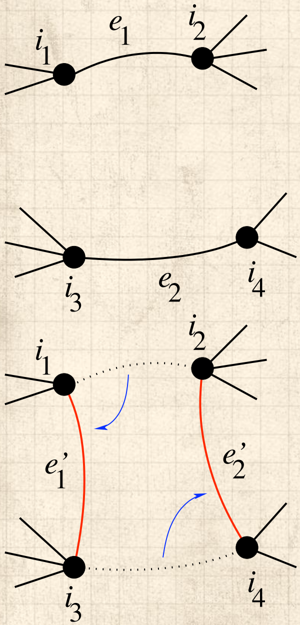

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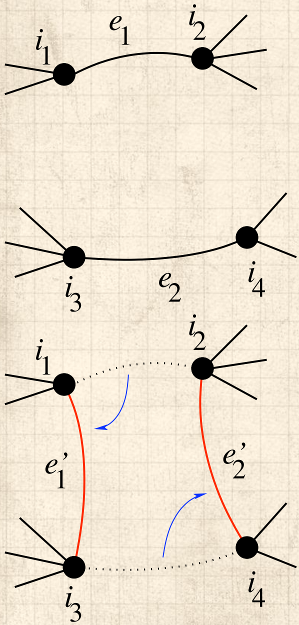

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- Same as finding on/off/on/off 4-cycles. and rotating them.

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- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [4].

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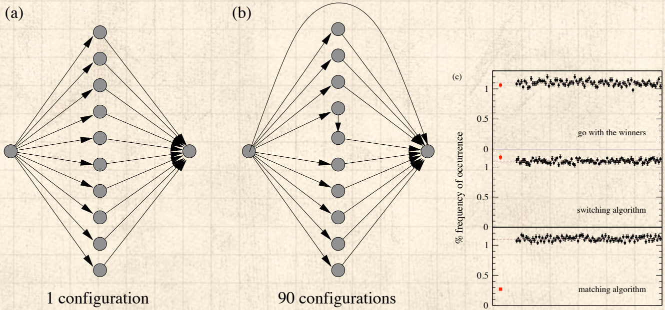
- Example from Milo et al. (2003) [4]:



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- Note: not all P_k will always give nodes that can be wired together.

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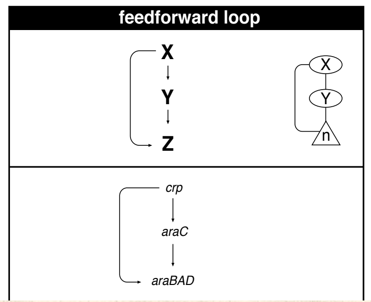
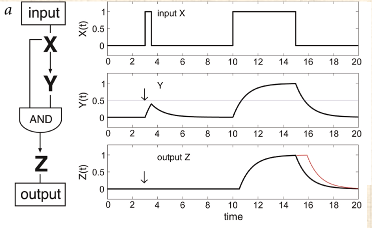
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

Network motifs





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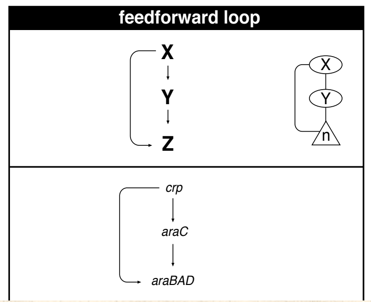

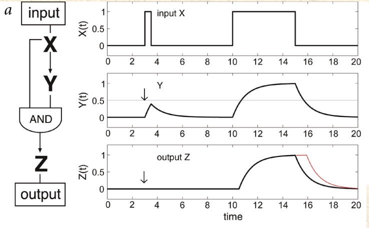

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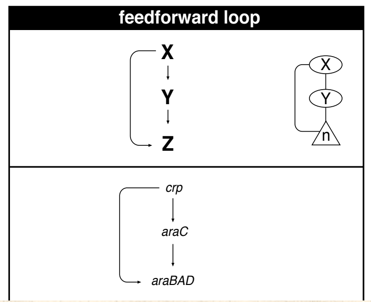

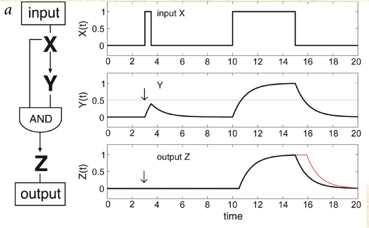

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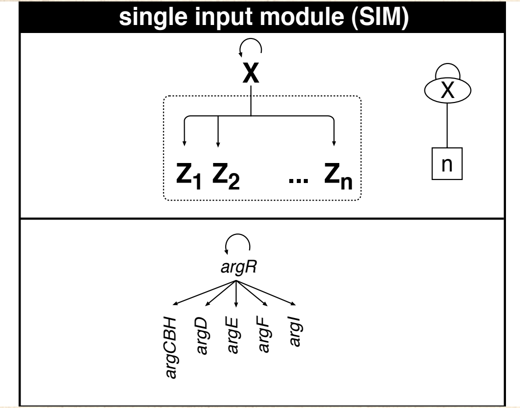
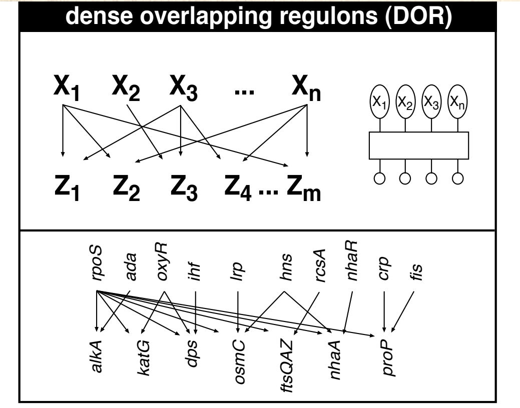
- Turning off X rapidly turns off Z.
- Analogy to elevator doors.

Network motifs



- Master switch.

Network motifs



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- For more, see work carried out by Wiggins *et al.* at Columbia.

Outline

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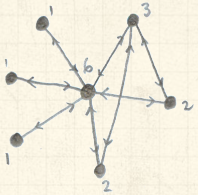
- Big deal: Rich-get-richer mechanism is built into this selection process.



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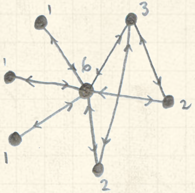

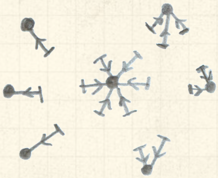

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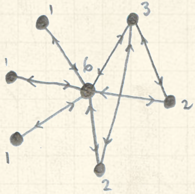

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- Natural question: what's the expected number of other friends that one friend has?

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- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...

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- #samesies.

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  4. See also: class size paradoxes (nod to: Gelman)

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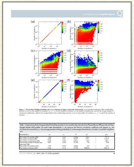
- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.



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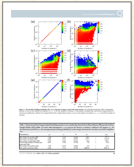

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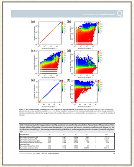

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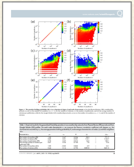

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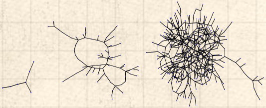
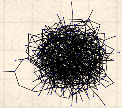
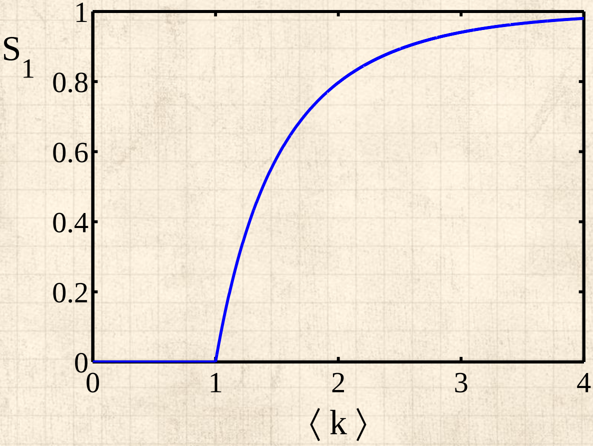
- Note: Component = Cluster

Outline

Giant component







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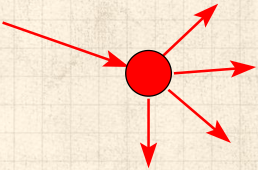

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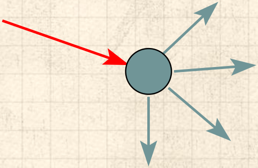

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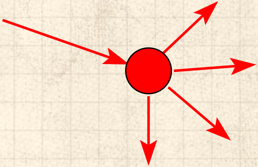

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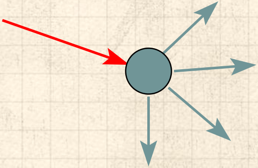

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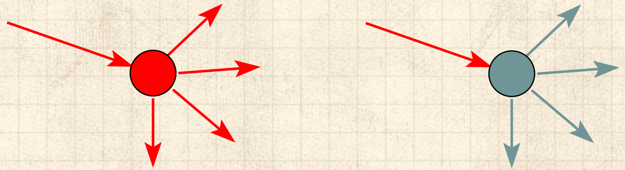

- Focus on binary case with edges and nodes either infected or not.
- First big question: for a given network and contagion process, can global spreading from a single seed occur?

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- Good: This is just our giant component condition again.

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- We say $\langle k \rangle = 1$ marks the critical point of the system.

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$$\propto x^{3-\gamma}\Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- So giant component always exists for these kinds of networks.
- Cutoff scaling is k^{-3}: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.
- How about $P_k = \delta_{kk_0}$?

Giant component

And how big is the largest component?

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- Substitute in Poisson distribution...

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- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

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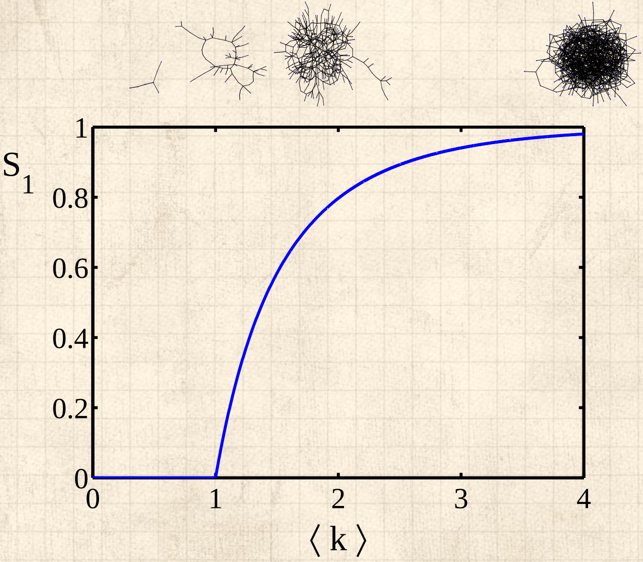
- Really a transcritical bifurcation. [8]

Giant component



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- We can sort many things out with sensible probabilistic arguments...
- More detailed investigations will profit from a spot of Generatingfunctionology. [9]

References I

[1] P. S. Dodds, K. D. Harris, and J. L. Payne.
Direct, phyiscally motivated derivation of the contagion condition for spreading processes on generalized random networks.
Phys. Rev. E, 83:056122, 2011. pdf

[2] Y.-H. Eom and H.-H. Jo.
Generalized friendship paradox in complex networks: The case of scientific collaboration.
Nature Scientific Reports, 4:4603, 2014. pdf

[3] S. L. Feld.
Why your friends have more friends than you do.
Am. J. of Sociol., 96:1464–1477, 1991. pdf

References II

[4] R. Milo, N. Kashtan, S. Itzkovitz, M. E. J. Newman, and U. Alon.
On the uniform generation of random graphs with prescribed degree sequences, 2003. pdf

[5] M. E. J. Newman.
Ego-centered networks and the ripple effect,.
Social Networks, 25:83–95, 2003. pdf

[6] M. E. J. Newman.
The structure and function of complex networks.
SIAM Rev., 45(2):167–256, 2003. pdf

[7] S. S. Shen-Orr, R. Milo, S. Mangan, and U. Alon.
Network motifs in the transcriptional regulation network of *Escherichia coli*.
Nature Genetics, 31:64–68, 2002. pdf

References III

[8] S. H. Strogatz.
Nonlinear Dynamics and Chaos.
Addison Wesley, Reading, Massachusetts, 1994.

[9] H. S. Wilf.
Generatingfunctionology.
A K Peters, Natick, MA, 3rd edition, 2006. pdf