# Mechanisms for Generating Power-Law Size Distributions, Part 1

Last updated: 2024/09/26, 14:29:32 EDT

## Principles of Complex Systems, Vols. 1, 2, & 3D
## CSYS/MATH 6701, 6713, & a pretend number, 2024–2025


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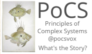
Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont









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# Outline

- Heavy-tailed distributions are **characters**.





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- Some of these distributions have power-law tails.


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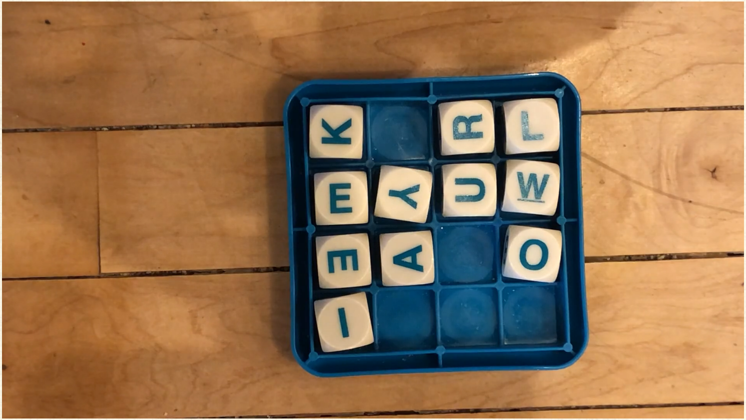
- Some of these distributions have power-law tails.
- Measured exponents ($\gamma$'s and $\alpha$'s) vary across systems (and measurers).
- What's their **origin story**?

## The Boggoracle Speaks:




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
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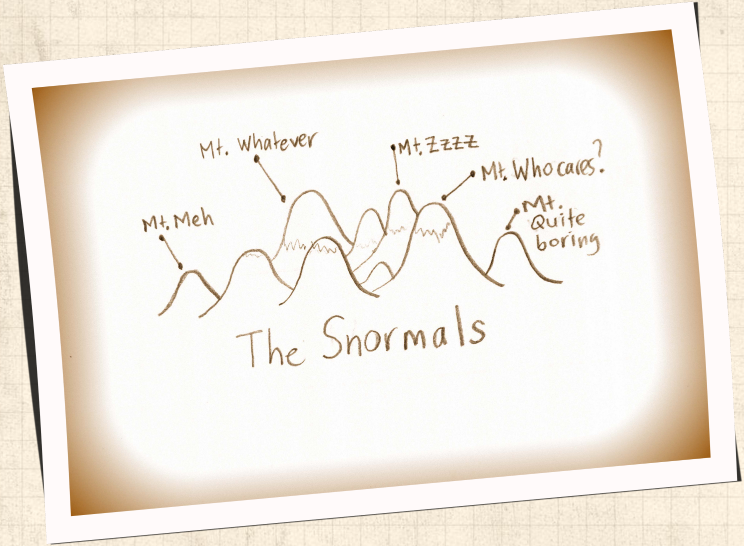
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS


MINISTRY OF RANDOM WALKS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Mt. Quite boring
The Snormals


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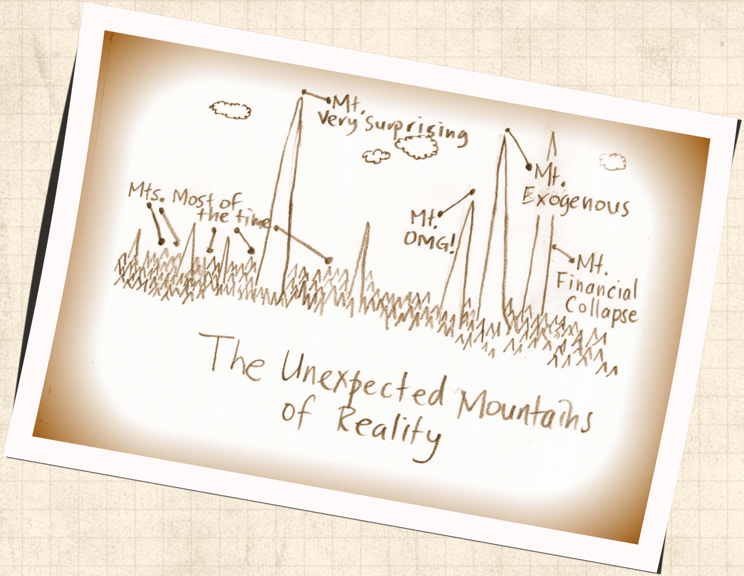
Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality


MINISTRY OF RANDOM WALKS

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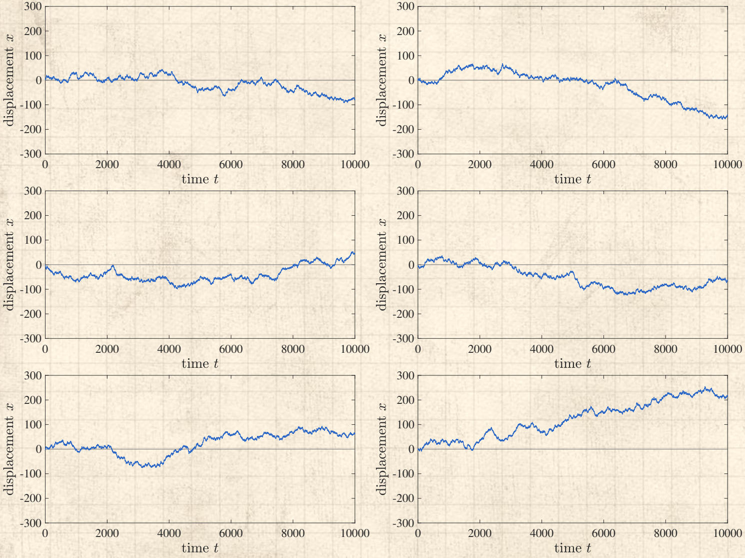
- Random walker (e.g., a zombie texter) starts at origin $x = 0$.
- Step at time $t$ is $\epsilon_t$:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

# A few random random walks:





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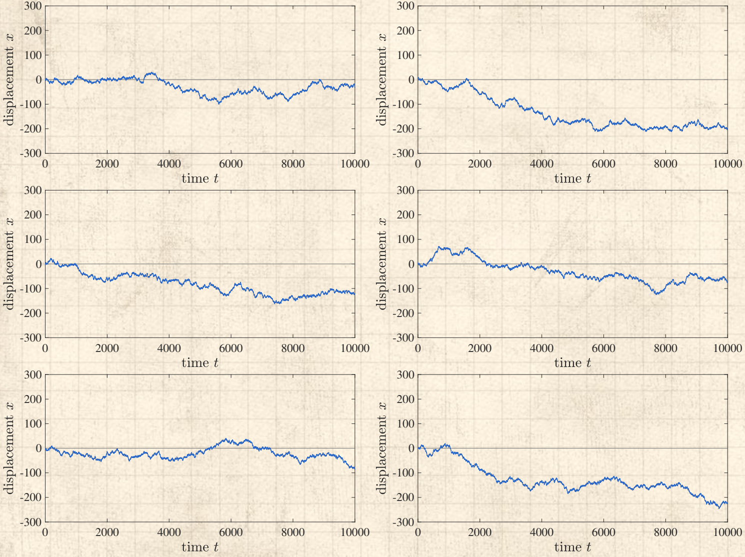




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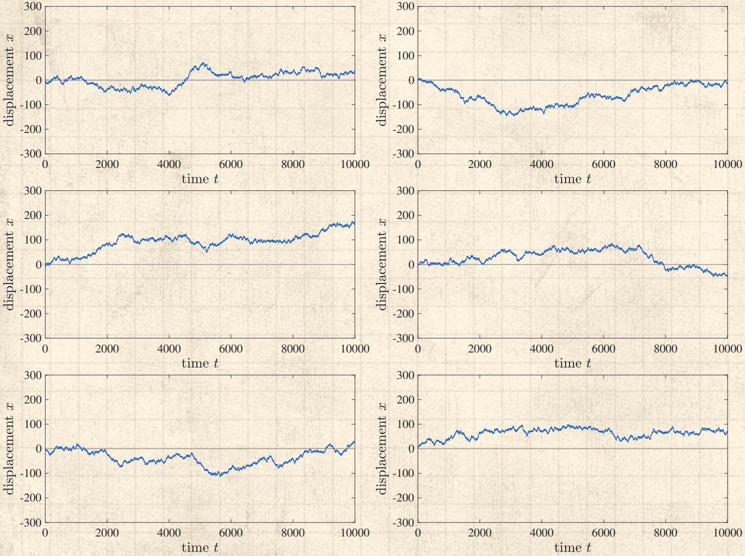






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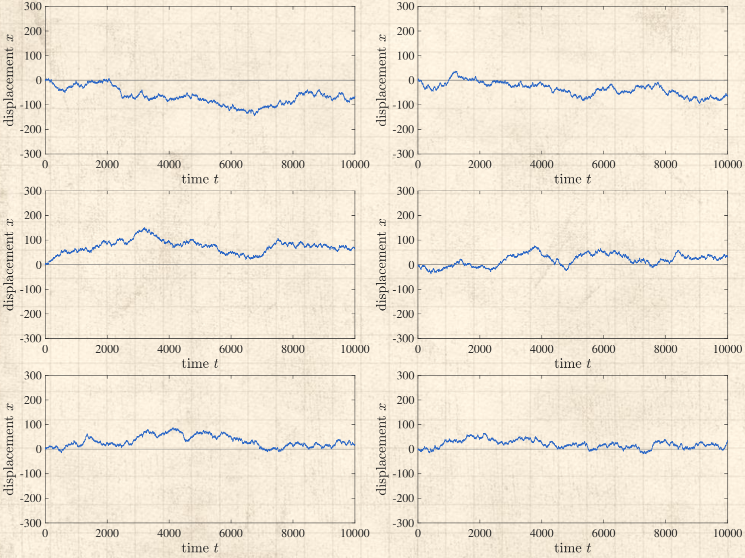


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- Obviously fails for odd number of steps...
- But as time goes on, the chance of our texting undead friend lurching back to $x$=0 must diminish, right?

## Variances sum:*

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- A non-trivial scaling law arises out of additive aggregation or accumulation.

## Great moments in Televised Random Walks:


Plinko! from the Price is Right.

- Plinko failure.
- Also known as the bean machine, the quincunx (simulation), and the Galton box.

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$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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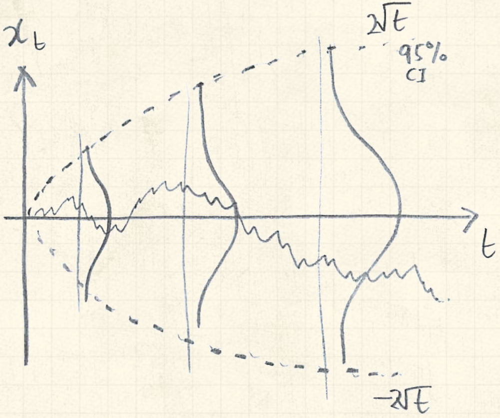
- See also: Stable Distributions

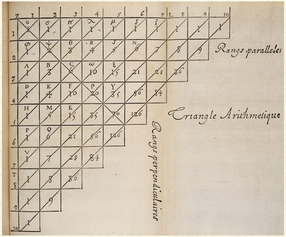
# Universality is also not left-handed:




- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.

# So many things are connected:

## Pascal's Triangle

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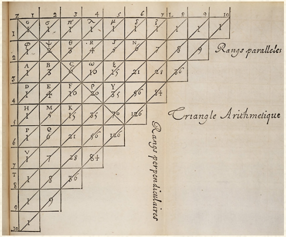


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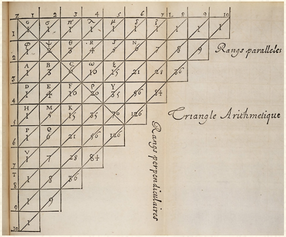


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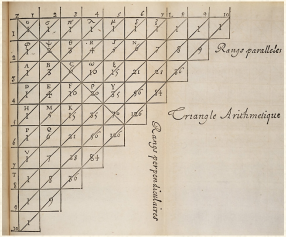


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MINISTRY OF
RANDOM WALKS


$t^{1/2}$
$t$
The Snormals
The Random Road
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MINISTRY OF
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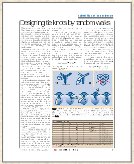
See Feller, Intro to Probability Theory, Volume I [5]

# Applied knot theory:




"Designing tie knots by random walks"
Fink and Mao,
Nature, **398**, 31–32, 1999. [6]

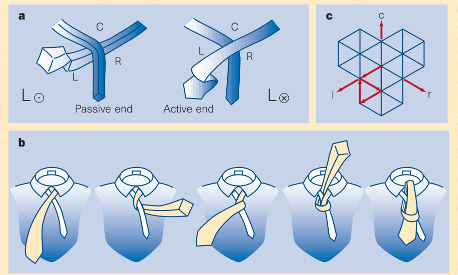



Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. **a,** The two ways of beginning a knot, $L_{\odot}$ and $L_{\otimes}$. For knots beginning with $L_{\odot}$, the tie must begin inside-out. **b,** The four-in-hand, denoted by the sequence $L_{\otimes} R_{\odot} L_{\otimes} C_{\odot} T$. **c,** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\hat{l}\,\hat{r}\,\hat{l}\,\hat{c}$.

Hexagons are the bestagons.





# Applied knot theory:

Table 1 **Aesthetic tie knots**

| $h$ | $\gamma$ | $\gamma/h$ | $K(h,\gamma)$ | $s$ | $b$ | Name | Sequence |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.33 | 1 | 0 | 0 | | $L_\odot R_\otimes C_\odot T$ |
| 4 | 1 | 0.25 | 1 | −1 | 1 | Four-in-hand | $L_\otimes R_\odot L_\otimes C_\odot T$ |
| 5 | 2 | 0.40 | 2 | −1 | 0 | Pratt knot | $L_\odot C_\otimes R_\odot L_\otimes C_\odot T$ |
| 6 | 2 | 0.33 | 4 | 0 | 0 | Half-Windsor | $L_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$ |
| 7 | 2 | 0.29 | 6 | −1 | 1 | | $L_\odot R_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$ |
| 7 | 3 | 0.43 | 4 | 0 | 1 | | $L_\odot C_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$ |
| 8 | 2 | 0.25 | 8 | 0 | 2 | | $L_\otimes R_\odot L_\otimes C_\odot R_\otimes L_\odot R_\otimes C_\odot T$ |
| 8 | 3 | 0.38 | 12 | −1 | 0 | Windsor | $L_\otimes C_\odot R_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$ |
| 9 | 3 | 0.33 | 24 | 0 | 0 | | $L_\odot R_\otimes C_\odot L_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$ |
| 9 | 4 | 0.44 | 8 | −1 | 2 | | $L_\odot C_\otimes R_\odot C_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$ |

Knots are characterized by half-winding number $h$, centre number $\gamma$, centre fraction $\gamma/h$, knots per class $K(h, \gamma)$, symmetry $s$, balance $b$, name and sequence.

- $h$ = number of moves
- $\gamma$ = number of center moves
- $K(h,\gamma) = 2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}$
- $s = \sum_{i=1}^{h} x_i$ where $x_i = -1$ for $L$ and $x_i = +1$ for $R$.
- $b = \frac{1}{2}\sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.

# Random walks

#crazytownbananapants

The problem of first return:





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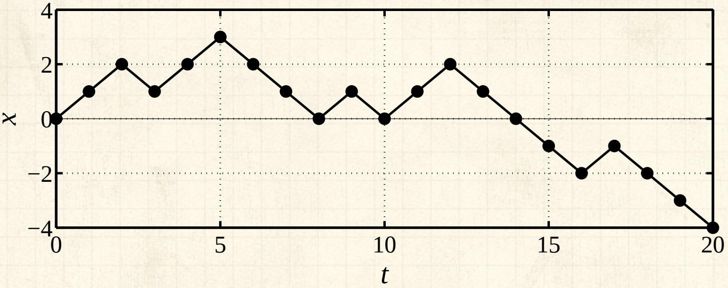
3. We'll start to see how different scalings relate to each other.

# For random walks in 1-$d$:



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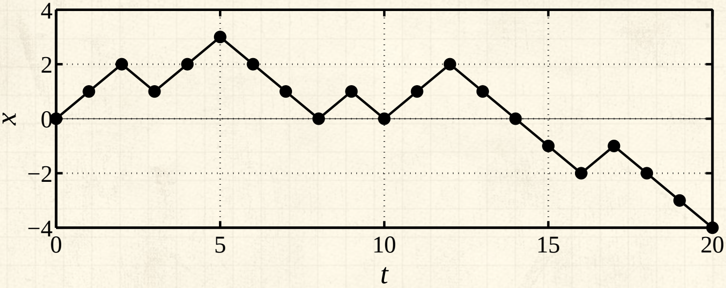


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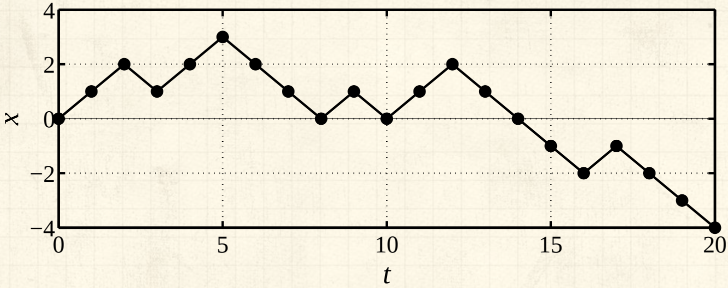


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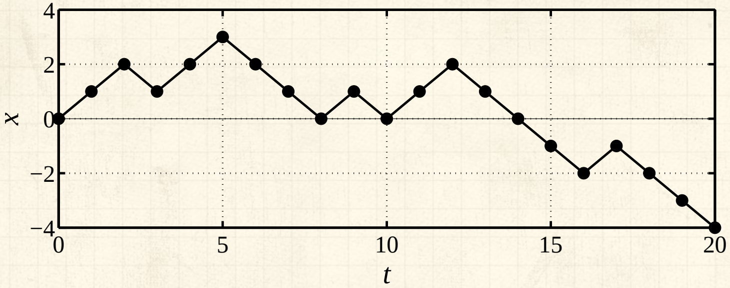


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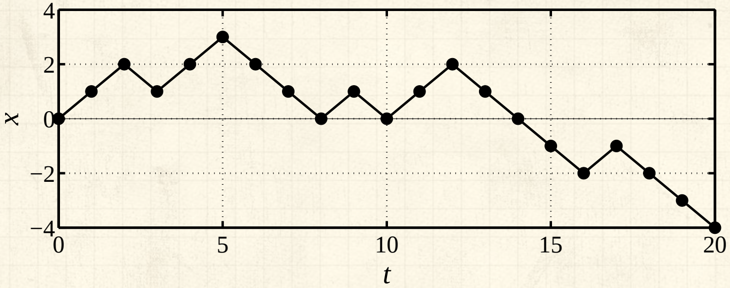


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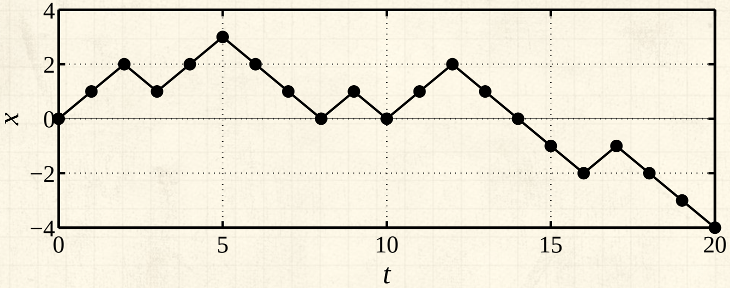


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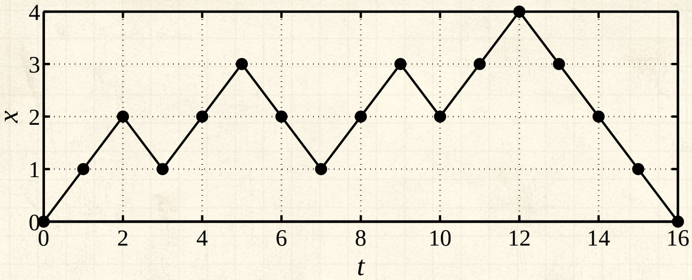
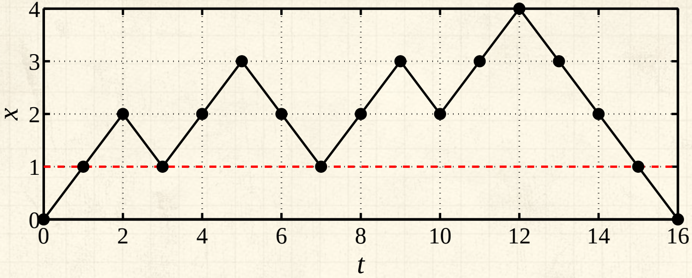
- Idea: Transform first return problem into an easier return problem.




Can assume zombie texter first lurches to $x = 1$.



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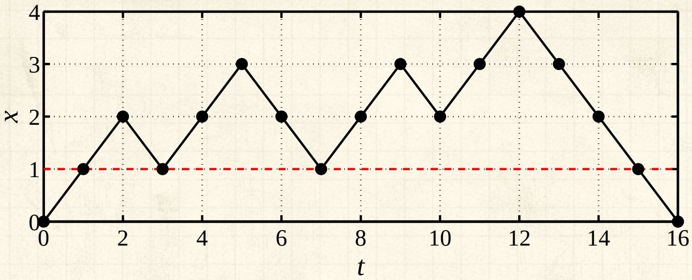


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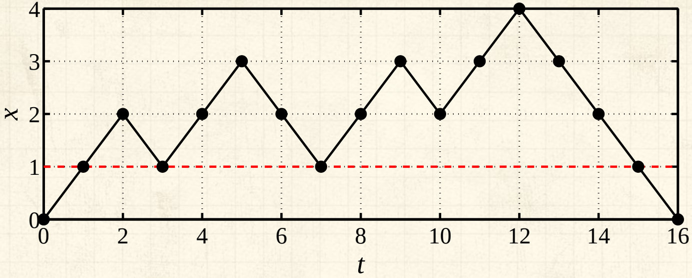


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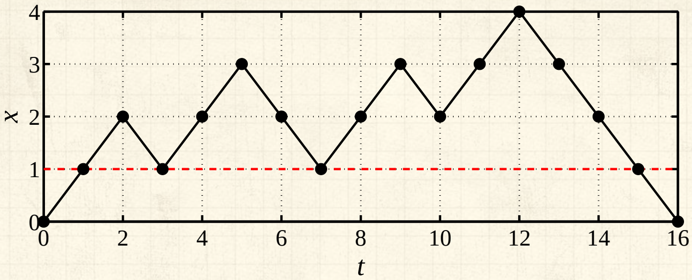


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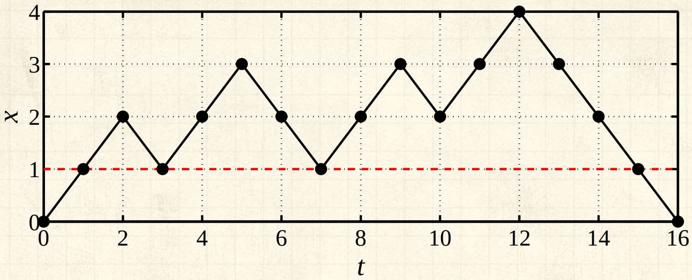


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- The 2 accounts for texters that first lurch to $x = -1$.





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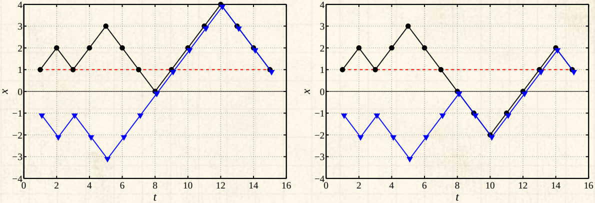
- We'll use a method of images to identify these excluded walks.

## Examples of excluded walks:



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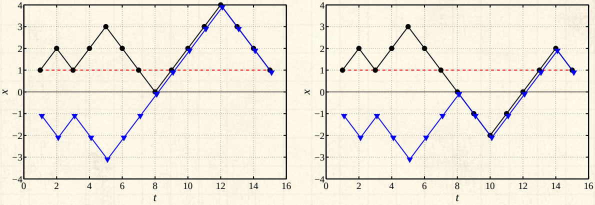


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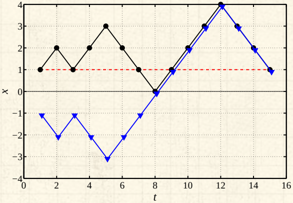


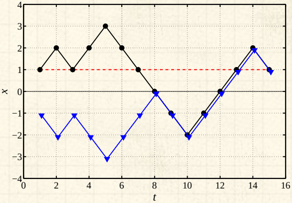


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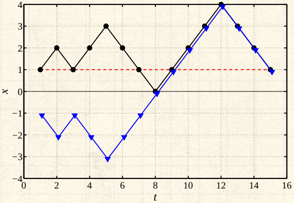


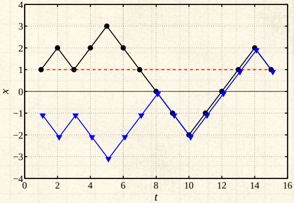


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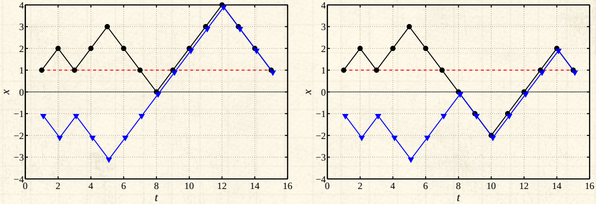


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  = $N(-1, +1, t)$

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- Overall:

$$N_{\text{fr}}(2n) = N_{\text{fr}}^{+}(2n) + N_{\text{fr}}^{-}(2n) = 2N_{\text{fr}}^{+}(2n)$$

$$= 2N(+1, +1, 2n-2) - 2N(-1, +1, 2n-2).$$

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MINISTRY OF
RANDOM WALKS

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## Higher dimensions:

- Walker in $d = 2$ dimensions must also return

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## Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
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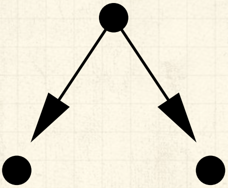
## On networks:

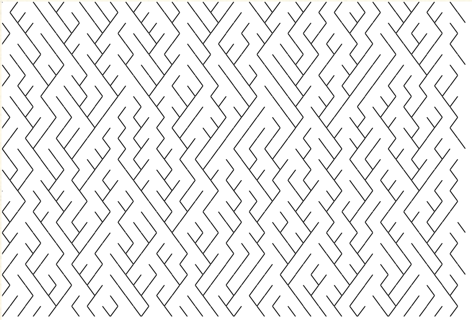
- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency.

#totallygroovy

# Scheidegger Networks [17, 4]







- Random directed network on triangular lattice.
- Toy model of real networks.
- ‘Flow’ is southeast or southwest with equal probability.

# Scheidegger networks

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- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

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- Plan: Redo calc with $\gamma$, $\tau$, and $h$.

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- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

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- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

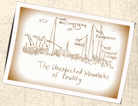
# Death ...

## Failure:

- A very simple model of failure/death
- $x_t$ = entity's 'health' at time $t$
- Start with $x_0 > 0$.
- Entity fails when $x$ hits 0.


Explaining mortality rate plateaus

"Explaining mortality rate plateaus"
Weitz and Fraser,
Proc. Natl. Acad. Sci., **98**, 15383–15386, 2001. [18]



# ... and the NBA:





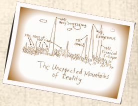
## Basketball and other sports [2]:

- Three arcsine laws (Lévy [12]) for continuous-time random walk lasting time $T$:

$$\frac{1}{\pi}\frac{1}{\sqrt{t(T-t)}}.$$

The arcsine distribution applies for:
(1) fraction of time positive,
(2) the last time the walk changes sign,
and (3) the time the maximum is achieved.

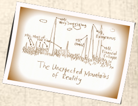
- Well approximated by basketball score lines [8, 2].
- Australian Rules Football has some differences [11].



# More than randomness

- Can generalize to Fractional Random Walks [15, 16, 14]





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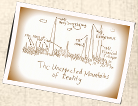

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  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation $\sigma$ scales as

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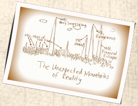

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$\alpha = 1/2$ — diffusive
$\alpha > 1/2$ — superdiffusive
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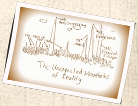

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  $\alpha = 1/2$ — diffusive
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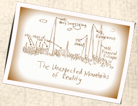
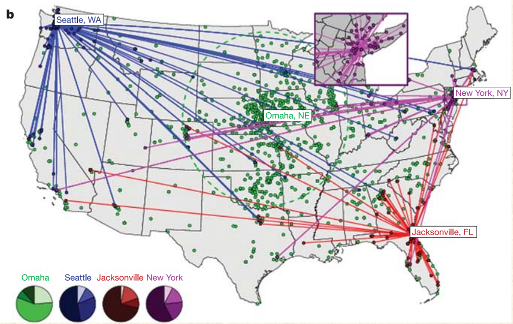
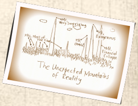
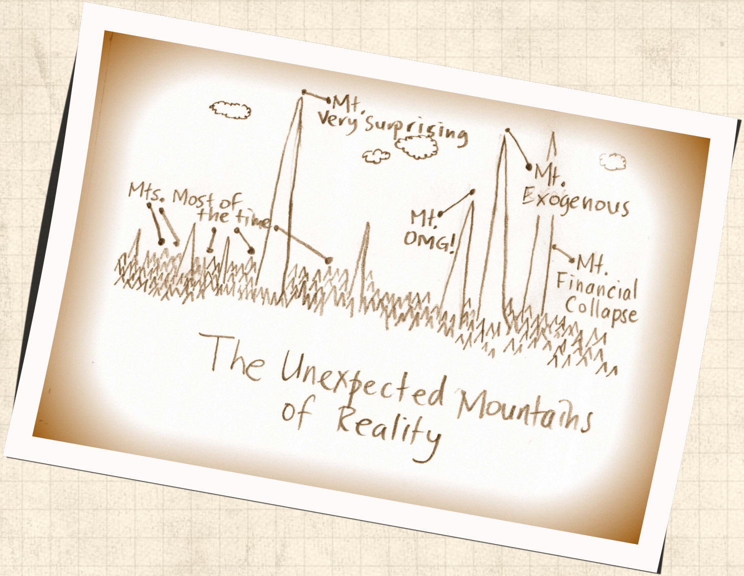
- Extensive memory of path now matters...






- First big studies of movement and interactions of people.
- Brockmann *et al.* [1] "Where's George" study.
- Beyond Lévy: Superdiffusive in space but with long waiting times.
- Tracking movement via cell phones [9] and Twitter [7].




Mt. Very Surprising
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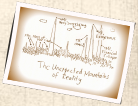

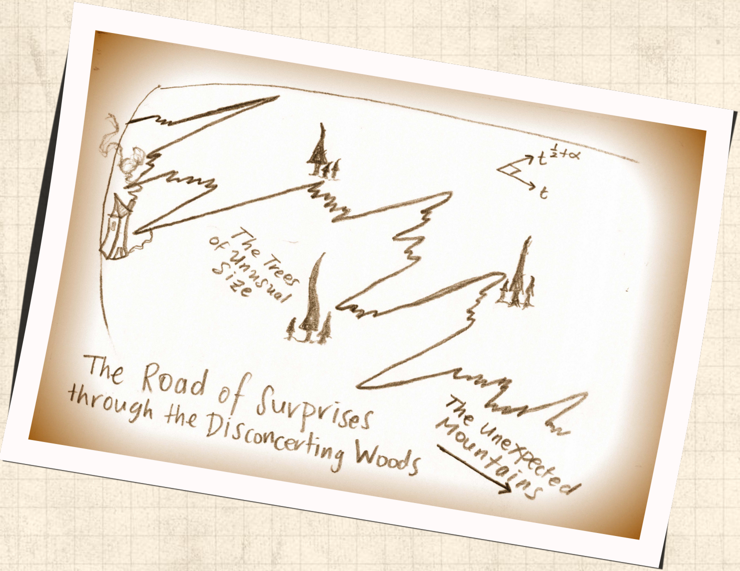
$t^{\frac{1}{2}+\alpha}$
$t$
The Trees of Unusual Size
The Road of Surprises through the Disconcerting Woods
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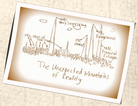
The Unexpected Mountains of Reality

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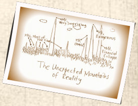

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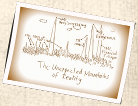

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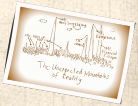

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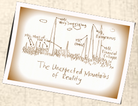

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