

# Optimal Supply Networks I: Branching

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Principles of Complex Systems, Vols. 1, 2, & 3D  
CSYS/MATH 6701, 6713, & a pretend number,  
2023–2024 | @pocsvox

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Santa Fe Institute | University of Vermont



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What's the best way to distribute stuff?



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What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...



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 **Some** fundamental network problems:



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1. Distribute stuff from a **single source** to **many sinks**





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## What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...


 **Some** fundamental network problems:


1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks



# Optimal supply networks

## What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...


 **Some** fundamental network problems:


1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks




# Optimal supply networks

## What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...

 **Some** fundamental network problems:

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2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks


 Supply and Collection are equivalent problems





# Single source optimal supply

Basic question for distribution/supply networks:

 How does flow behave given cost:

$$C = \sum_j I_j^\gamma Z_j$$

where

$I_j$  = current on link  $j$


and

$Z_j$  = link  $j$ 's impedance.



# Single source optimal supply

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 How does flow behave given cost:


$$C = \sum_j I_j^\gamma Z_j$$

where

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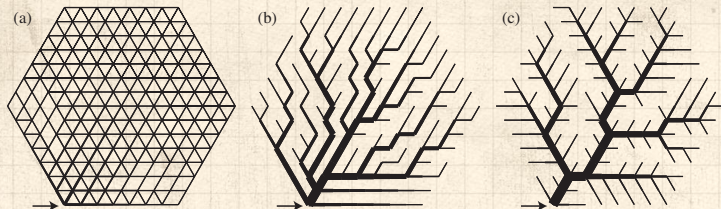
and

$Z_j$  = link  $j$ 's impedance.

 Example:  $\gamma = 2$  for electrical networks.




# Single source optimal supply



(a)  $\gamma > 1$ : Braided (bulk) flow

(b)  $\gamma < 1$ : Local minimum: Branching flow

(c)  $\gamma < 1$ : Global minimum: Branching flow

 Note: This is a single source supplying a region.

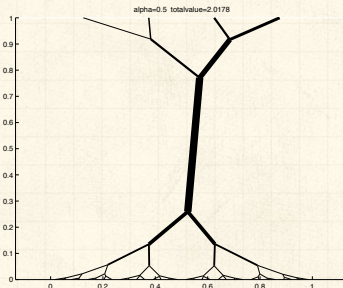
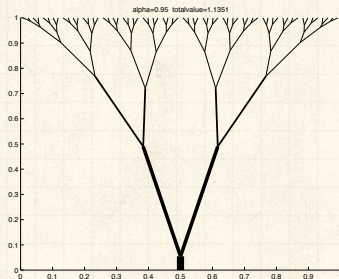
From Bohn and Magnasco <sup>[3]</sup>

See also Banavar *et al.* <sup>[1]</sup>: “Topology of the Fittest Transportation Network”; focus is on presence or absence of loops—same story



# Single source optimal supply

Optimal paths related to transport (Monge)  
problems ↗:



“Optimal paths related to transport  
problems” ↗

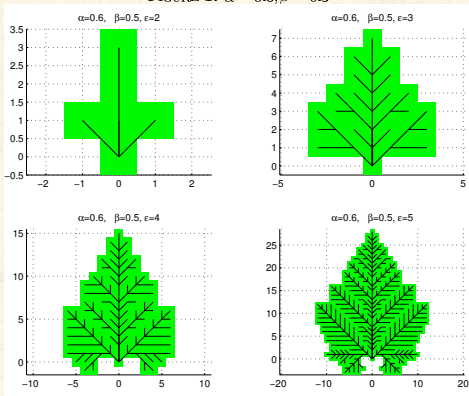
Qinglan Xia,  
Communications in Contemporary  
Mathematics, **5**, 251–279, 2003. <sup>[20]</sup>







# Growing networks—two parameter model: [21]

FIGURE 1.  $\alpha = 0.6, \beta = 0.5$



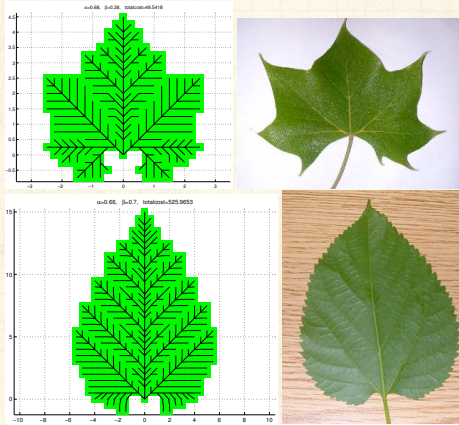
 Parameters control impedance ( $0 \leq \alpha < 1$ ) and angles of junctions ( $0 < \beta$ )


 For this example:  $\alpha = 0.6$  and  $\beta = 0.5$



# Growing networks: [21]

FIGURE 3. A maple leaf




 Top:  $\alpha = 0.66$ ,  $\beta = 0.38$ ; Bottom:  $\alpha = 0.66$ ,  $\beta = 0.70$



# Single source optimal supply



An immensely controversial issue ...

 The form of natural branching networks:  
Random, optimal, or some  
combination? [6, 19, 2, 5, 4]



# Single source optimal supply



An immensely controversial issue ...

-  The form of natural branching networks:  
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-  River networks, blood networks, trees, ...



# Single source optimal supply

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-  The form of natural branching networks:  
Random, optimal, or some  
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Two observations:




# Single source optimal supply

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-  The form of natural branching networks:  
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## Two observations:

-  Self-similar networks appear everywhere in nature  
for single source supply/single sink collection.



# Single source optimal supply

## An immensely controversial issue ...

- 🧱 The form of natural branching networks:  
Random, optimal, or some  
combination? [6, 19, 2, 5, 4]
- 🧱 River networks, blood networks, trees, ...

## Two observations:

- 🧱 Self-similar networks appear everywhere in nature  
for single source supply/single sink collection.
- 🧱 Real networks differ in details of scaling but  
reasonably agree in scaling relations.



# River network models

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Optimality:

 Optimal channel networks<sup>[13]</sup>


 Thermodynamic analogy<sup>[14]</sup>






# River network models


## Optimality:


 Optimal channel networks <sup>[13]</sup>

 Thermodynamic analogy <sup>[14]</sup>

versus ...

## Randomness:

 Scheidegger's directed random networks

 Undirected random networks





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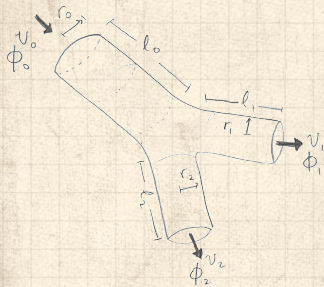
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# Optimization—Murray's law



Murray's law (1926)  
connects branch radii at  
forks: [11, 10, 12, 7, 17]



$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$

where  $r_{\text{parent}}$  = radius of  
'parent' branch, and  
 $r_{\text{offspring1}}$  and  $r_{\text{offspring2}}$  are  
radii of the two 'offspring'  
sub-branches.



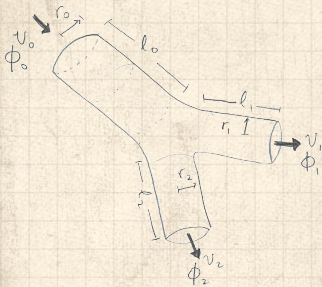
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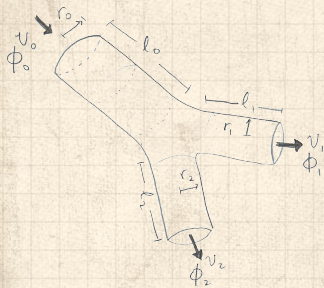
Holds up well for outer branchings of blood  
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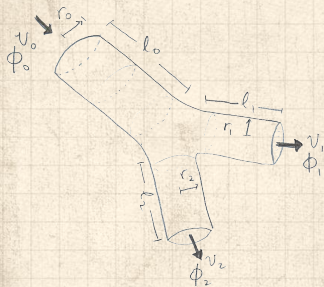
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Also found to hold for trees [12, 8] when xylem is  
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-  Holds up well for outer branchings of blood networks [15].
-  Also found to hold for trees [12, 8] when xylem is not a supporting structure [9].
-  See D'Arcy Thompson's "On Growth and Form" for background and general inspiration [16, 17].

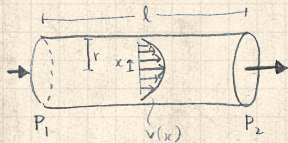




Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where  $\Delta p$  = pressure difference,  $\Phi$  = flux.



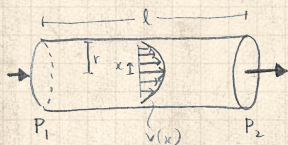




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Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius  $r$  and length  $l$ :

$$Z = \frac{8\eta l}{\pi r^4}$$



$\eta$  = dynamic viscosity (units:  $ML^{-1}T^{-1}$ ).

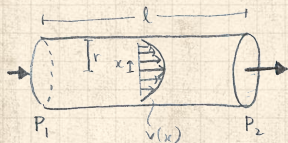




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Power required to overcome impedance:

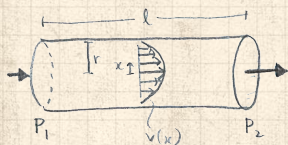
$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$



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Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

Also have rate of energy expenditure in maintaining blood given metabolic constant  $c$ :

$$P_{\text{metabolic}} = cr^2 l$$



# Optimization—Murray's law

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Aside on  $P_{\text{drag}}$



# Optimization—Murray's law


Aside on  $P_{\text{drag}}$


 Work done =  $F \cdot d$  = energy transferred by force  $F$



# Optimization—Murray's law

Aside on  $P_{\text{drag}}$


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# Optimization—Murray's law

## Aside on $P_{\text{drag}}$

 Work done =  $F \cdot d$  = energy transferred by force  $F$


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  $\Delta p$  = Pressure differential = Force per unit area





# Optimization—Murray's law

## Aside on $P_{\text{drag}}$

 Work done =  $F \cdot d$  = energy transferred by force  $F$

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  $\Phi$  = Volume flow per unit time (current)  
= cross-sectional area  $\cdot$  velocity








# Optimization—Murray's law


## Aside on $P_{\text{drag}}$

 Work done =  $F \cdot d$  = energy transferred by force  $F$

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  $\Delta p$  = Pressure differential = Force per unit area


  $\Phi$  = Volume flow per unit time (current)  
= cross-sectional area  $\cdot$  velocity

 So  $\Phi \Delta p$  = Force  $\cdot$  velocity



# Optimization—Murray's law

Murray's law:


 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}}$$



# Optimization—Murray's law

Murray's law:


 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$




# Optimization—Murray's law

Murray's law:

 Total power (cost):


$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

 Observe power increases linearly with  $\ell$





# Optimization—Murray's law

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 But  $r$ 's effect is nonlinear:





# Optimization—Murray's law


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 But  $r$ 's effect is nonlinear:

 increasing  $r$  makes flow easier **but increases metabolic cost** (as  $r^2$ )





# Optimization—Murray's law



## Murray's law:

 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

 Observe power increases linearly with  $\ell$


 But  $r$ 's effect is nonlinear:

-  increasing  $r$  makes flow easier **but increases metabolic cost** (as  $r^2$ )
-  decreasing  $r$  decrease metabolic cost **but impedance goes up** (as  $r^{-4}$ )



# Optimization—Murray's law

Murray's law:

 Minimize  $P$  with respect to  $r$ :


$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left( \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$






# Optimization—Murray's law

## Murray's law:

 Minimize  $P$  with respect to  $r$ :

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left( \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$

 Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches



# Optimization—Murray's law

Murray's law:

 Find:

$$\Phi = kr^3$$





# Optimization—Murray's law

Murray's law:

 Find:

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 Insert assignment question 






# Optimization—Murray's law

Murray's law:

 Find:

$$\Phi = kr^3$$

 Insert assignment question 

 All of this means we have a groovy cube-law:

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$



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## Murray meets Tokunaga:

  $\Phi_\omega$  = volume rate of flow into an order  $\omega$  vessel segment

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## Murray meets Tokunaga:


  $\Phi_\omega$  = volume rate of flow into an order  $\omega$  vessel segment


 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$




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 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$


 Using  $\phi_\omega = kr_\omega^3$


$$(r_\omega)^3 = 2(r_{\omega-1})^3 + \sum_{k=1}^{\omega-1} T_k (r_{\omega-k})^3$$






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 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

 Using  $\phi_\omega = kr_\omega^3$


$$(r_\omega)^3 = 2(r_{\omega-1})^3 + \sum_{k=1}^{\omega-1} T_k (r_{\omega-k})^3$$

 Same form as:

$$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$



Murray meets Tokunaga:

 Find Horton ratio for vessel radius  $R_r = r_\omega / r_{\omega-1}$ .



## Murray meets Tokunaga:

- Find Horton ratio for vessel radius  $R_r = r_\omega / r_{\omega-1}$ .
- Find  $R_r^3$  satisfies same equation as  $R_n$  and  $R_v$  ( $v$  is for volume):

$$R_r^3 = R_n = R_v$$



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- Find Horton ratio for vessel radius  $R_r = r_\omega / r_{\omega-1}$ .
- Find  $R_r^3$  satisfies same equation as  $R_n$  and  $R_v$  ( $v$  is for volume):

$$R_r^3 = R_n = R_v$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?



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Murray meets Tokunaga:

 Isometry:  $V_\omega \propto l_\omega^3$



# Optimization

## Murray meets Tokunaga:

 Isometry:  $V_\omega \propto l_\omega^3$

 Gives

$$R_\ell^3 = R_r^3 = R_n^3 = R_v^3$$



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Isometry:  $V_\omega \propto l_\omega^3$

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We need one more constraint ...



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We need one more constraint ...

West *et al.* (1997)<sup>[19]</sup> achieve similar results following Horton's laws (but this work is a disaster).





## Murray meets Tokunaga:

Isometry:  $V_\omega \propto \ell_\omega^3$

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$$R_\ell^3 = R_r^3 = R_n = R_v$$




We need one more constraint ...

West *et al.* (1997)<sup>[19]</sup> achieve similar results following Horton's laws (but this work is a disaster).

So does Turcotte *et al.* (1998)<sup>[18]</sup> using Tokunaga (sort of).






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