Generating Functions and Networks

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Principles of Complex Systems, Vols. 1, 2, & 3D CSYS/MATH 6701, 6713, & a pretend number, 2023–2024| @pocsvox

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Computational Story Lab | Vermont Complex Systems Center
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Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
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THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
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THE STORIES CONTAGIOUS
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THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
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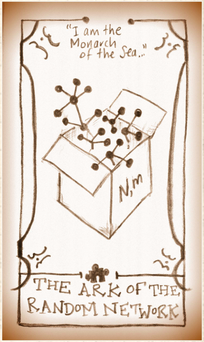
"I am the Monarch of the Sea.."
N,m
THE ARK OF THE RANDOM NETWORK


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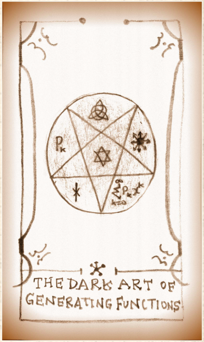
THE DARK ART OF GENERATING FUNCTIONS

Outline

Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

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- Related to Fourier, Laplace, Mellin, ...

Simple examples:

Rolling dice and flipping coins:

- $p_k^{(⚂)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

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- We'll come back to these simple examples as we derive various delicious properties of generating functions.

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- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \Big|_{x=0}$$

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  3. Add a coin flip to one die roll.

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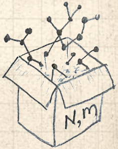
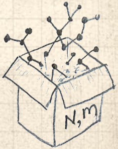


Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$



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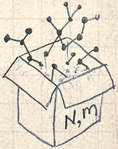

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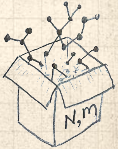

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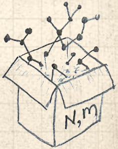

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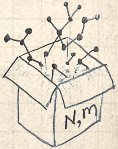

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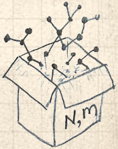

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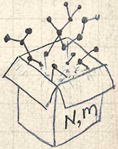

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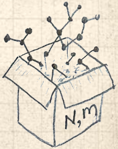
- Now find how F_R is related to F_P ...



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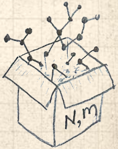

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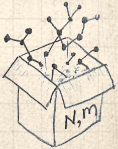

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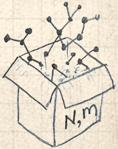

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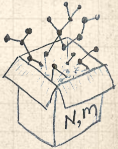

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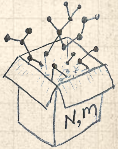

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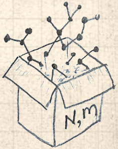

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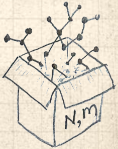

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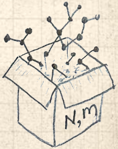

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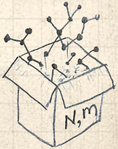
Finally, since $\langle k \rangle = F_P'(1)$,

$$\boxed{F_R(x) = \frac{F_P'(x)}{F_P'(1)}}$$



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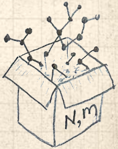

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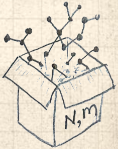

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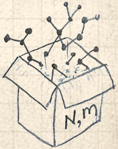
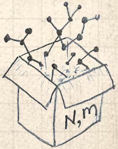


Outline



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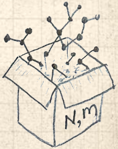
To figure out the size of the largest component (S_1), we need more resolution on component sizes.



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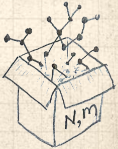

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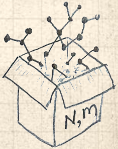

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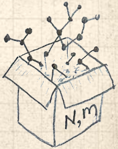
Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors $\Leftrightarrow$ components



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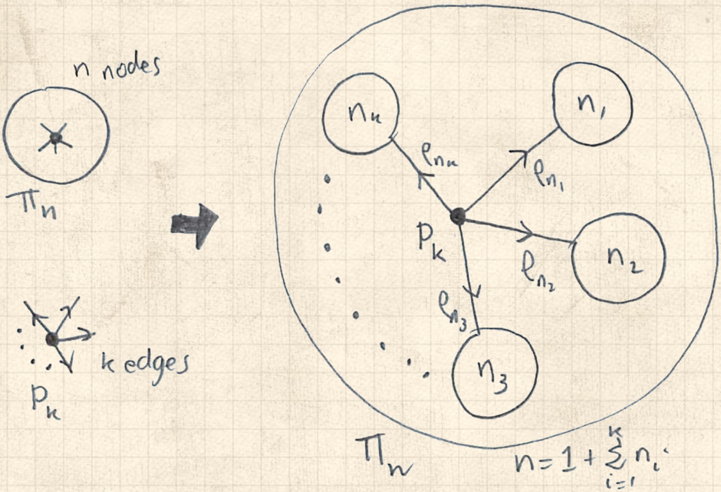
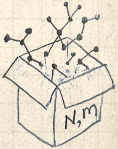


- Markov property of random networks connects π_n, ρ_n, and P_k.



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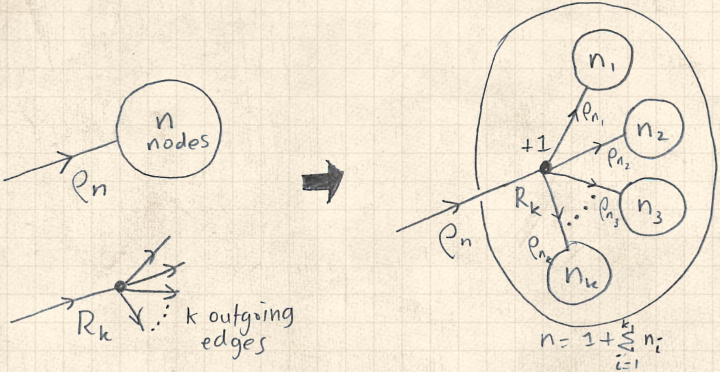

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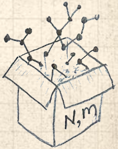

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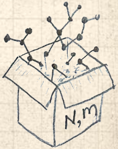

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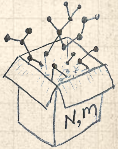

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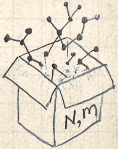

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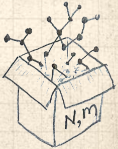
Our mission, which we accept:

- Determine and connect the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$



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THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
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THE ARKS OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
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THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
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THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
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THE SOCIAL WILD
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THE RATIO OF GAIN
THE MECHANICS OF FAME
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THE ADJACENT NARRATIVES
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With some concentration, observe:

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- Alternate, groovier proof in the accompanying assignment.

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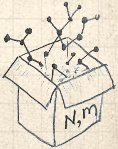
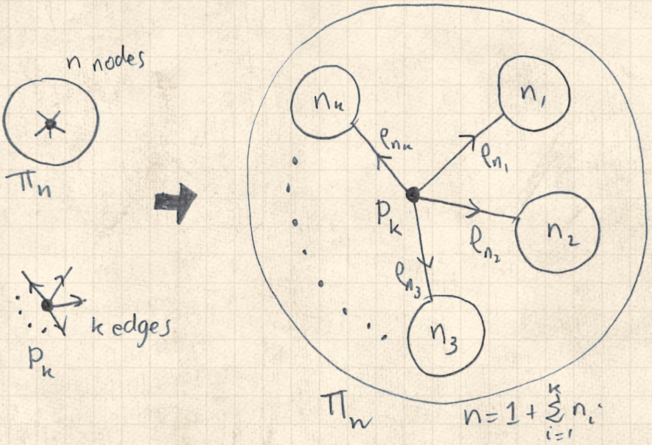
Outline

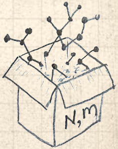


Connecting generating functions:

- **Goal:** figure out forms of the component generating functions, F_π and F_ρ.



- Relate π_n to P_k and ρ_n through one step of recursion.



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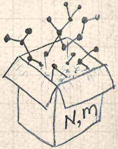

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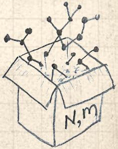

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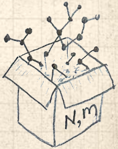

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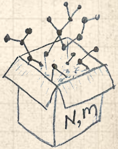

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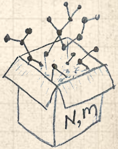

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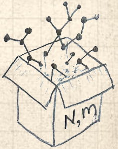

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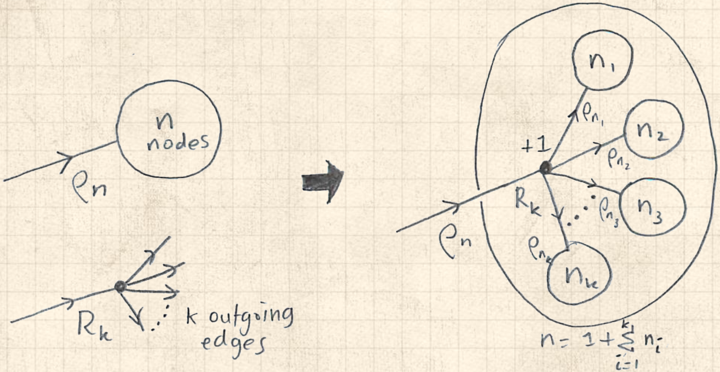

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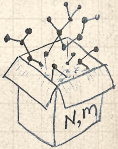

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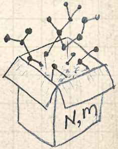

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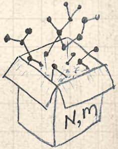

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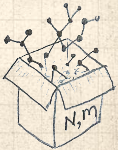

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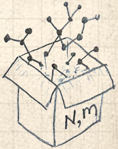

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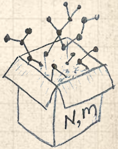

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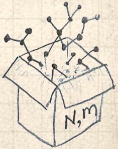

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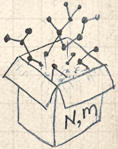

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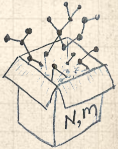

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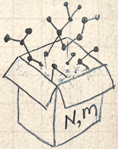

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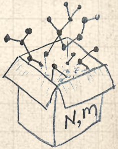

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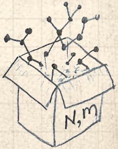
- The first equation then immediately gives us F_π in terms of F_ρ and F_R.



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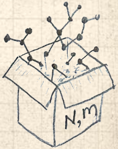

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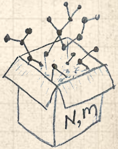

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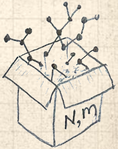

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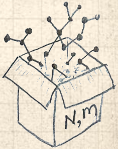

Component sizes

- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.
- Set $x = 1$ in our two equations:

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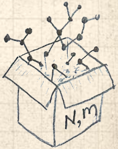
- Solve second equation numerically for $F_\rho(1)$.
- Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.



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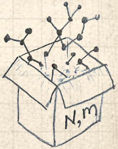

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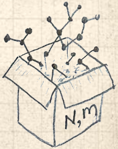

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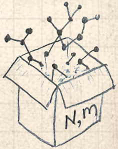

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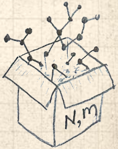

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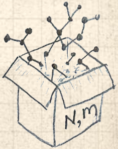

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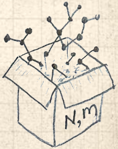

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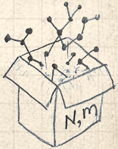

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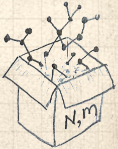

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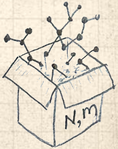

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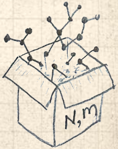

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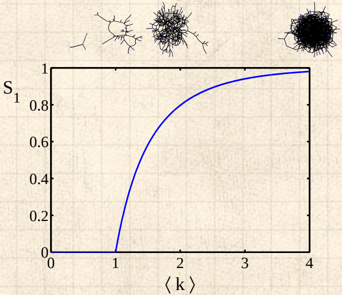

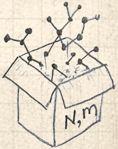

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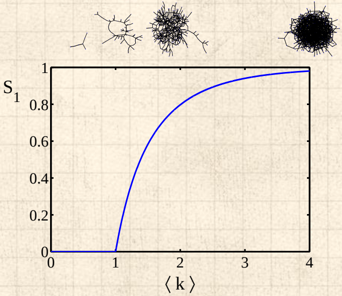

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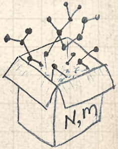

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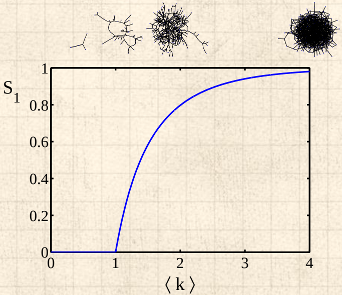

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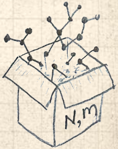
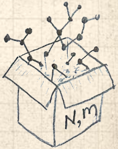
- Again, we (usually) have to resort to numerics ...



Outline



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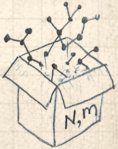

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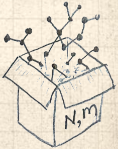

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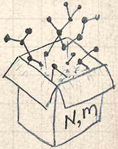

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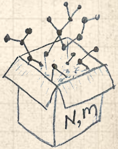

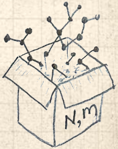

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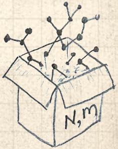

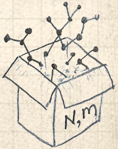

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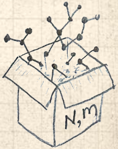

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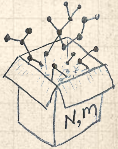

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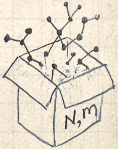

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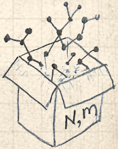

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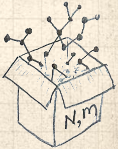

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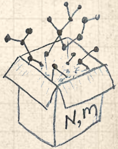
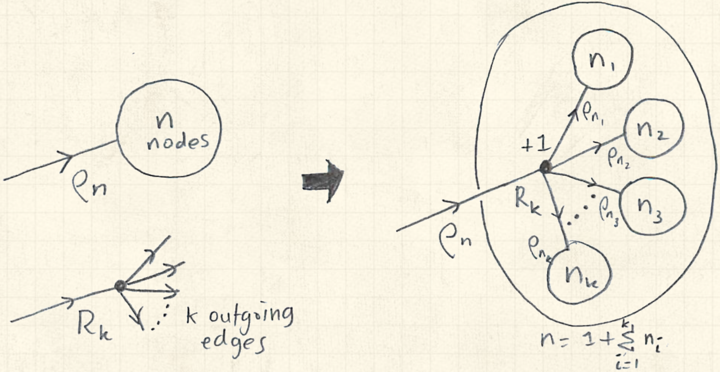
  - $F'_P(1) = \langle k \rangle_P = 2$ and $F'_R(1) = \langle k \rangle_R = \frac{3}{2}$.
- Things to figure out: Component size generating functions for π_n and ρ_n, and the size of the giant component.



Find $F_\rho(x)$ first:

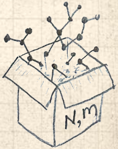
- We know:

$$F_\rho(x) = xF_R\left(F_\rho(x)\right).$$



Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$



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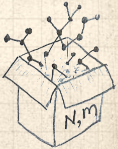
$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$



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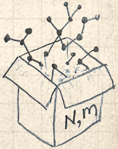
- The promise: non-negative powers of x with non-negative coefficients.



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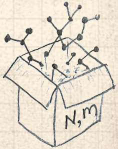
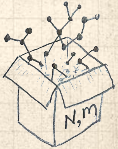
- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.
- First: which sign do we take?



Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.



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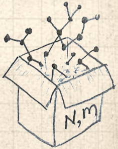
- So we must have:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$

- We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \ldots$$



- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

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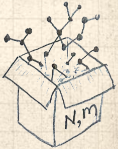

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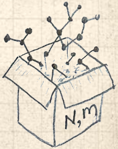
where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.



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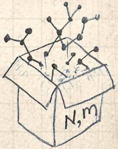
- Note: $(1+z)^\theta \sim 1 + \theta z$ always.



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$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

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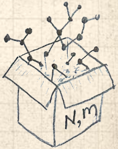
- Do odd powers make sense?

We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$



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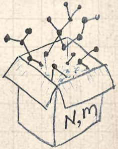

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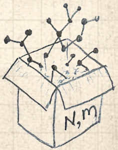
- Delicious.



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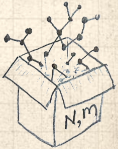
- In principle, we can now extract all the π_n.



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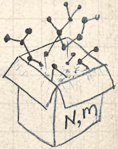
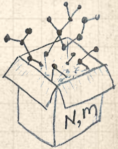
- Delicious.
- In principle, we can now extract all the π_n.
- But let's just find the size of the giant component.



First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$



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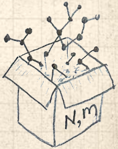

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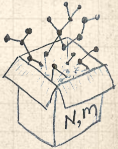
$$F_\pi(1) = 1 \cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right)$$



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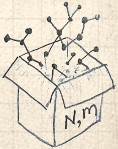

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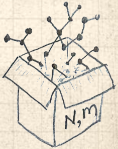

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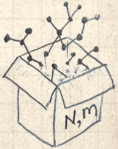
- This is the probability that a random chosen node belongs to a finite component.
- Finally, we have

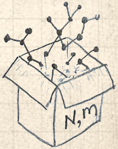
$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$

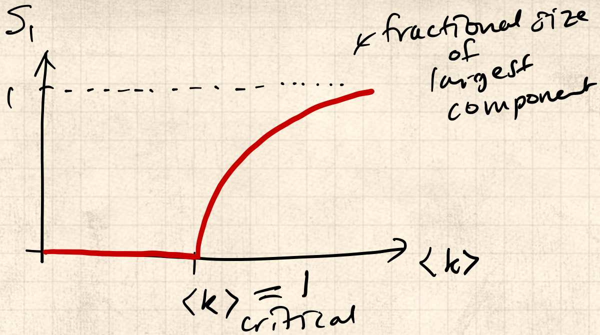


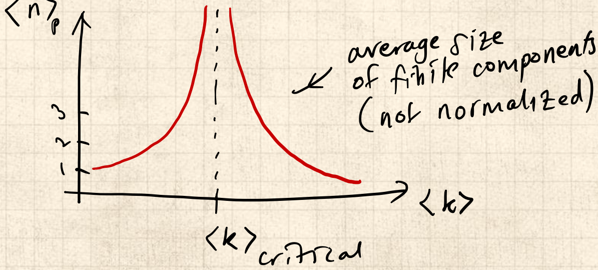
Outline

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References




S_1
1
fractional size of largest component
$\langle k \rangle$
$\langle k \rangle_{\text{critical}} = 1$


$\langle n \rangle_p$
3
2
1
average size of finite components (not normalized)
$\langle k \rangle$
$\langle k \rangle_{\text{critical}}$

Average component size

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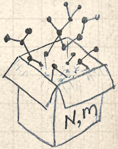

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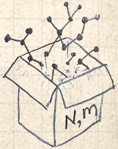

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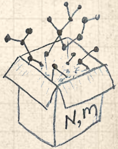

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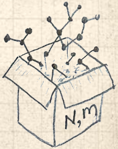

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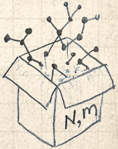

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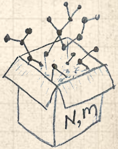

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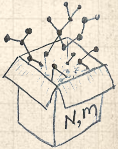
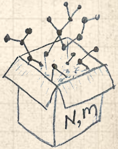


Average component size

Example: Standard random graphs.



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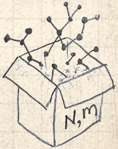

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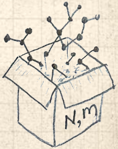

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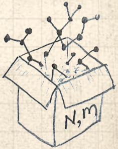

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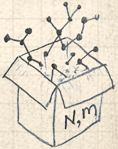

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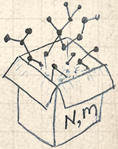

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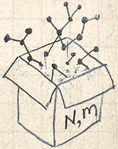

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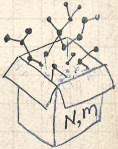

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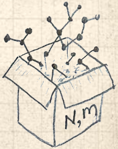

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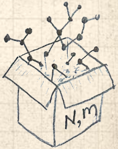

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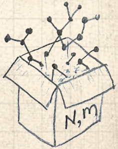

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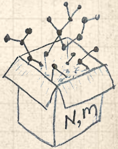

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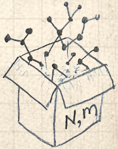

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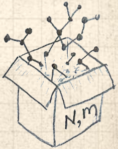

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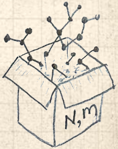

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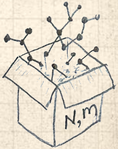

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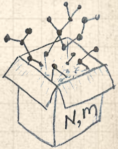

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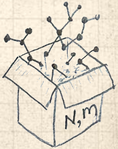

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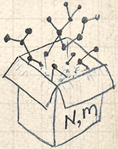

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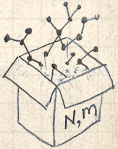
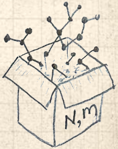


- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.



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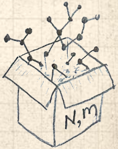
- Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$



We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

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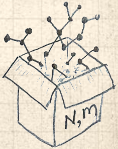

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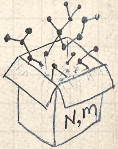

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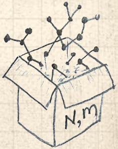

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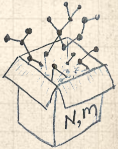

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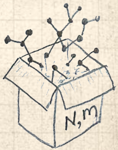

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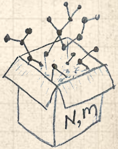
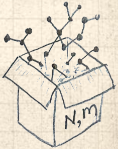
- So, kinda small.



Nutshell

- Generating functions allow us to strangely calculate features of random networks.



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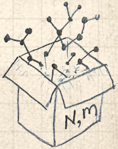

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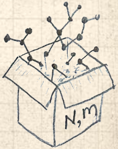

Nutshell

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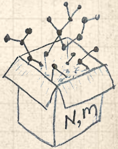
- They're a bit scary and magical.
- Generating functions can be useful for contagion.
- But: For the big results, more direct, physics-bearing calculations are possible.



Neural reboot (NR):

Elevation:

https://www.youtube.com/watch?v=bGBoZbT7cR8?rel=0

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