

# Branching Networks II

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Principles of Complex Systems, Vols. 1, 2, & 3D  
CSYS/MATH 6701, 6713, & a pretend number,  
2023–2024 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center  
Santa Fe Institute | University of Vermont



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Networks II  
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Tokunaga

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Scaling relations

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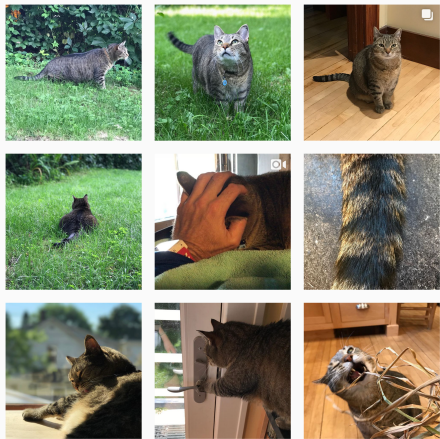
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## Special Guest Executive Producer



 On Instagram at [pratchett\\_the\\_cat](https://www.instagram.com/pratchett_the_cat) 

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# Outline

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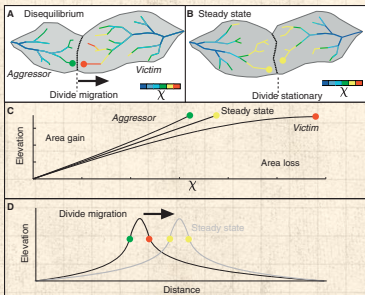
# Piracy on the high $\chi$ 's:



“Dynamic Reorganization of River Basins” 

Willett et al.,

Science, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x, t)}{\partial t} = U - K A^m \left| \frac{\partial z(x, t)}{\partial x} \right|^n$$

$$z(x) = z_b + \left( \frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_b}^x \left( \frac{A_0}{A(x')} \right)^{m/n} dx'$$

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[https://www.youtube.com/watch?v=FnroL1\\_-l2c?rel=0](https://www.youtube.com/watch?v=FnroL1_-l2c?rel=0)

More: [How river networks move across a landscape](#)   
(Science Daily)









# Can Horton and Tokunaga be happy?

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Horton and Tokunaga seem different:

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Horton and Tokunaga seem different:

- 📦 In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.



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Horton and Tokunaga seem different:

-  In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
-  Oddly, Horton's laws have **four** parameters and Tokunaga has **two** parameters.



# Can Horton and Tokunaga be happy?


## Horton and Tokunaga seem different:


- ❏ In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
  - ❏ Oddly, Horton's laws have **four** parameters and Tokunaga has **two** parameters.
  - ❏  $R_n, R_a, R_\ell,$  and  $R_s$  **versus**  $T_1$  and  $R_T$ . One simple redundancy:  $R_\ell = R_s$ .
- Insert assignment question 





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Insert assignment question 


 To make a connection, clearest approach is to start with Tokunaga's law ...





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Insert assignment question 

 To make a connection, clearest approach is to start with Tokunaga's law ...

 Known result: Tokunaga  $\rightarrow$  Horton <sup>[18, 19, 20, 9, 2]</sup>



# Let us make them happy

We need one more ingredient:

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# Let us make them happy

We need one more ingredient:

Space-fillingness

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# Let us make them happy

We need one more ingredient:

## Space-fillingness

 A network is **space-filling** if the average distance between adjacent streams is roughly constant.

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# Let us make them happy

We need one more ingredient:

## Space-fillingness




-  A network is **space-filling** if the average distance between adjacent streams is roughly constant.
-  Reasonable for river and cardiovascular networks



# Let us make them happy

We need one more ingredient:

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-  A network is **space-filling** if the average distance between adjacent streams is roughly constant.
-  Reasonable for river and cardiovascular networks
-  For river networks:  
**Drainage density**  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.



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-  For river networks:  
**Drainage density**  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.
-  In terms of basin characteristics:





$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}}$$



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-  A network is **space-filling** if the average distance between adjacent streams is roughly constant.
-  Reasonable for river and cardiovascular networks
-  For river networks:  
**Drainage density**  $\rho_{dd}$  = inverse of typical distance between channels in a landscape.
-  In terms of basin characteristics:

$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_{\omega} \bar{s}_{\omega}}{a_{\Omega}}$$



# More with the happy-making thing

Start with Tokunaga's law:  $T_k = T_1 R_T^{k-1}$

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# More with the happy-making thing

Start with Tokunaga's law:  $T_k = T_1 R_T^{k-1}$



Start looking for Horton's stream number law:


$$n_\omega / n_{\omega+1} = R_n.$$






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 Estimate  $n_\omega$ , the number of streams of order  $\omega$  in terms of other  $n_{\omega'}, \omega' > \omega$ .





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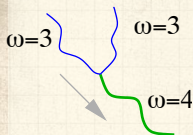
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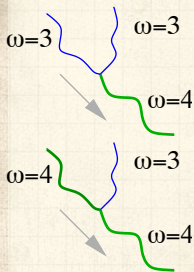
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2. Running into and being absorbed by a stream of higher order  $\omega' > \omega$  ...





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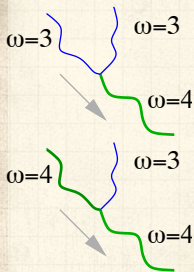
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▶  $2n_{\omega+1}$  streams of order  $\omega$  do this

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# More with the happy-making thing

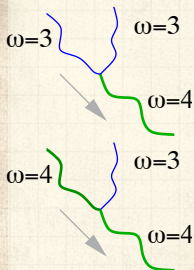
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...

▶  $2n_{\omega+1}$  streams of order  $\omega$  do this

2. Running into and being absorbed by a stream of higher order  $\omega' > \omega$  ...

▶  $n_{\omega'} T_{\omega'-\omega}$  streams of order  $\omega$  do this



# More with the happy-making thing

Putting things together:



$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} +$$

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# More with the happy-making thing

Putting things together:



$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}}_{\text{absorption}} n_{\omega'}$$

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 Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain  $R_n$ .

 Insert assignment question 






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 Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain  $R_n$ .

 Insert assignment question 

 Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)

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# Finding other Horton ratios

Connect Tokunaga to  $R_s$

 Now use uniform drainage density  $\rho_{dd}$ .

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# Finding other Horton ratios

## Connect Tokunaga to $R_s$

- Now use uniform drainage density  $\rho_{dd}$ .
- Assume side streams are roughly separated by distance  $1/\rho_{dd}$ .



# Finding other Horton ratios

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- For an order  $\omega$  **stream segment**, expected length is

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + \sum_{k=1}^{\omega-1} T_k \right)$$



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$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + \sum_{k=1}^{\omega-1} T_k \right)$$

- Substitute in Tokunaga's law  $T_k = T_1 R_T^{k-1}$ :

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right)$$



# Finding other Horton ratios

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- Now use uniform drainage density  $\rho_{dd}$ .
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$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^\omega$$



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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T$$





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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



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Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



Recall  $R_\ell = R_s$  so

$$R_\ell = R_s = R_T$$



# Horton and Tokunaga are happy

Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



Recall  $R_\ell = R_s$  so

$$R_\ell = R_s = R_T$$



And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$



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Some observations:

  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .



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## Some observations:

  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .

 Seems that  $R_a$  must as well ...



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## Some observations:

  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .

 Seems that  $R_a$  must as well ...

 Suggests Horton's laws must contain some redundancy



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## Some observations:

-   $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
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-  Suggests Horton's laws must contain some redundancy
-  We'll in fact see that  $R_a = R_n$ .



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## Some observations:

- 🧱  $R_n$  and  $R_\ell$  depend on  $T_1$  and  $R_T$ .
- 🧱 Seems that  $R_a$  must as well ...
- 🧱 Suggests Horton's laws must contain some redundancy
- 🧱 We'll in fact see that  $R_a = R_n$ .
- 🧱 Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]





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## The other way round

 Note: We can invert the expressions for  $R_n$  and  $R_\ell$  to find Tokunaga's parameters in terms of Horton's parameters.



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$$R_T = R_\ell,$$



$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$



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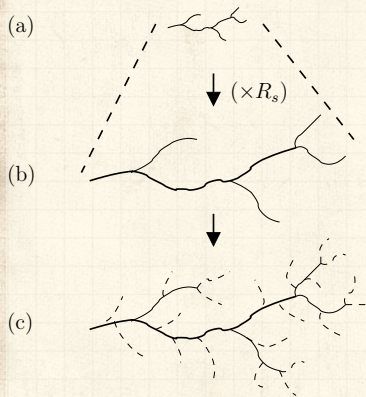
$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$

 Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...



# Horton and Tokunaga are friends

From Horton to Tokunaga [2]



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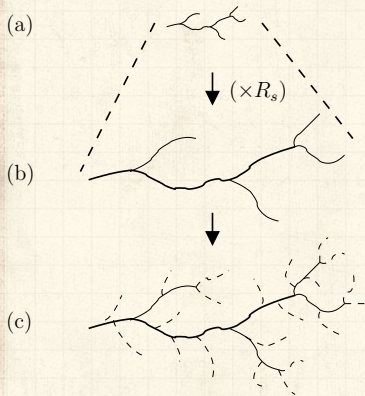


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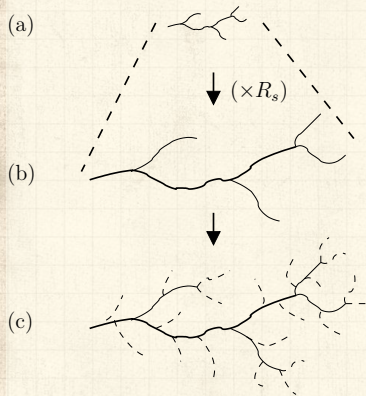


Assume Horton's laws hold for number and length



# Horton and Tokunaga are friends

From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length

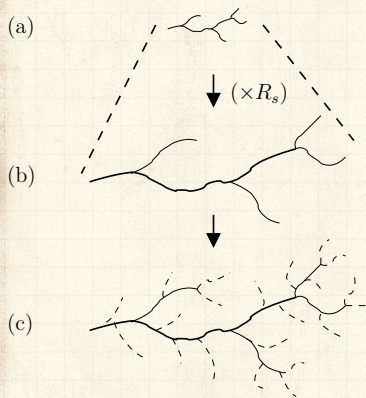


Start with picture showing an order  $\omega$  stream and order  $\omega - 1$  generating and side streams.



# Horton and Tokunaga are friends

## From Horton to Tokunaga [2]



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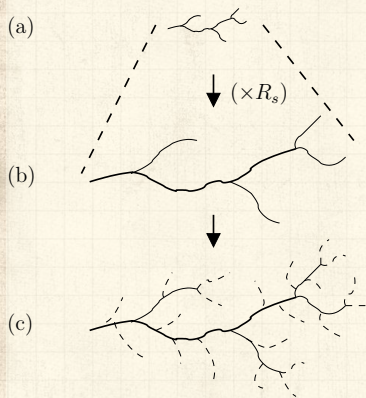


Scale up by a factor of  $R_\ell$ , orders increment to  $\omega + 1$  and  $\omega$ .



# Horton and Tokunaga are friends

## From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length



Start with picture showing an order  $\omega$  stream and order  $\omega - 1$  generating and side streams.



Scale up by a factor of  $R_e$ , orders increment to  $\omega + 1$  and  $\omega$ .




Maintain drainage density by adding new order  $\omega - 1$  streams





# Horton and Tokunaga are friends

...and in detail:

 Must retain same drainage density.

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...and in detail:



Must retain same drainage density.



Add an extra  $(R_\ell - 1)$  first order streams for each original tributary.



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...and in detail:

- Must retain same drainage density.
- Add an extra  $(R_\ell - 1)$  first order streams for each original tributary.
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- For large  $\omega$ , Tokunaga's law is the solution—let's check ...



# Horton and Tokunaga are friends

Just checking:

 Substitute Tokunaga's law  $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$  into

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$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{k-1}}{R_\ell - 1} = T_1 R_\ell^{k-1} \quad \dots\text{yep.}$$



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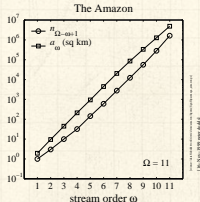
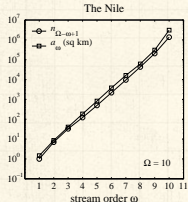
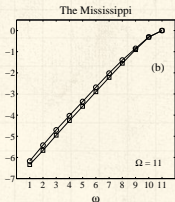
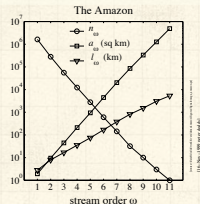
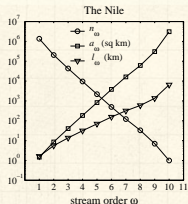
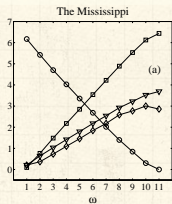
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In bottom plots, stream number graph has been flipped vertically.



Highly suggestive that  $R_n \equiv R_a \dots$



# Measuring Horton ratios is tricky:

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How robust are our estimates of ratios?



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How robust are our estimates of ratios?



Rule of thumb: discard data for two smallest and two largest orders.



# Mississippi:

$\omega$ range	$R_n$	$R_a$	$R_\ell$	$R_s$	$R_a/R_n$
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3, 8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7, 8]	4.16	4.67	2.41	2.56	1.12
mean $\mu$	4.69	4.85	2.40	2.33	1.04
std dev $\sigma$	0.21	0.13	0.04	0.07	0.03
$\sigma/\mu$	0.045	0.027	0.015	0.031	0.024



# Amazon:

$\omega$ range	$R_n$	$R_a$	$R_\ell$	$R_s$	$R_a/R_n$
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean $\mu$	4.42	4.53	2.25	2.10	1.02
std dev $\sigma$	0.17	0.10	0.10	0.09	0.02
$\sigma/\mu$	0.038	0.023	0.045	0.042	0.019



# Reducing Horton's laws:

Rough first effort to show  $R_n \equiv R_a$ :

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# Reducing Horton's laws:


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  $a_\Omega \propto$  sum of all stream segment lengths in a order  $\Omega$  basin (assuming uniform drainage density)



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 So:

$$\begin{aligned} a_\Omega &\simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{dd} \\ &\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \hat{1}^{n_\Omega}}_{n_\omega} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_\omega} \\ &= \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^\omega \end{aligned}$$



# Reducing Horton's laws:

Continued ...



$$a_{\Omega} \propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^{\omega}$$

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# Reducing Horton's laws:

Continued ...



$$\begin{aligned} a_{\Omega} &\propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^{\omega} \\ &= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)} \end{aligned}$$

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# Reducing Horton's laws:

Continued ...



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# Reducing Horton's laws:

Continued ...



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 So,  $a_{\Omega}$  is growing like  $R_n^{\Omega}$  and therefore:

$$R_n \equiv R_a$$

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Not quite:

 ...But this only a rough argument as Horton's laws do not imply a strict hierarchy



# Reducing Horton's laws:

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Not quite:





 ...But this only a rough argument as Horton's laws do not imply a strict hierarchy

 Need to account for sidebranching.



# Reducing Horton's laws:

## Not quite:

-  ...But this only a rough argument as Horton's laws do not imply a strict hierarchy
-  Need to account for sidebranching.
-  Insert assignment question 



# Equipartitioning:

Intriguing division of area:



Observe: Combined area of basins of order  $\omega$  independent of  $\omega$ .

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# Equipartitioning:


## Intriguing division of area:


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- Not obvious: basins of low orders not necessarily contained in basin on higher orders.




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$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$







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 Reason:

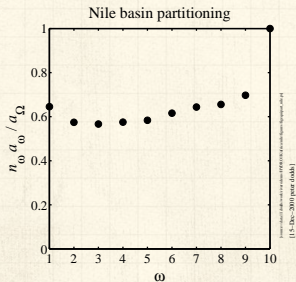
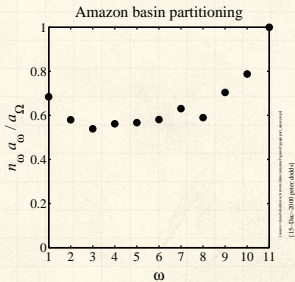
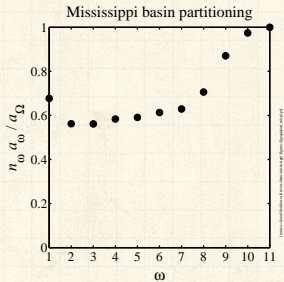
$$n_\omega \propto (R_n)^{-\omega}$$

$$\bar{a}_\omega \propto (R_a)^\omega \propto n_\omega^{-1}$$



# Equipartitioning:

Some examples:



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# Neural Reboot: Fwoompf

<https://www.youtube.com/watch?v=5mUs70SqD4o?rel=0>



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# Scaling laws

The story so far:

 Natural branching networks are **hierarchical**, **self-similar** structures

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The story so far:




 Natural branching networks are **hierarchical**,  
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 Hierarchy is **mixed**



# Scaling laws

The story so far:

-  Natural branching networks are **hierarchical**, **self-similar** structures
-  Hierarchy is **mixed**
-  Tokunaga's law describes detailed architecture:  
$$T_k = T_1 R_T^{k-1}.$$



# Scaling laws

The story so far:

- 🧱 Natural branching networks are **hierarchical**, **self-similar** structures
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$$T_k = T_1 R_T^{k-1}.$$
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# Scaling laws







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- 🧱 Natural branching networks are **hierarchical**, **self-similar** structures
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- 🧱 Only two Horton laws are independent ( $R_n = R_a$ )



# Scaling laws

## The story so far:

-  Natural branching networks are **hierarchical**, **self-similar** structures
-  Hierarchy is **mixed**
-  Tokunaga's law describes detailed architecture:  
$$T_k = T_1 R_T^{k-1}.$$
-  We have connected Tokunaga's and Horton's laws
-  Only two Horton laws are independent ( $R_n = R_a$ )
-  Only **two** parameters are **independent**:  
$$(T_1, R_T) \Leftrightarrow (R_n, R_s)$$



# Scaling laws

A little further ...

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A little further ...

 Ignore stream ordering for the moment



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A little further ...



Ignore stream ordering for the moment






Pick a random location on a branching network  $p$ .



# Scaling laws





## A little further ...

-  Ignore stream ordering for the moment
-  Pick a random location on a branching network  $p$ .
-  Each point  $p$  is associated with a basin and a longest stream length



# Scaling laws






## A little further ...

-  Ignore stream ordering for the moment
-  Pick a random location on a branching network  $p$ .
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-  **Q:** What is probability that the  $p$ 's drainage basin has area  $a$ ?



# Scaling laws

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-  **Q:** What is probability that the longest stream from  $p$  has length  $\ell$ ?





# Scaling laws






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-  Each point  $p$  is associated with a basin and a longest stream length
-  **Q:** What is probability that the  $p$ 's drainage basin has area  $a$ ?  $P(a) \propto a^{-\tau}$  for large  $a$
-  **Q:** What is probability that the longest stream from  $p$  has length  $\ell$ ?



# Scaling laws







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# Scaling laws

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-  **Q:** What is probability that the  $p$ 's drainage basin has area  $a$ ?  $P(a) \propto a^{-\tau}$  for large  $a$
-  **Q:** What is probability that the longest stream from  $p$  has length  $\ell$ ?  $P(\ell) \propto \ell^{-\gamma}$  for large  $\ell$
-  Roughly observed:  $1.3 \lesssim \tau \lesssim 1.5$  and  $1.7 \lesssim \gamma \lesssim 2.0$



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## Probability distributions with power-law decays



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## Probability distributions with power-law decays

 We see them everywhere:



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## Probability distributions with power-law decays



We see them everywhere:



Earthquake magnitudes (Gutenberg-Richter law)



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## Probability distributions with power-law decays



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-  Earthquake magnitudes (Gutenberg-Richter law)
-  City sizes (Zipf's law)



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## Probability distributions with power-law decays



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-  Earthquake magnitudes (Gutenberg-Richter law)
-  City sizes (Zipf's law)
-  Word frequency (Zipf's law) <sup>[22]</sup>









# Scaling laws

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-  Word frequency (Zipf's law) <sup>[22]</sup>
-  Wealth (maybe not—at least heavy tailed)



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-  Word frequency (Zipf's law)<sup>[22]</sup>
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-  Statistical mechanics (phase transitions)<sup>[5]</sup>



## Probability distributions with power-law decays



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A big part of the story of complex systems



## Probability distributions with power-law decays



We see them everywhere:

-  Earthquake magnitudes (Gutenberg-Richter law)
-  City sizes (Zipf's law)
-  Word frequency (Zipf's law) <sup>[22]</sup>
-  Wealth (maybe not—at least heavy tailed)
-  Statistical mechanics (phase transitions) <sup>[5]</sup>



A big part of the story of complex systems








Arise from **mechanisms**: growth, randomness, optimization, ...



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A big part of the story of complex systems



Arise from **mechanisms**: growth, randomness, optimization, ...



Our task is always to illuminate the mechanism ...



# Scaling laws

## Connecting exponents

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# Scaling laws

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We have the detailed picture of branching networks (Tokunaga and Horton)



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-  Plan: Derive  $P(a) \propto a^{-\tau}$  and  $P(\ell) \propto \ell^{-\gamma}$  starting with Tokunaga/Horton story <sup>[17, 1, 2]</sup>





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Finding  $\gamma$ :

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# Scaling laws

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Often useful to work with **cumulative distributions**, especially when dealing with power-law distributions.





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 Also known as the exceedance probability.



# Scaling laws

Finding  $\gamma$ :

 The connection between  $P(x)$  and  $P_{>}(x)$  when  $P(x)$  has a power law tail is simple:

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$$\propto l_*^{-(\gamma-1)} \quad \text{for } l_{\max} \gg l_*$$





# Scaling laws

Finding  $\gamma$ :

 **Aim:** determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$

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# Scaling laws




## Finding $\gamma$ :

-  **Aim:** determine probability of randomly choosing a point on a network with main stream length  $> \ell_*$
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$$P_{>}(l_*) = \frac{N_{>}(l_*; \Delta)}{N_{>}(0; \Delta)}.$$

where  $N_{>}(l_*; \Delta)$  is the number of sites with main stream length  $> l_*$ .




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-  Use Horton's law of stream segments:  
 $\bar{s}_\omega / \bar{s}_{\omega-1} = R_s \dots$



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Finding  $\gamma$ :

 Set  $l_* = \bar{l}_\omega$  for some  $1 \ll \omega \ll \Omega$ .

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$$P_{>}(\bar{l}_\omega) = \frac{N_{>}(\bar{l}_\omega; \Delta)}{N_{>}(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}$$






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 Denominator is  $a_{\Omega} \rho_{dd}$ , a constant.




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## Finding $\gamma$ :


 Set  $l_* = \bar{l}_\omega$  for some  $1 \ll \omega \ll \Omega$ .



$$P_>(\bar{l}_\omega) = \frac{N_>(\bar{l}_\omega; \Delta)}{N_>(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}$$

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 Denominator is  $a_\Omega \rho_{dd}$ , a constant.


 So ...using Horton's laws ...

$$P_>(\bar{l}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})$$




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 Cleaning up irrelevant constants:

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 Change summation order by substituting  $\omega'' = \Omega - \omega'$ .

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$$P_{>}(\bar{\ell}_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left( \frac{R_s}{R_n} \right)^{\Omega-\omega''}$$



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# Scaling laws

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again using  $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$




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# Scaling laws

Finding  $\gamma$ :


 Nearly there:

$$P_{>}(\bar{l}_\omega) \propto \left( \frac{R_n}{R_s} \right)^{-\omega}$$



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 Need to express right hand side in terms of  $\bar{\ell}_{\omega}$ .





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 Recall that  $\bar{\ell}_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$ .





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$$\bar{\ell}_{\omega} \propto R_{\ell}^{\omega} = R_s^{\omega} = e^{\omega \ln R_s}$$



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Finding  $\gamma$ :


 Therefore:

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$$P_{>}(\bar{\ell}_{\omega}) \propto e^{-\omega \ln(R_n/R_s)} = (e^{\omega \ln R_s})^{-\ln(R_n/R_s)/\ln(R_s)}$$

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 Proceeding in a similar fashion, we can show


$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert assignment question 




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 Such connections between exponents are called **scaling relations**




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Insert assignment question 

 Such connections between exponents are called **scaling relations**

 Let's connect to one last relationship: Hack's law



# Scaling laws

Hack's law: [6]



$$l \propto a^h$$

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# Scaling laws

Hack's law: <sup>[6]</sup>



$$l \propto a^h$$

 Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .





# Scaling laws

Hack's law: <sup>[6]</sup>



$$l \propto a^h$$

 Typically observed that  $0.5 \lesssim h \lesssim 0.7$ .

 Use Horton laws to connect  $h$  to Horton ratios:

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$$\propto (R_n^\omega)^{\ln R_s / \ln R_n}$$



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Horton  $\Leftrightarrow$   
Tokunaga

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# We mentioned there were a good number of 'laws': [2]

**Relation:**    **Name or description:**

$T_k = T_1 (R_T)^{k-1}$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\ell_{\omega+1} / \ell_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law





# Connecting exponents

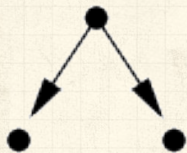
Only 3 parameters are independent:  
e.g., take  $d$ ,  $R_n$ , and  $R_s$

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	$d$
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$ $R_T = R_s$
$n_\omega/n_{\omega+1} = R_n$	$R_n$
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



# Scheidegger's model

## Directed random networks <sup>[11, 12]</sup>



$$P(\searrow) = P(\swarrow) = 1/2$$



Functional form of all scaling laws exhibited but exponents differ from real world <sup>[15, 16, 14]</sup>



Useful and interesting test case



# A toy model—Scheidegger's model

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Tokunaga

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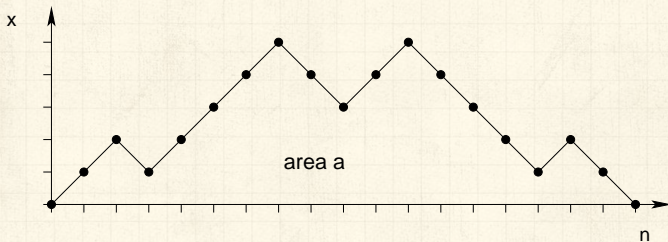
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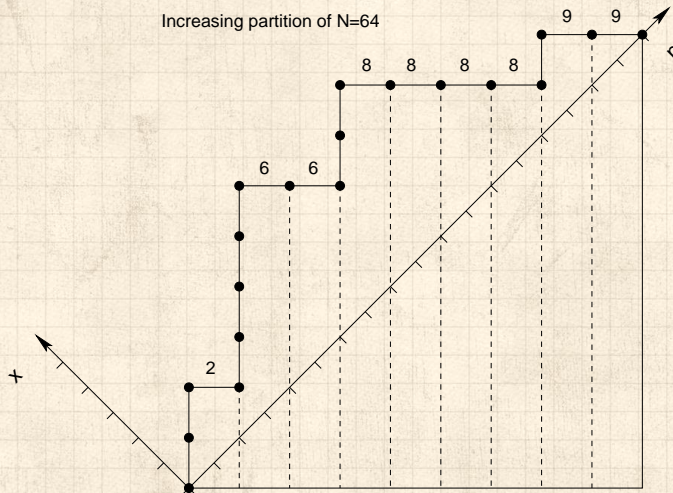
References

## Random walk basins:

 Boundaries of basins are random walks



# Scheidegger's model



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# Scheidegger's model

Prob for first return of a random walk in  $(1+1)$  dimensions (from CSYS/MATH 300):

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$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so  $P(\ell) \propto \ell^{-3/2}$ .




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Find  $\tau = 4/3$ ,  $h = 2/3$ ,  $\gamma = 3/2$ ,  $d = 1$ .





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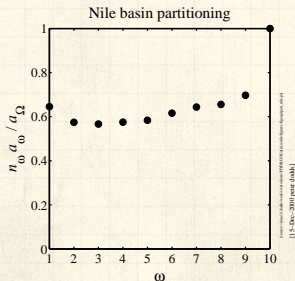
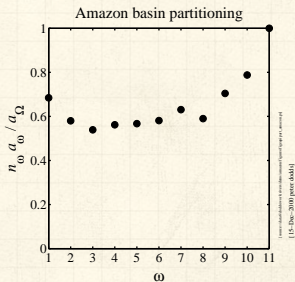
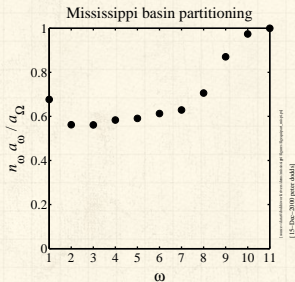


$R_n$  and  $R_\ell$  have not been derived analytically.



# Equipartitioning reexamined:

Recall this story:



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# Equipartitioning

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What about


$$P(a) \sim a^{-\tau} \quad ?$$



# Equipartitioning

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 Since  $\tau > 1$ , suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$



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# Equipartitioning


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  $P(a)$  overcounts basins within basins ...

 while stream ordering separates basins ...



# Fluctuations

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Moving beyond the mean:





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Moving beyond the mean:

 Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$



# Fluctuations

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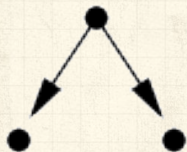
$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$

- Natural generalization to consider relationships between **probability distributions**
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



# A toy model—Scheidegger's model

## Directed random networks <sup>[11, 12]</sup>



$$P(\searrow) = P(\swarrow) = 1/2$$



Flow is directed downwards



# Generalizing Horton's laws

$$\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$$

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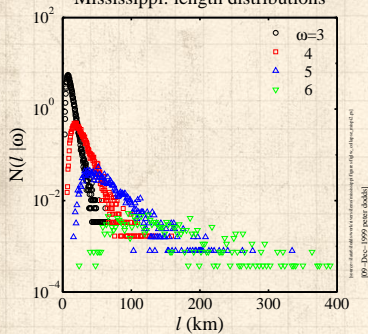


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Mississippi: length distributions

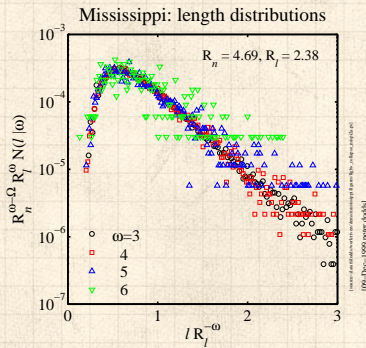
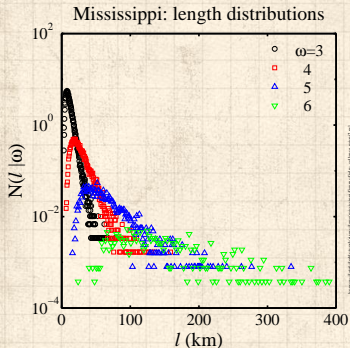




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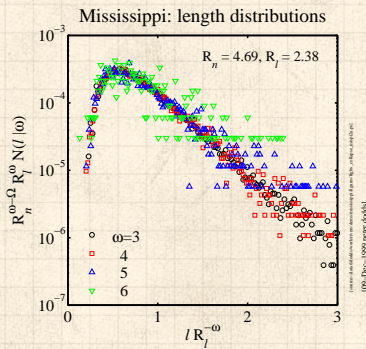
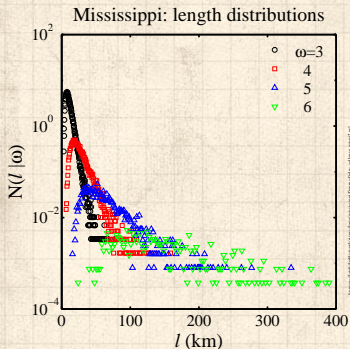
Scaling collapse works well for intermediate orders



# Generalizing Horton's laws

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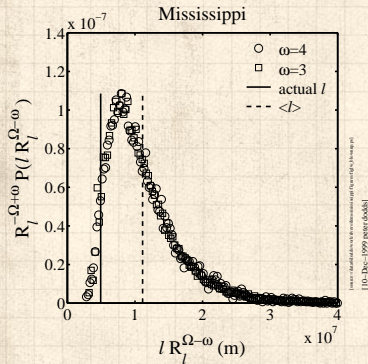
Scaling collapse works well for intermediate orders

All **moments** grow exponentially with order




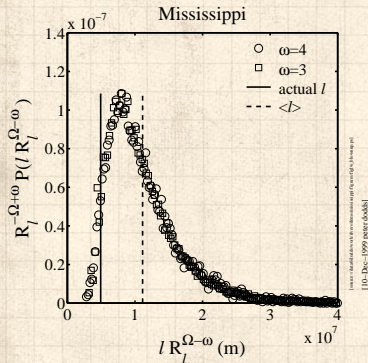
# Generalizing Horton's laws


 How well does overall basin fit internal pattern?



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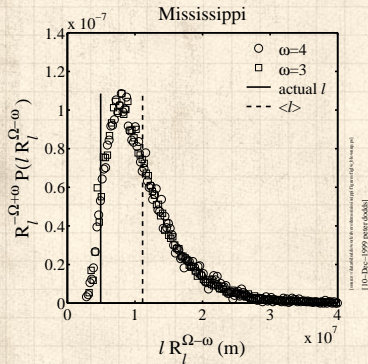


 Actual length = 4920  
km (at 1 km res)



# Generalizing Horton's laws

How well does overall basin fit internal pattern?




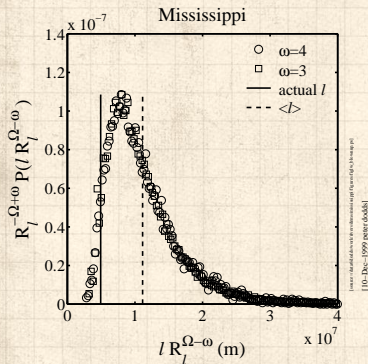
Actual length = **4920 km** (at 1 km res)


Predicted Mean length = **11100 km**




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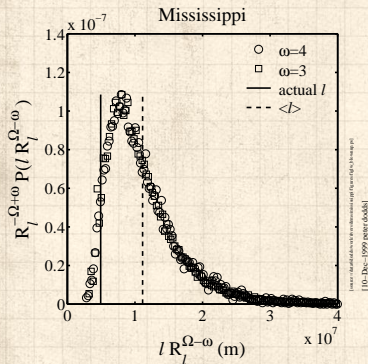
 Predicted Mean length = **11100 km**


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# Generalizing Horton's laws


 How well does overall basin fit internal pattern?



 Actual length = **4920 km** (at 1 km res)


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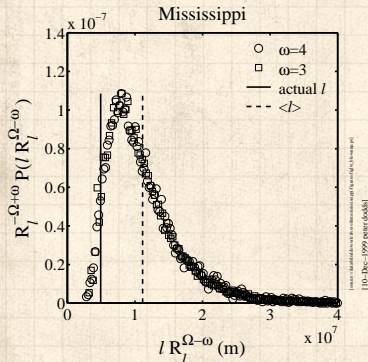
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 Actual length/Mean length = **44 %**




# Generalizing Horton's laws


 How well does overall basin fit internal pattern?




 Actual length = **4920 km** (at 1 km res)

 Predicted Mean length = **11100 km**

 Predicted Std dev = **5600 km**

 Actual length/Mean length = **44 %**

 Okay.





# Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in  $10^3$  km):

basin:	$l_\Omega$	$\bar{l}_\Omega$	$\sigma_l$	$l_\Omega/\bar{l}_\Omega$	$\sigma_l/\bar{l}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	$a_\Omega$	$\bar{a}_\Omega$	$\sigma_a$	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



# Combining stream segments distributions:

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Tokunaga

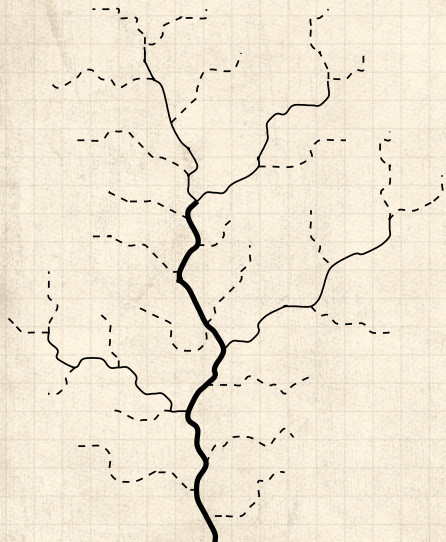
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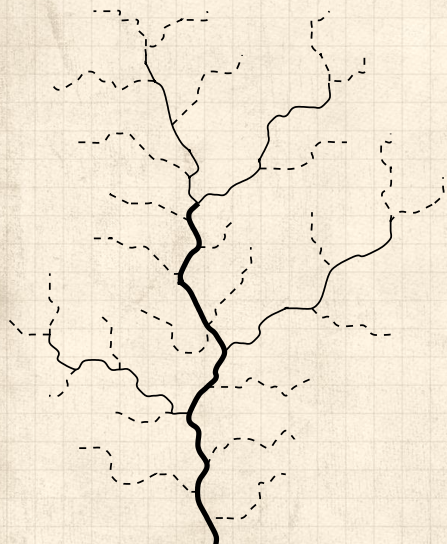
Stream segments  
sum to give main  
stream lengths



$$l_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$



# Combining stream segments distributions:



Stream segments  
sum to give main  
stream lengths



$$l_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$



$P(l_{\omega})$  is a  
convolution of  
distributions for  
the  $s_{\omega}$



# Generalizing Horton's laws



Sum of variables  $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$  leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \dots * N(s|\omega)$$

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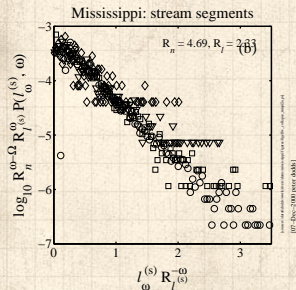
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$$N(s|\omega) = \frac{1}{R_n^\omega R_l^\omega} F(s/R_l^\omega)$$

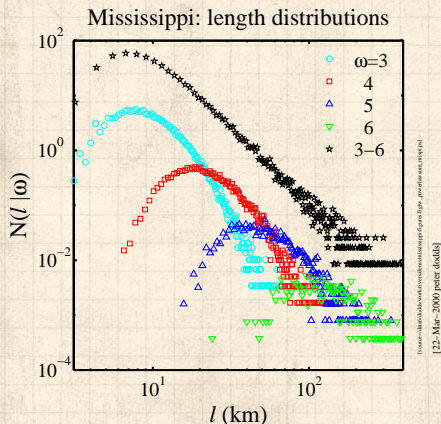
$$F(x) = e^{-x/\xi}$$


Mississippi:  $\xi \simeq 900$  m.



# Generalizing Horton's laws


 Next level up: Main stream length distributions must combine to give overall distribution for stream length

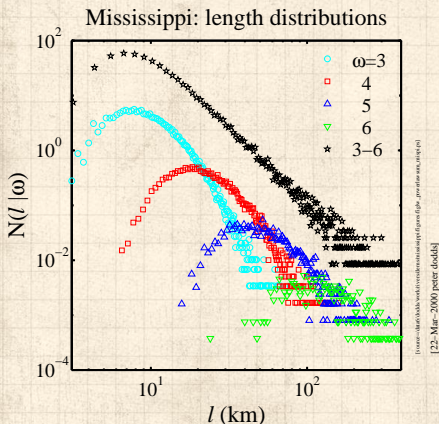



  $P(l) \sim l^{-\gamma}$





# Generalizing Horton's laws

 Next level up: Main stream length distributions must combine to give overall distribution for stream length




  $P(l) \sim l^{-\gamma}$


 Another round of convolutions <sup>[3]</sup>

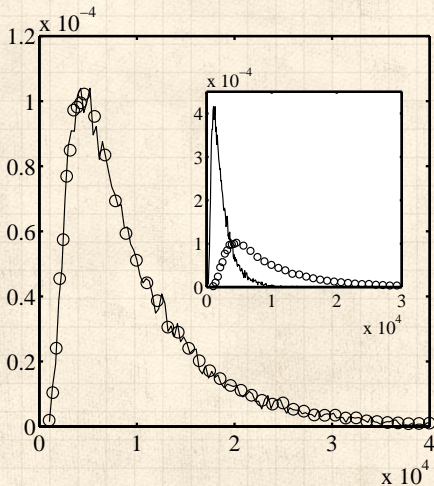
 Interesting ...



# Generalizing Horton's laws

 Number and area distributions for the Scheidegger model [3]

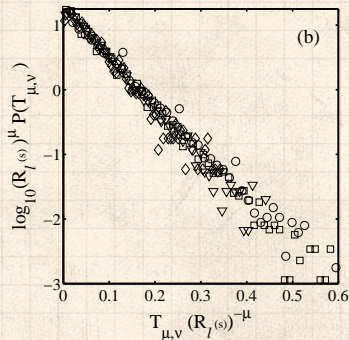
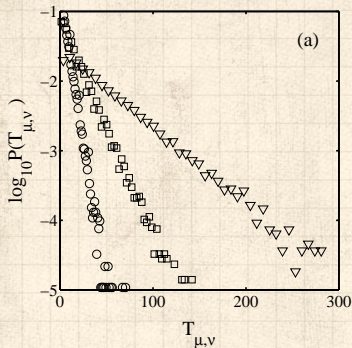
  $P(n_{1,6})$  versus  $P(a_6)$  for a randomly selected  $\omega = 6$  basin.







# Generalizing Tokunaga's law

Scheidegger:



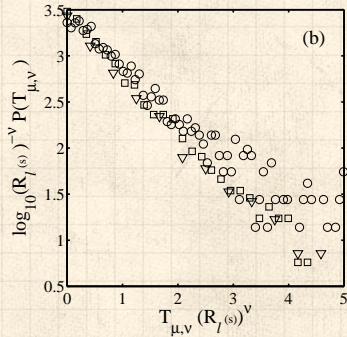
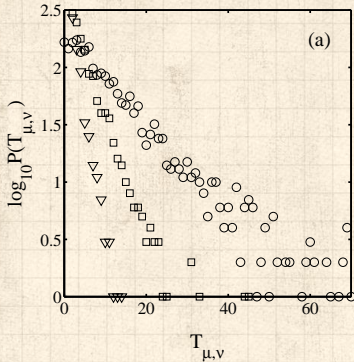
 Observe exponential distributions for  $T_{\mu,\nu}$

 Scaling collapse works using  $R_s$



# Generalizing Tokunaga's law

Mississippi:



Same data collapse for Mississippi ...



# Generalizing Tokunaga's law


So


$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t [T_{\mu,\nu}/(R_s)^{\mu-\nu-1}]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})$$

 Exponentials arise from randomness.

 Look at joint probability  $P(s_\mu, T_{\mu,\nu})$ .



# Generalizing Tokunaga's law

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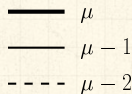
## Network architecture:



Inter-tributary  
lengths  
exponentially  
distributed



Leads to random  
spatial  
distribution of  
stream segments



# Generalizing Tokunaga's law

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 Follow streams segments down stream from their beginning



# Generalizing Tokunaga's law

- Follow stream segments down stream from their beginning
- Probability (or rate) of an order  $\mu$  stream segment terminating is **constant**:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1} \xi_s$$



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- Inter-tributary lengths exponentially distributed
- $\Rightarrow$  random spatial distribution of stream segments




# Generalizing Tokunaga's law

 Joint distribution for generalized version of Tokunaga's law:


$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

where

  $p_\nu$  = probability of absorbing an order  $\nu$  side stream





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  $p_\nu$  = probability of absorbing an order  $\nu$  side stream

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 Approximation: depends on distance units of  $s_\mu$

 In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



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 Now deal with this thing:


$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$



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 Set  $(x, y) = (s_{\mu}, T_{\mu, \nu})$  and  $q = 1 - p_{\nu} - \tilde{p}_{\mu}$ , approximate liberally.




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 Set  $(x, y) = (s_\mu, T_{\mu,\nu})$  and  $q = 1 - p_\nu - \tilde{p}_\mu$ , approximate liberally.

 Obtain


$$P(x, y) = Nx^{-1/2} [F(y/x)]^x$$

where

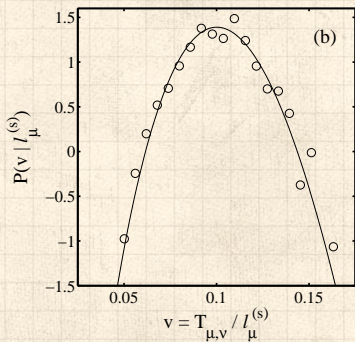
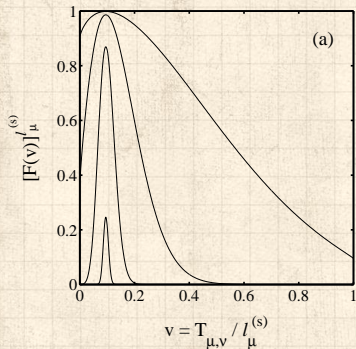
$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$



# Generalizing Tokunaga's law


 Checking form of  $P(s_\mu, T_{\mu,\nu})$  works:

Scheidegger:

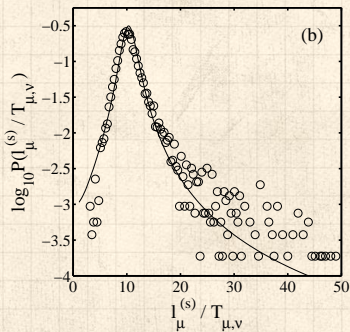
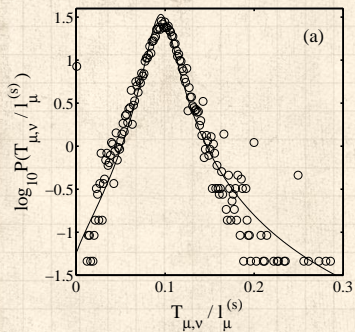




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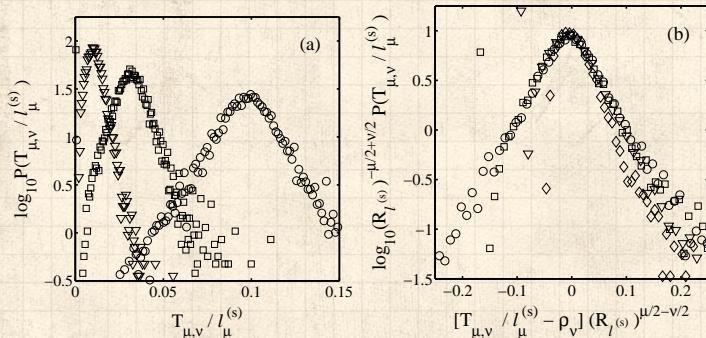
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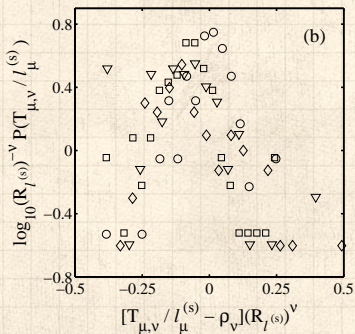
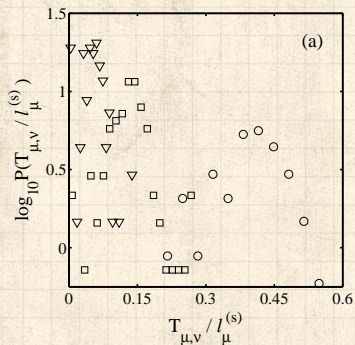
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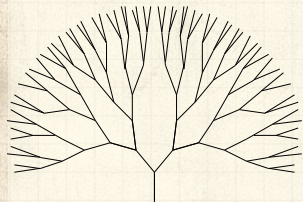
Mississippi:





# Models

## Random subnetworks on a Bethe lattice <sup>[13]</sup>



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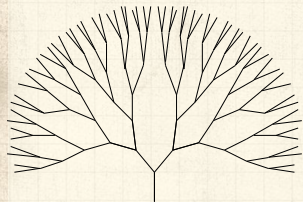
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## Random subnetworks on a Bethe lattice <sup>[13]</sup>

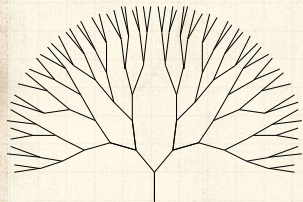


Dominant theoretical concept for several decades.

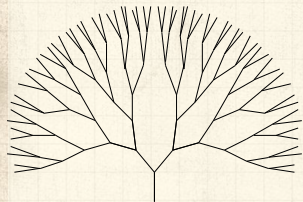





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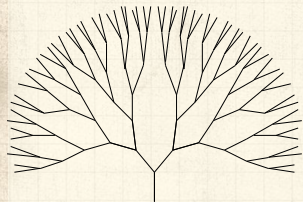






-  Dominant theoretical concept for several decades.
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-  Led to idea of “Statistical inevitability” of river network statistics <sup>[7]</sup>





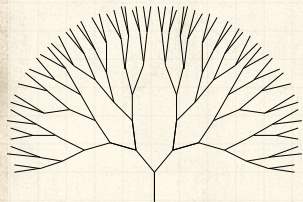
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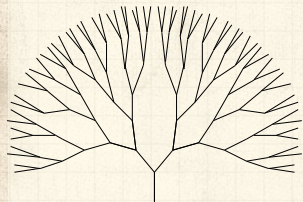
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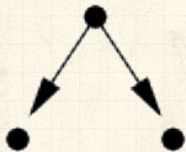


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-  In fact, Bethe lattices  $\simeq$  infinite dimensional spaces (oops).
-  So let's move on ...



# Scheidegger's model

## Directed random networks [11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$



Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]



# Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. <sup>[10]</sup>

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 **But:** numerical method used matters.

 **And:** Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network <sup>[8]</sup>



# Theoretical networks

## Summary of universality classes:


<b>network</b>	<b>h</b>	<b>d</b>
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$$h \Rightarrow \ell \propto a^h \text{ (Hack's law).}$$

$$d \Rightarrow \ell \propto L_{\parallel}^d \text{ (stream self-affinity).}$$





## Branching networks II Key Points:

 Horton's laws and Tokunaga's law all fit together.






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- 🧱 For scaling laws, only  $h = \ln R_\ell / \ln R_n$  and  $d$  are needed.
- 🧱 Laws can be extended nicely to laws of distributions.
- 🧱 Numerous models of branching network evolution exist: nothing rock solid yet ...?



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