Random Networks Nutshell

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2024–2025

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Computational Story Lab | Vermont Complex Systems Center
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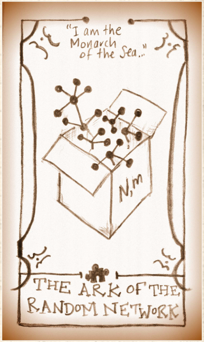
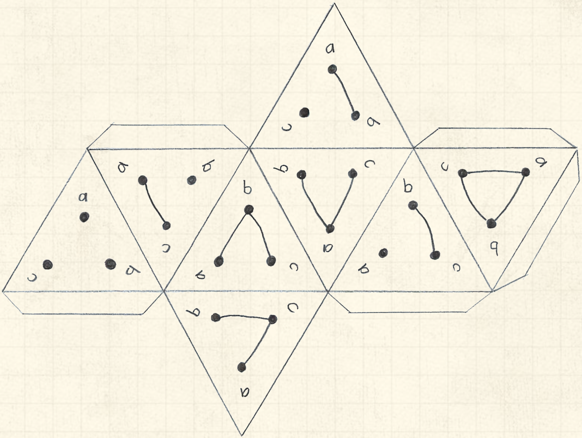
Outline


"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK

Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

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Random networks

Pure, abstract random networks:

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- Consider set of all networks with N labelled nodes and m edges.
- Standard random network =
  one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

Random networks—basic features:

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- Real world: links are usually costly so real networks are almost always sparse.

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   - 1 and 2 are effectively equivalent for large N.

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- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions
Generalized Random Networks
Configuration model
How to build in practice
Motifs
Strange friends
Largest component
References

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Random networks: examples

Next slides:

Example realizations of random networks

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Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples for N=500

m = 100
$\langle k \rangle$ = 0.4

m = 200
$\langle k \rangle$ = 0.8

m = 230
$\langle k \rangle$ = 0.92

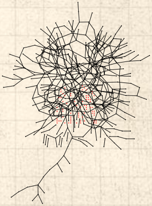
m = 240
$\langle k \rangle$ = 0.96

m = 250
$\langle k \rangle$ = 1

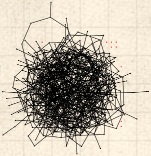
m = 260
$\langle k \rangle$ = 1.04

m = 280
$\langle k \rangle$ = 1.12

m = 300
$\langle k \rangle$ = 1.2

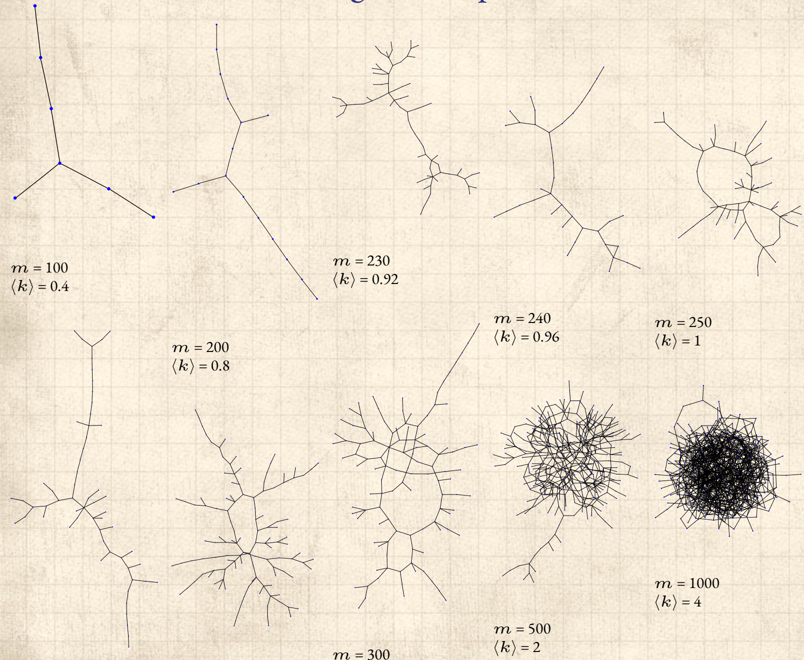


m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4

Random networks: largest components



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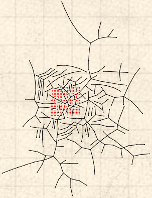

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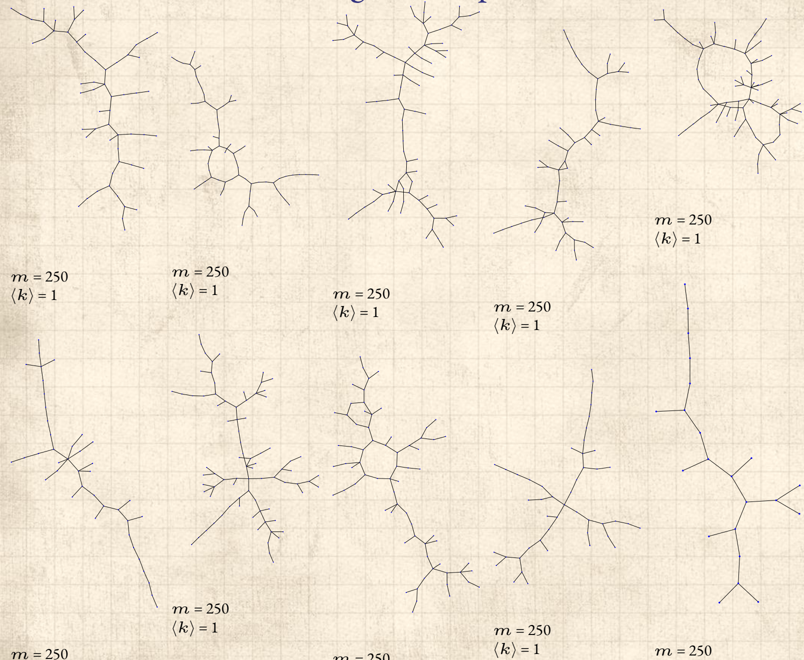
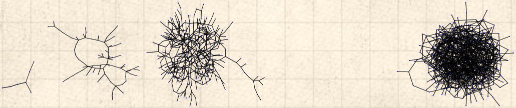
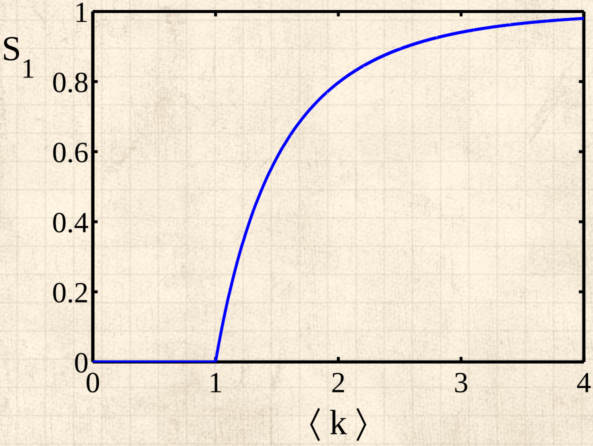


Giant component





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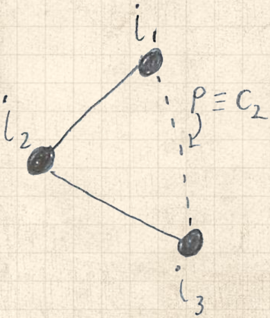

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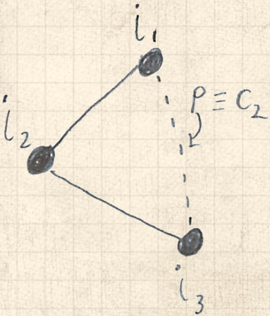

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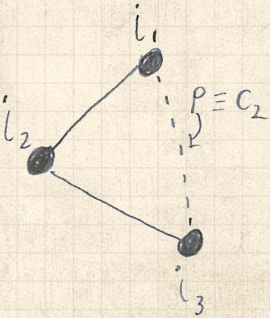

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- For standard random networks, we have simply that

$$C_2 = p.$$

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- Each connection occurs with probability p, each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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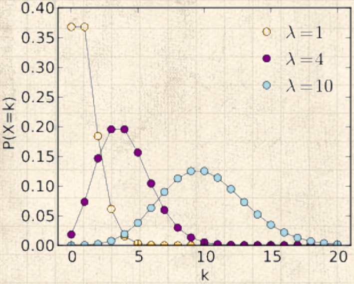
- This is a Poisson distribution with mean $\langle k \rangle$.

Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.:
  phone calls/minute,
  horse-kick deaths.
- 'Law of small numbers'

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- Note: This is a special property of Poisson distribution and can trip us up...

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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

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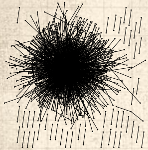
- Apart from degree distribution, wiring is random.

Random networks: examples for N=1000



$\gamma = 2.1$
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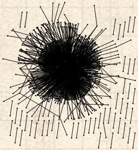

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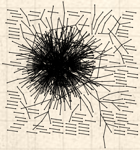

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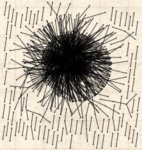

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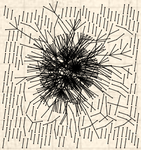

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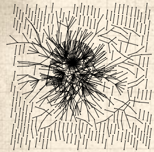

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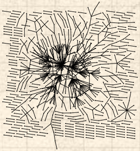

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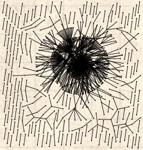

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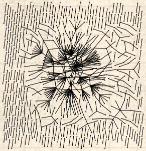

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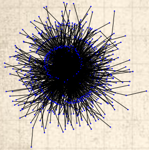
Random networks: largest components



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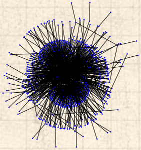

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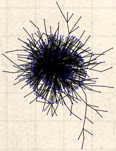

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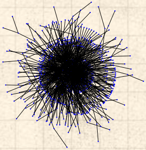

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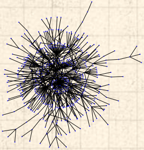

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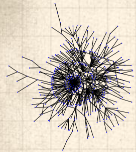

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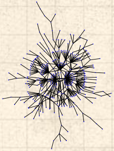

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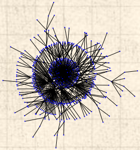

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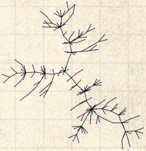

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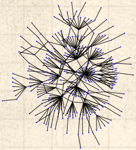

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Outline

Models

Generalized random networks:

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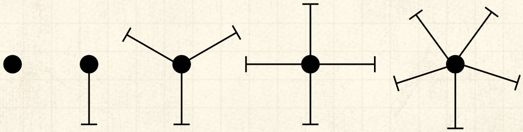
- **Idea:** start with a soup of unconnected nodes with stubs (half-edges):



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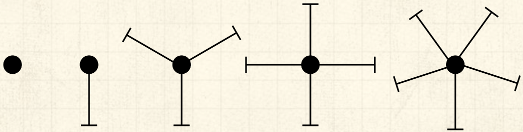

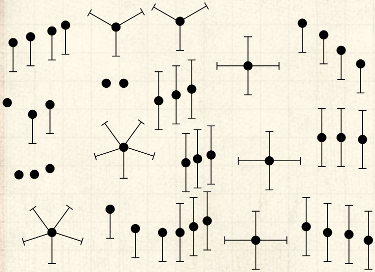

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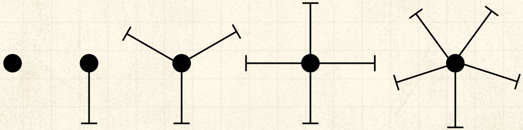

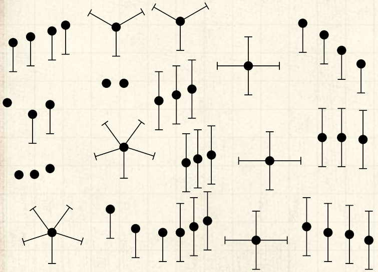

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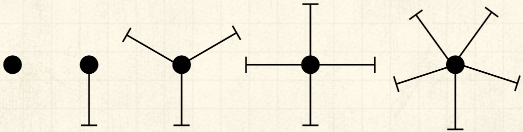

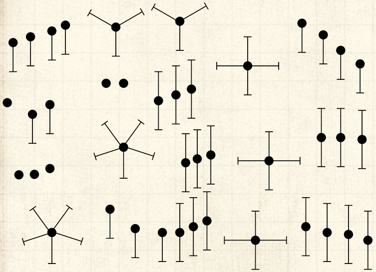

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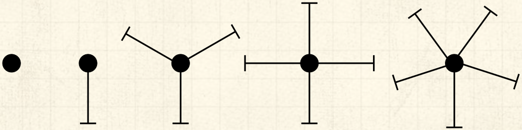

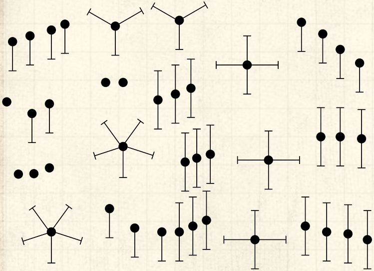

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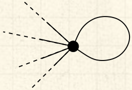
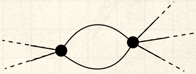
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.

(A)  (B) 

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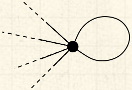
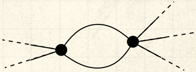
(A)  (B) 

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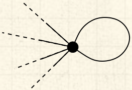
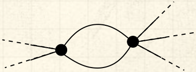
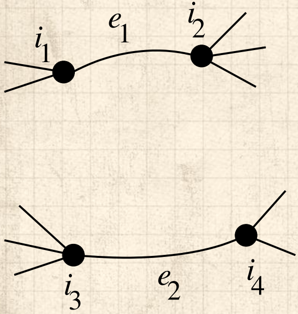
(A)  (B) 

- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

General random rewiring algorithm



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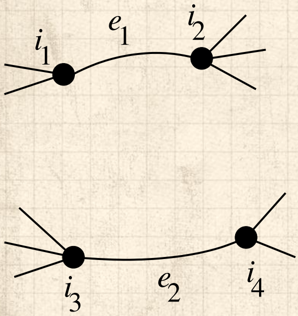

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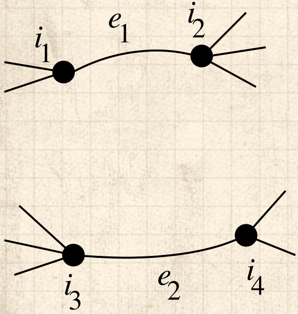

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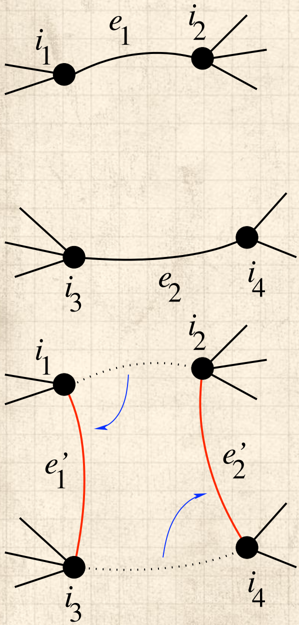

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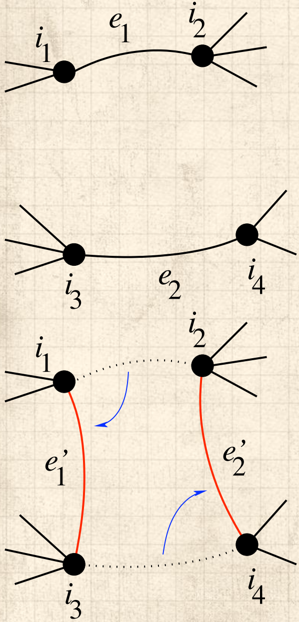

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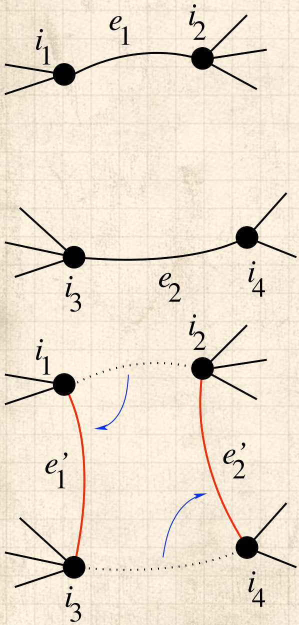

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- Same as finding on/off/on/off 4-cycles. and rotating them.

Sampling random networks

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Sampling random networks

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- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [4].

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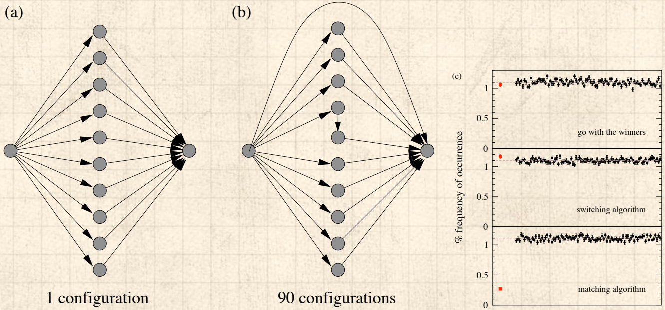
- Example from Milo et al. (2003) [4]:



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- Note: not all P_k will always give nodes that can be wired together.

Outline

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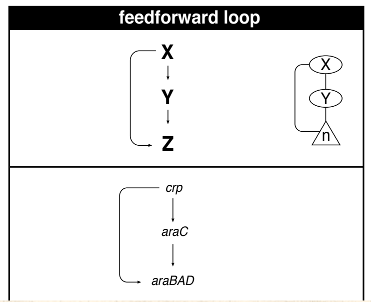
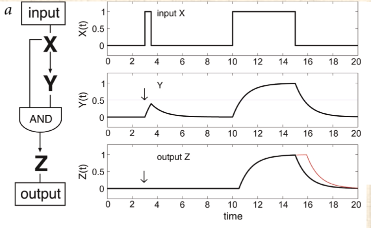
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

Network motifs





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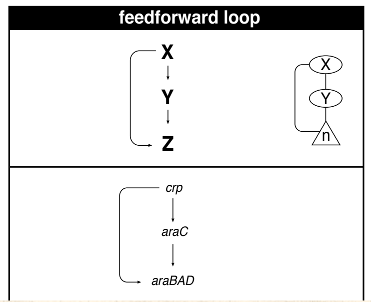

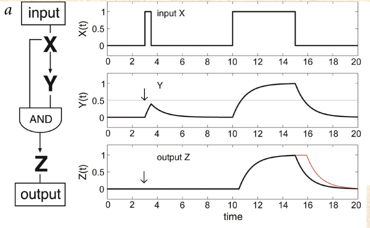

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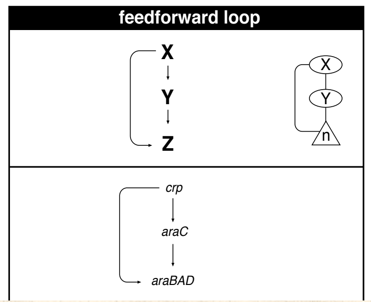

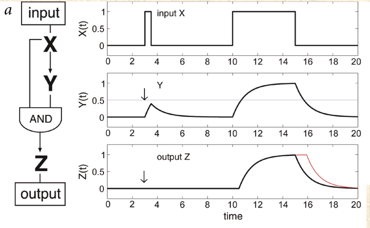

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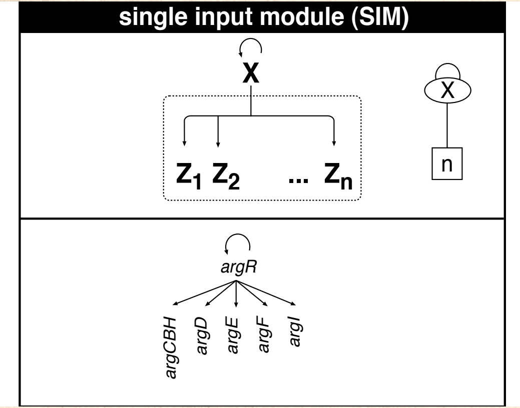
- Analogy to elevator doors.

Network motifs



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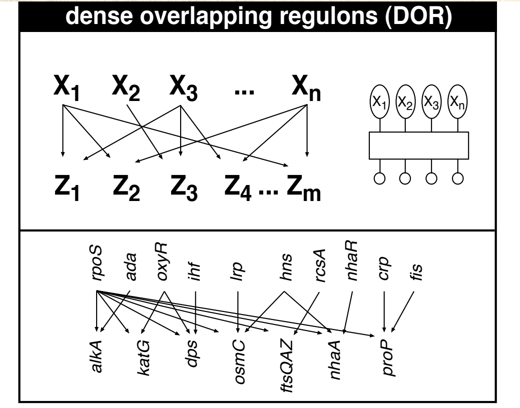

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- For more, see work carried out by Wiggins *et al.* at Columbia.

Outline

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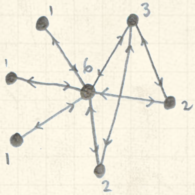
- Big deal: Rich-get-richer mechanism is built into this selection process.



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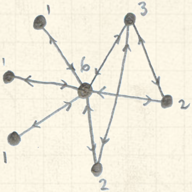

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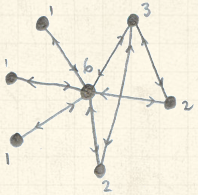

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- Equivalent to friend having degree $k+1$.
- Natural question: what's the expected number of other friends that one friend has?

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  4. See also: class size paradoxes (nod to: Gelman)

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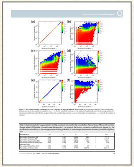
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.



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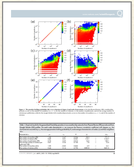

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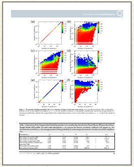

"Generalized friendship paradox in complex networks: The case of scientific collaboration"
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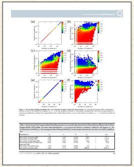

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- The hope: Maybe they have more enemies and diseases too.

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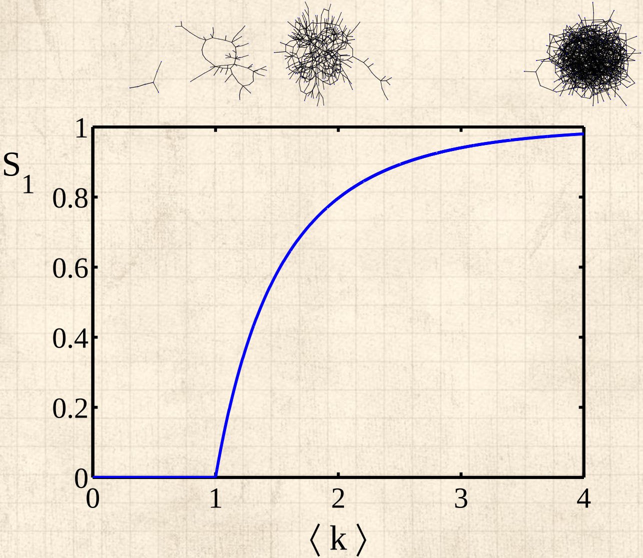
- Note: Component = Cluster

Outline

Giant component



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- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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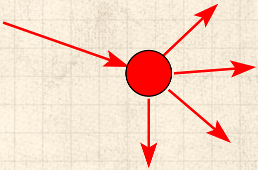

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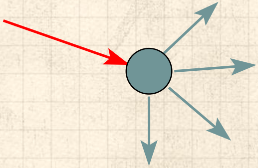

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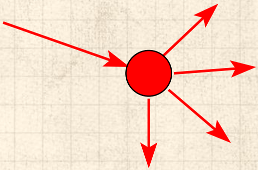

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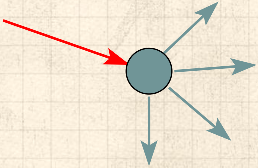

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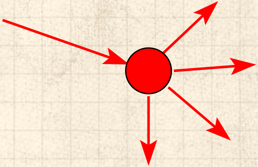

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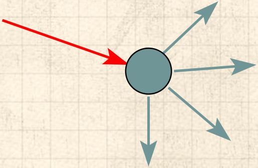

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- First big question: for a given network and contagion process, can global spreading from a single seed occur?

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- Good: This is just our giant component condition again.

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- We say $\langle k \rangle = 1$ marks the critical point of the system.

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- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

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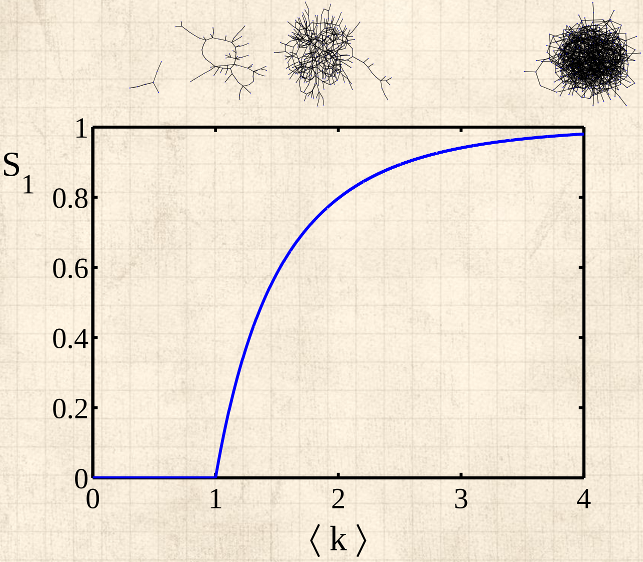
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation. [8]

Giant component



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- We can sort many things out with sensible probabilistic arguments...
- More detailed investigations will profit from a spot of Generatingfunctionology. [9]

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