Power-Law Mechanisms: Variable Transformation

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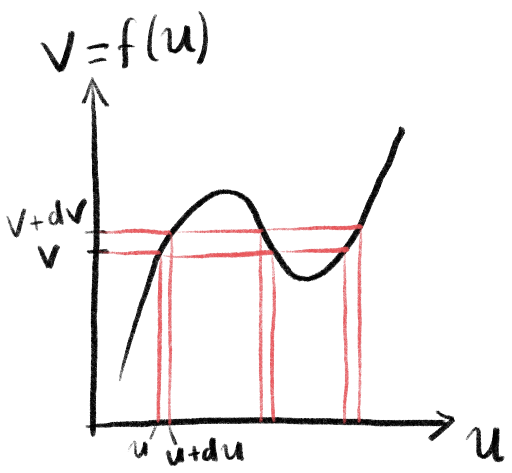
Outline

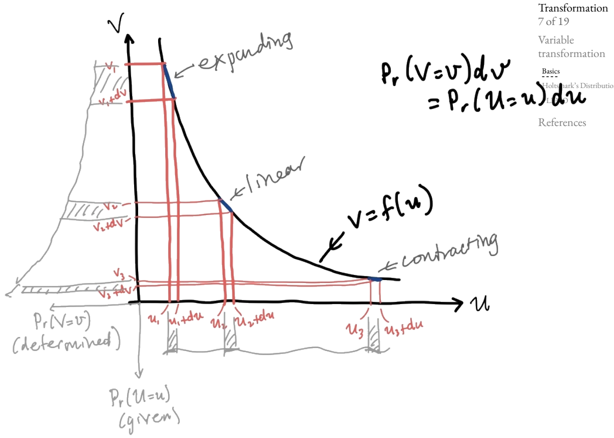


Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...



Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}, \alpha > 0$
- Look at y large and x small
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$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

$$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x\left(\overbrace{\left(\frac{y}{c}\right)^{-1/\alpha}}^{(x)}\right)\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

Gravity

- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtsmark who was thinking about electrostatics and plasma [1].
- Again, the humans naming things after humans, poorly.[1]

[1] Stigler's Law of Eponymy.

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

- Connect differentials: $\mathrm{d}r \propto \mathrm{d}F^{-\frac{1}{2}} \propto F^{-\frac{3}{2}}\mathrm{d}F$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F.$$

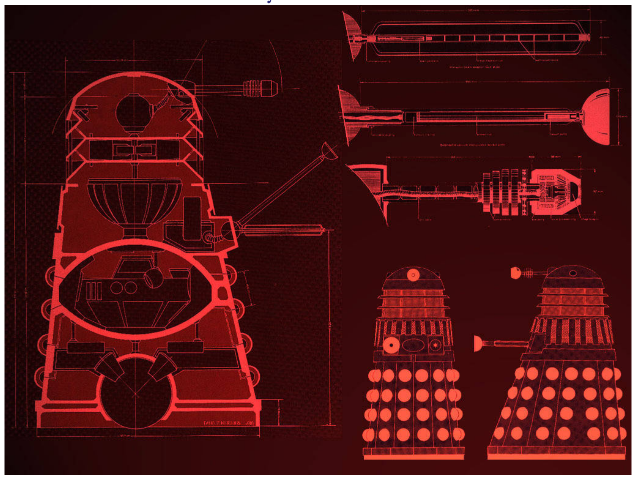
Gravity:

- $$P_F(F) = F^{-5/2}\mathrm{d}F$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

☐ Todo: Build Dalek army.



Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument ...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

References I

[1] J. Holtsmark.
Über die verbreiterung von spektrallinien.
Ann. Phys., 58:577–630, 1919. pdf

[2] D. Sornette.
Critical Phenomena in Natural Sciences.
Springer-Verlag, Berlin, 1st edition, 2003.