Branching Networks I

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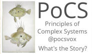
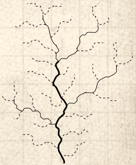
Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2024–2025

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont





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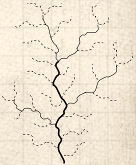

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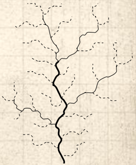

Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
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THE POURING DATA
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THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
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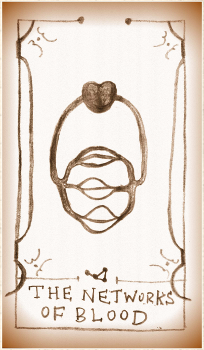
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THE TREES OF REALITY

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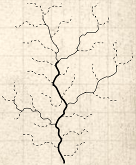

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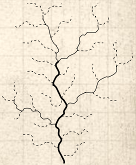

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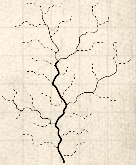

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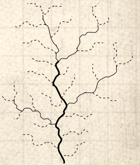

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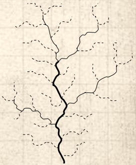

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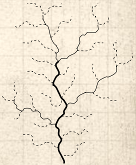

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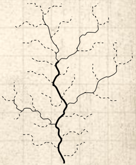
- Evolutionary trees



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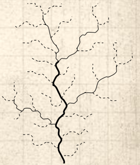
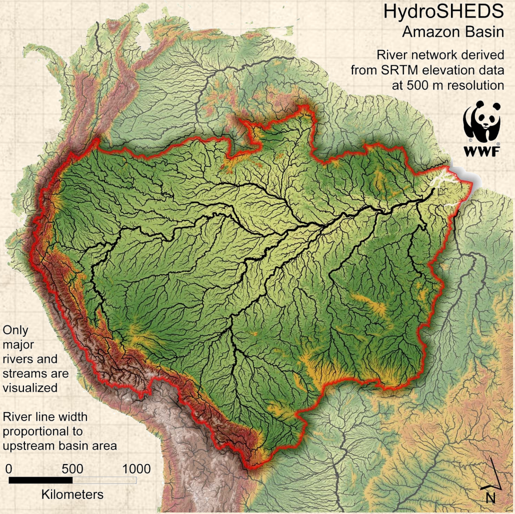
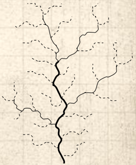
- Organizations (only in theory ...)



Branching networks are everywhere ...



http://hydrosheds.cr.usgs.gov/



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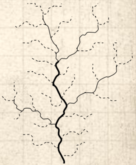
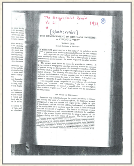
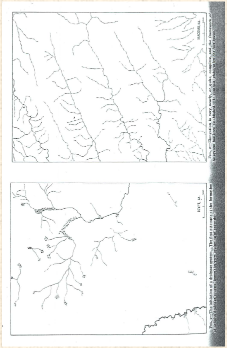
http://en.wikipedia.org/wiki/Image:Applebox.JPG



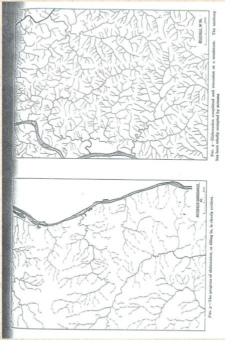
An early thought piece: Extension and Integration



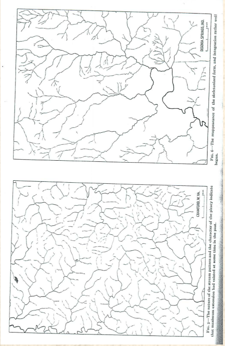
"The Development of Drainage Systems: A Synoptic View"
Waldo S. Glock,
The Geographical Review, **21**, 475–482, 1931. [2]



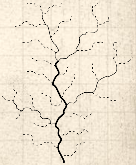
Initiation, Elongation



Elaboration, Piracy.



Abstraction, Absorption.



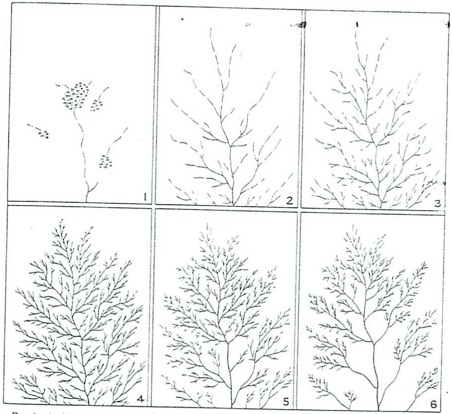
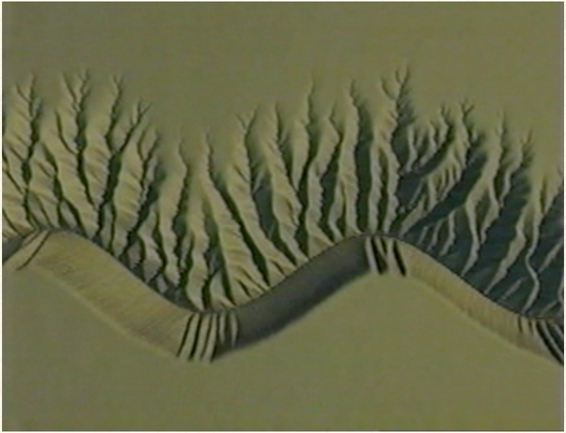


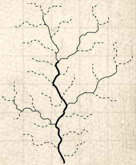
FIG. 8—An ideal diagrammatic summary of the development of a drainage system given for purposes of comparison only. The first four parts show extension, thus: 1, initiation; 2, elongation; 3, elaboration; and 4, maximum extension. Parts 5 and 6 represent steps during integration.

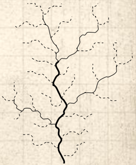
The sequential stages recognized in the evolution of a drainage system are "extension" and "integration"; the first, a stage of increasing complexity; the second, of simplification.

Shaw and Magnasco's beautiful erosion simulations



- Unpublished.
- Though to be destroyed and lost.
- The VHS.





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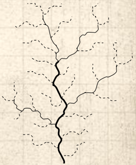
Geomorphological networks

Definitions

- **Drainage basin** for a point p is the complete region of land from which overland flow drains through p.



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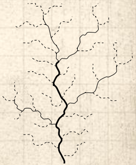

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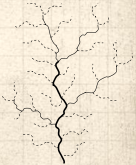

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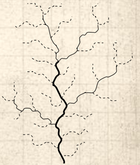

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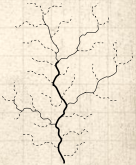

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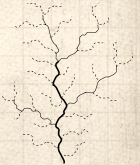

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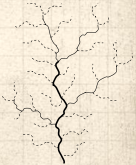

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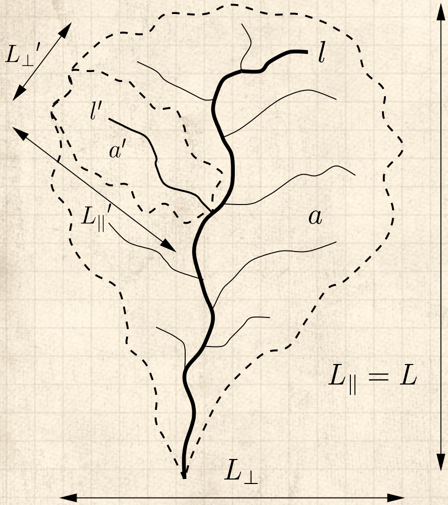

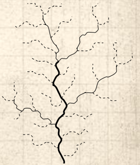

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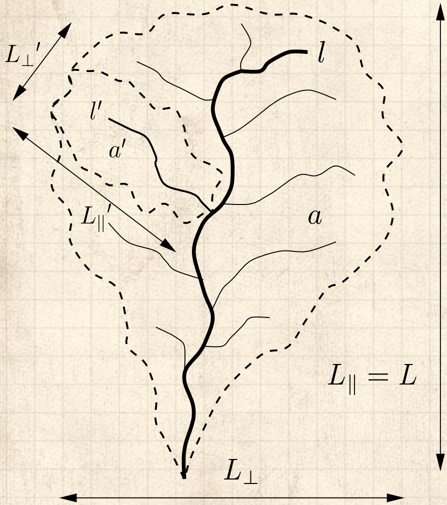

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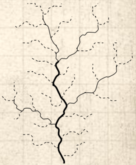

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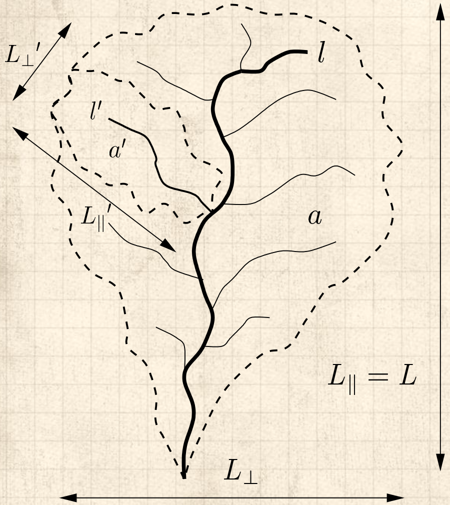
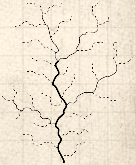
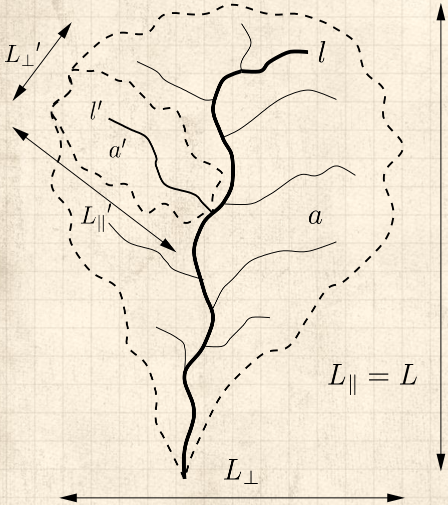


- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)



Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:



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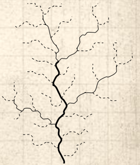

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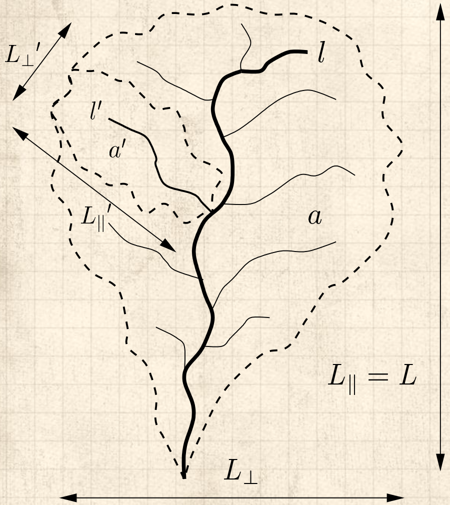

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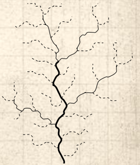
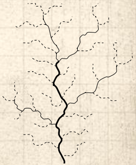


Outline



Allometry

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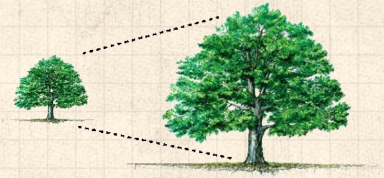

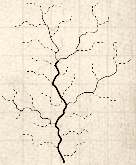

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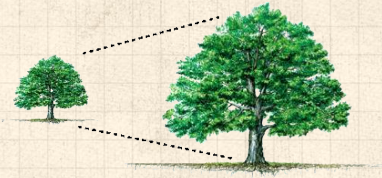

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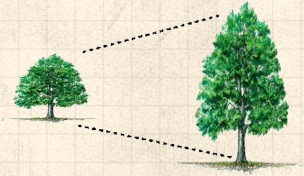

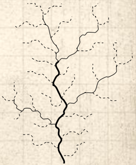

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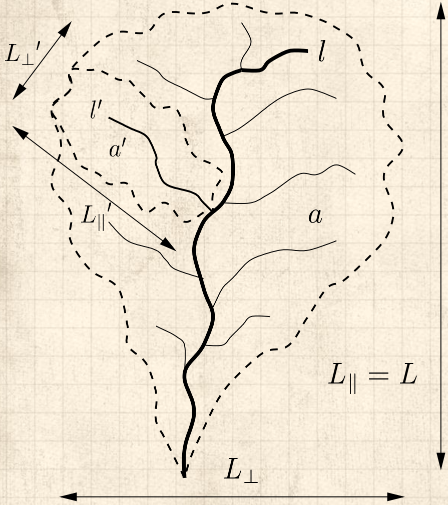

Allometric relationships:

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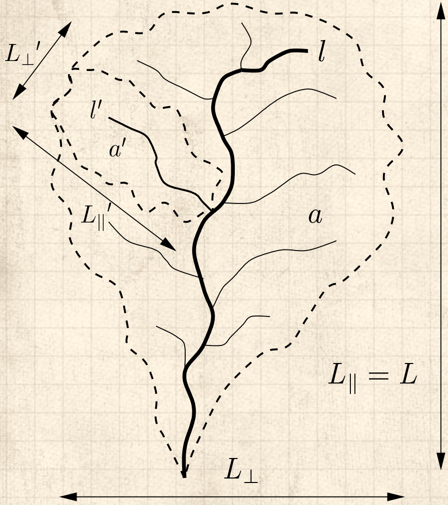

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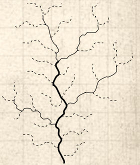

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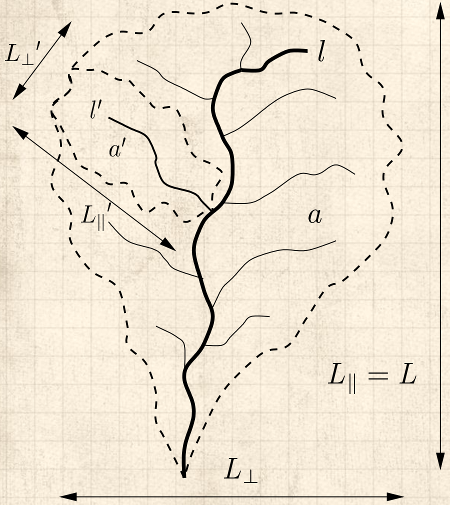

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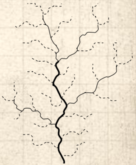

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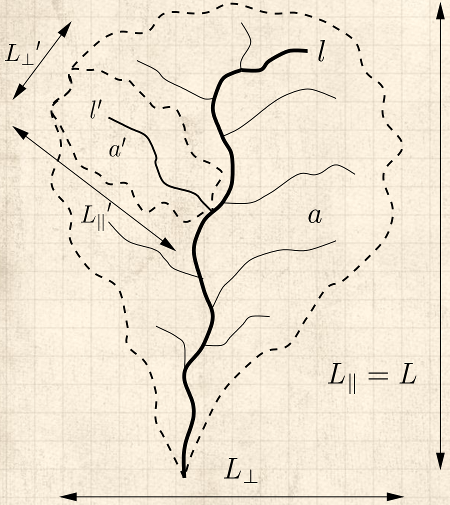
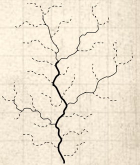


Allometric relationships:

- $$\ell \propto a^h$$
- $$\ell \propto L^d$$
- Combine above:

$$a \propto L^{d/h} \equiv L^D$$



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- Basin allometry:

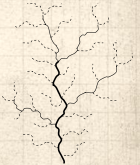
$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

$D < 2 \rightarrow$ basins elongate.

Outline

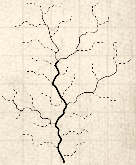
There are a few more 'laws': [1]

Relation:	Name or description:
$T_k = T_1 (R_T)^{k-1}$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law



Reported parameter values: [1]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05



Kind of a mess ...

Order of business:

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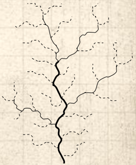

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1. Find out how these relationships are connected.

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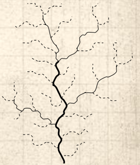

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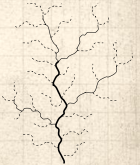

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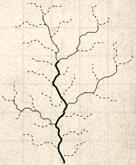
For (3): Many attempts: not yet sorted out ...




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
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THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
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THE BRAID OR THE BRANCH
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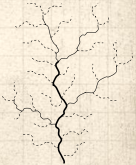
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS


THE ORDERS OF STREAMS

Stream Ordering:

Method for describing network architecture:

Stream Ordering:



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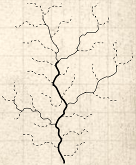

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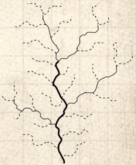

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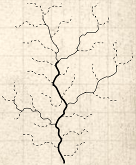
- Can be seen as iterative trimming of a network.



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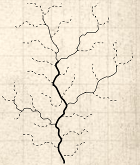
Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.



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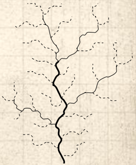

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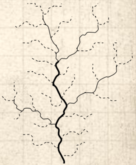

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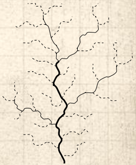
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.



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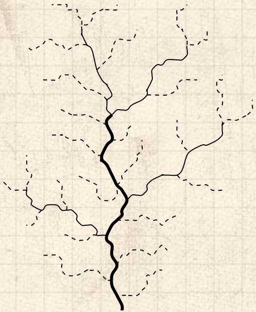

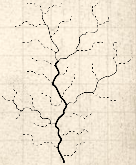

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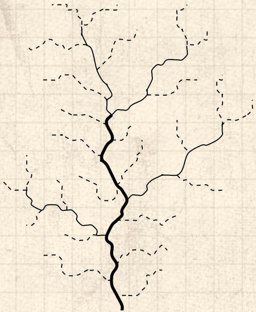

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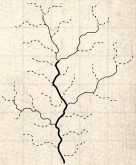

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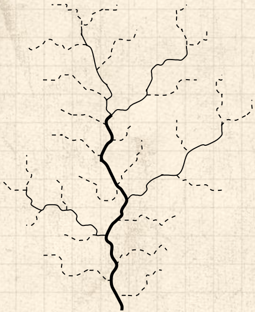

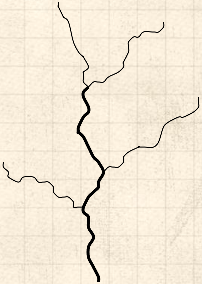

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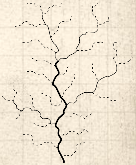

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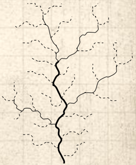

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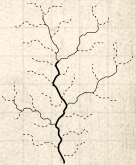

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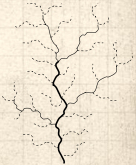

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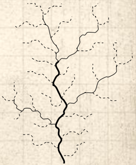

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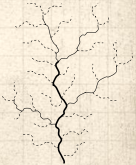
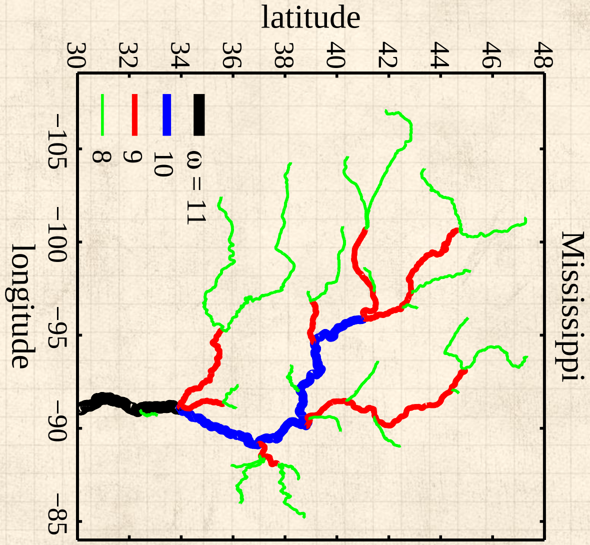
5. Example above is a basin of order $\Omega = 3$.

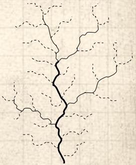


Stream Ordering—A large example:



[source=/data6/dodds/work/rivers/dems/mississippi/figures/figorder_paths_mispi10.ps]

[21−Mar−2000 peter dodds]



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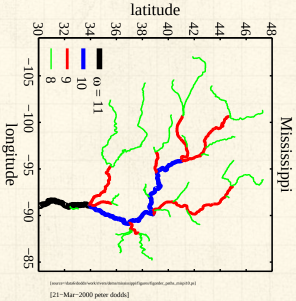

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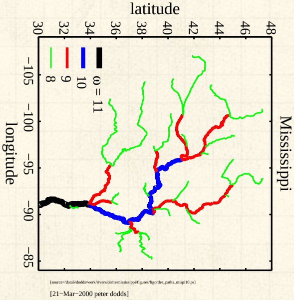

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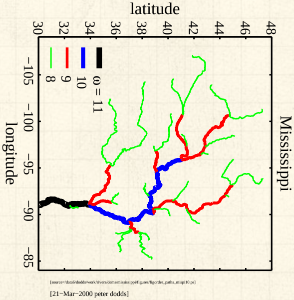

Stream Ordering:

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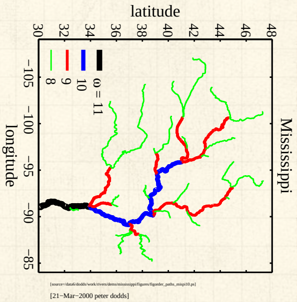
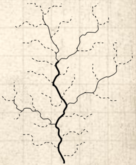
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).





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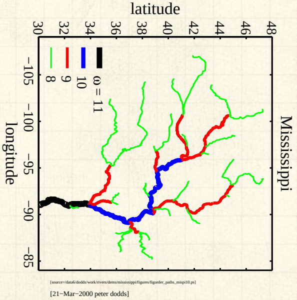

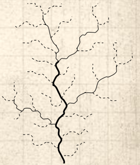

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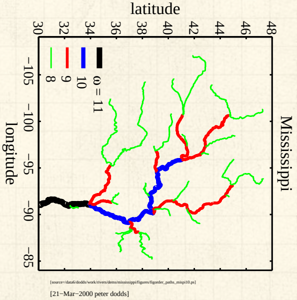
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1,\omega_2}$$

where δ is the Kronecker delta.



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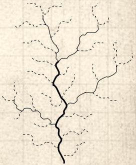

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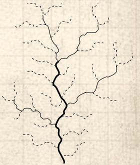

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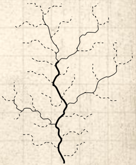

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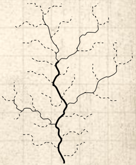

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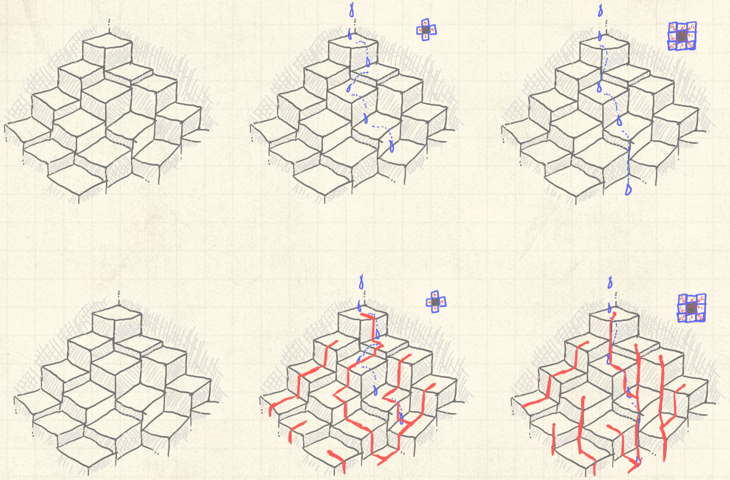
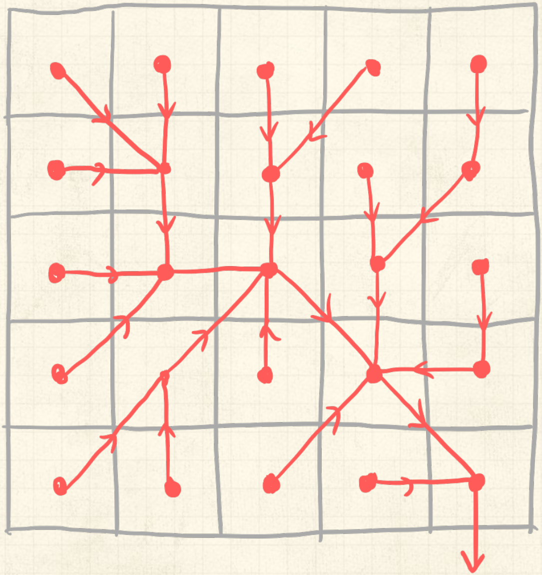
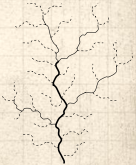
- Goal: understand network architecture

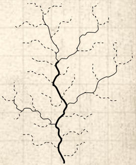
Basic algorithm for extracting networks from Digital Elevation Models (DEMs):

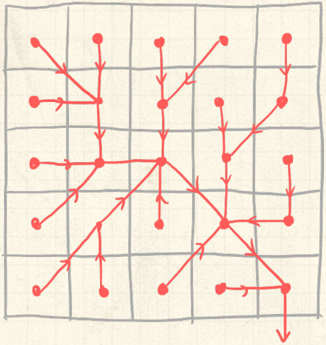


Also:
`/Users/dodds/work/rivers/1998dems/kevinlakewaster.c`

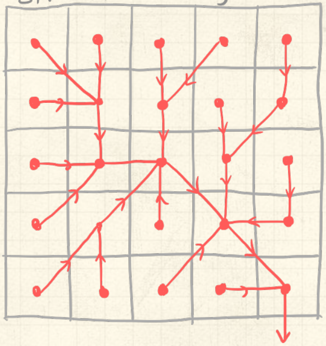




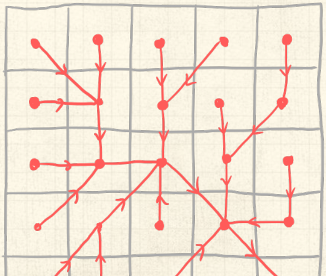




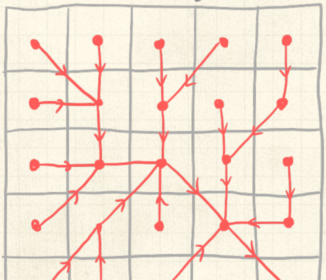
stream ordering ω:



basin area a:



main stream length ℓ:



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Resultant definitions:

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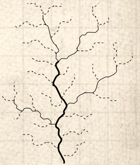

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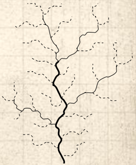

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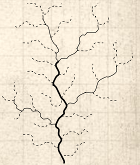

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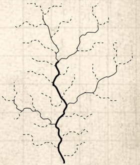

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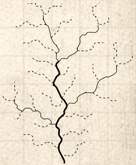

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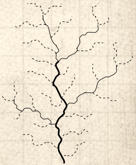


THE LAWS OF
BRANCHING

Horton's laws

Self-similarity of river networks

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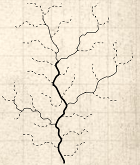

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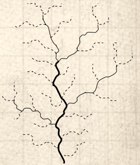

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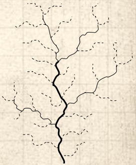

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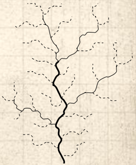

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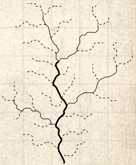
- Horton's laws describe exponential decay or growth:

$$\begin{aligned} n_\omega &= n_{\omega-1}/R_n \\ &= n_{\omega-2}/R_n^2 \\ &\ \vdots \\ &= n_1/R_n^{\omega-1} \\ &= n_1 e^{-(\omega-1)\ln R_n} \end{aligned}$$



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- $$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$
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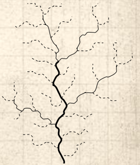
- As stream order increases, number drops and area and length increase.



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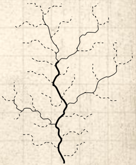

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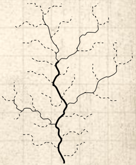

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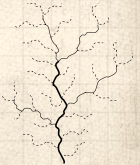

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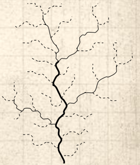
- But we need one other piece of information ...



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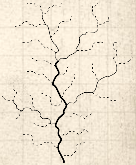

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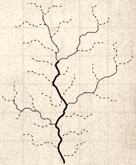

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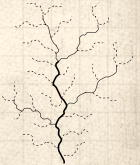
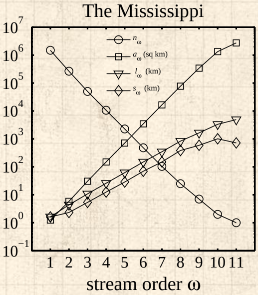
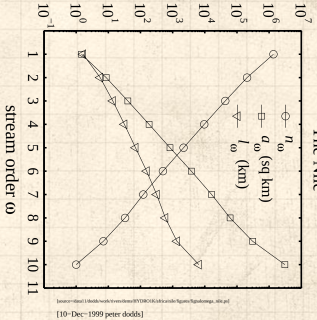
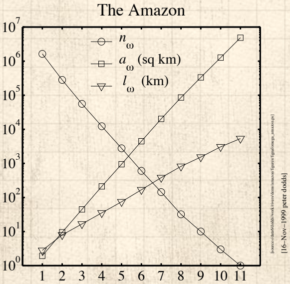
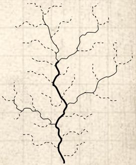


Horton's laws in the real world:



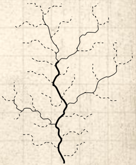






Horton's laws-at-large

Blood networks:



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Blood networks:

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- Measuring such networks is tricky and messy ...

Horton's laws-at-large

Blood networks:

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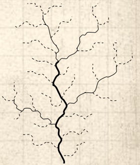
- Vessel diameters obey an analogous Horton's law.

Data from real blood networks

Network	R_n	R_r	R_ℓ	$-\frac{\ln R_r}{\ln R_n}$	$-\frac{\ln R_\ell}{\ln R_n}$	α
West *et al.*	–	–	–	1/2	1/3	3/4
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73
cat (PAT) [11]	3.67	1.71	1.78	0.41	0.44	0.79
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90
pig (LCX)	3.57	1.89	2.20	0.50	0.62	0.62
pig (RCA)	3.50	1.81	2.12	0.47	0.60	0.65
pig (LAD)	3.51	1.84	2.02	0.49	0.56	0.65
human (PAT)	3.03	1.60	1.49	0.42	0.36	0.83
human (PAT)	3.36	1.56	1.49	0.37	0.33	0.94



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Observations:

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R_n	3.0–5.0
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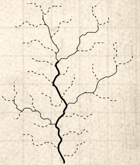

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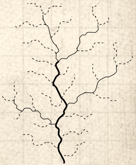
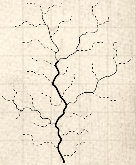
- ...but they don't explain how networks are structured.



Tokunaga's law

Delving deeper into network architecture:



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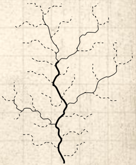
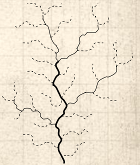
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.



Tokunaga's law

Delving deeper into network architecture:

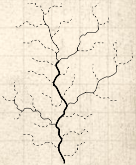
- Tokunaga (1968) identified a clearer picture of network structure [8, 9, 10]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
- Tokunaga's law is also a law of averages.



Network Architecture

Definition:

- $T_{\mu,\nu}$ = the average number of side streams of order ν that enter as tributaries to streams of order μ



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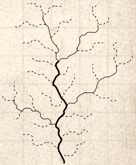

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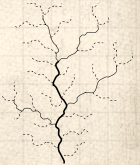

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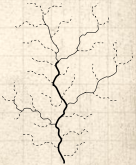

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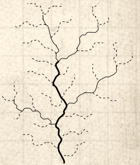
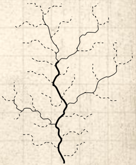
- These generating streams are not considered side streams.



Network Architecture

Tokunaga's law [8, 9, 10]



Network Architecture

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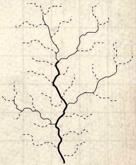
- Property 1: Scale independence—depends only on difference between orders:



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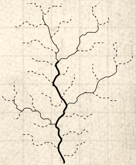

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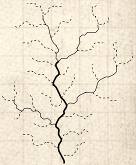

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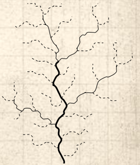

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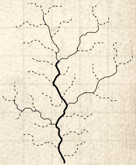
- We usually write Tokunaga's law as:

$$\boxed{T_k = T_1 (R_T)^{k-1}} \quad \text{where } R_T \simeq 2$$

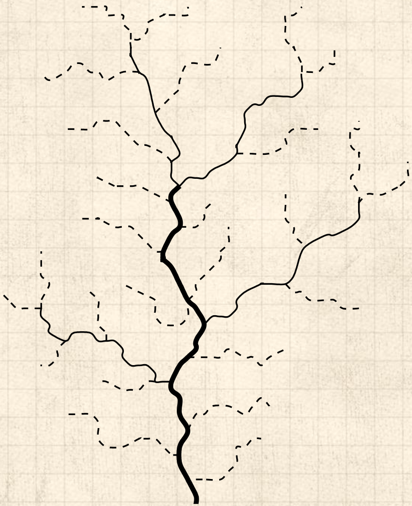
.

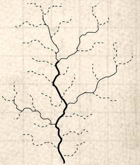


Tokunaga's law—an example:

$T_1 \simeq 2$

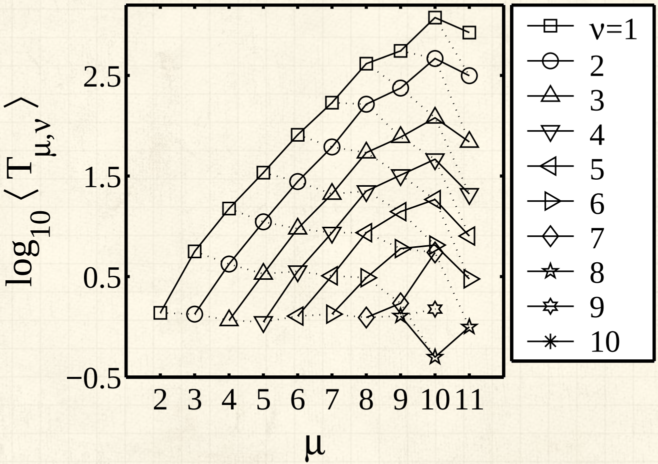
$R_T \simeq 4$

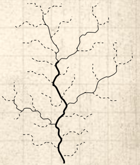




The Mississippi

A Tokunaga graph:





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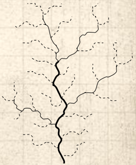

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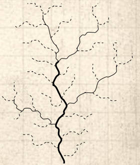

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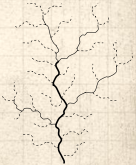

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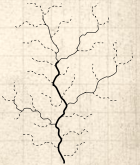

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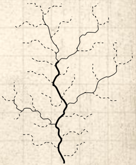

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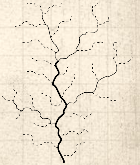
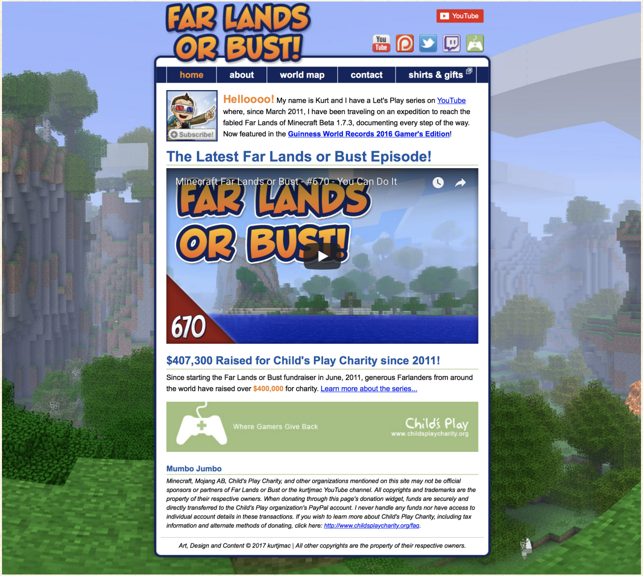
- Surprisingly:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$



Crafting landscapes—Far Lands or Bust:



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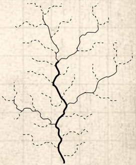

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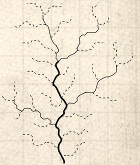

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