Structure detection methods

Last updated: 2021/10/02, 00:15:03 EDT

Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

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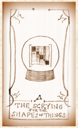
Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



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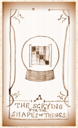

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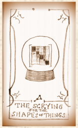


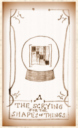
Outline

Overview

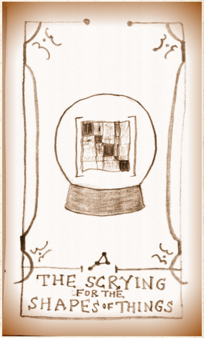
Methods

- Hierarchy by aggregation
- Hierarchy by division
- Hierarchy by shuffling
- Spectral methods
- Hierarchies & Missing Links
- Overlapping communities
- Link-based methods
- General structure detection

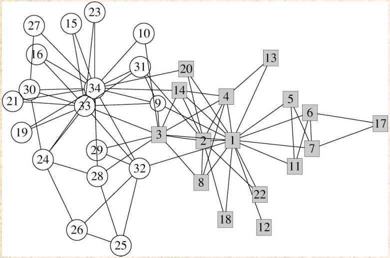
References




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END

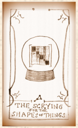

THE SCRYING FOR THE SHAPES OF THINGS

Structure detection



▲ Zachary's karate club [19, 12]

The issue:
how do we elucidate the internal structure of large networks across many scales?



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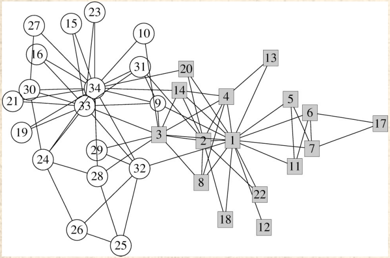

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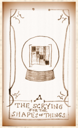
- **Possible substructures:** hierarchies, cliques, rings, ...



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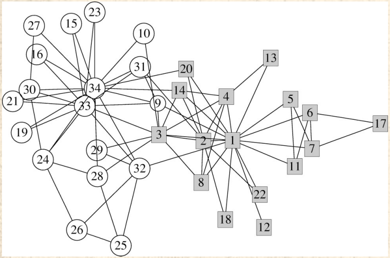

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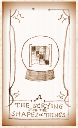

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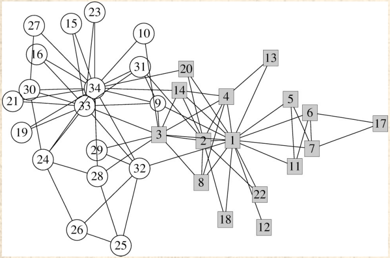

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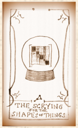
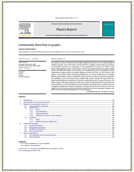
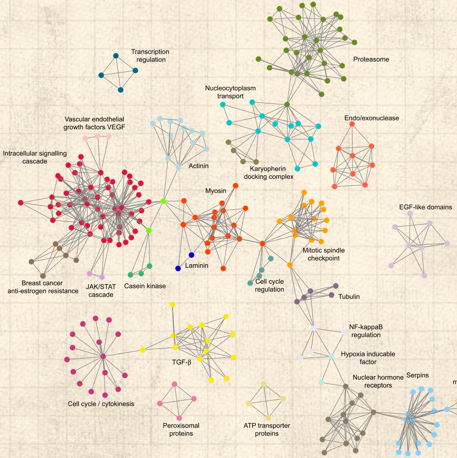
- Possible substructures: hierarchies, cliques, rings, ...
- Plus: All combinations of substructures.
- Much focus on hierarchies...

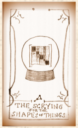




“Community detection in graphs”

Santo Fortunato,
Physics Reports, **486**, 75–174, 2010. [6]





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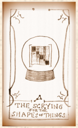

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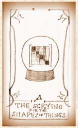
- **Idea:** Extract hierarchical classification scheme for N objects by an agglomeration process.



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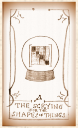

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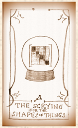

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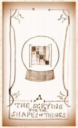

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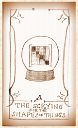

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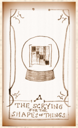

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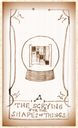

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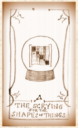

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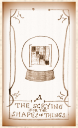
- Clusters gradually emerge, likely with clusters inside of clusters.
- Call above property Modularity.
- Works well for data sets where a distance between all objects can be specified (e.g., Aussie Rules [9]).



Hierarchy by aggregation

Bottom up problems:

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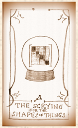
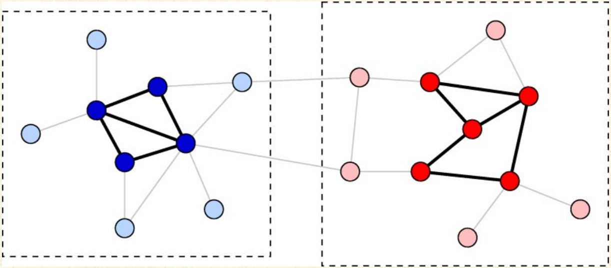
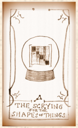


Hierarchy by aggregation

Bottom up problems:

- Tend to plainly not work on data sets representing networks with known modular structures.
- Good at finding cores of well-connected (or similar) nodes... but fail to cope well with peripheral, in-between nodes.





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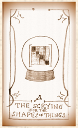
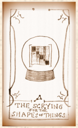


Hierarchy by division

Top down:

- **Idea:** Identify global structure first and recursively uncover more detailed structure.



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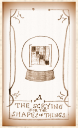

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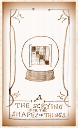

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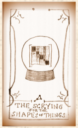

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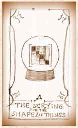

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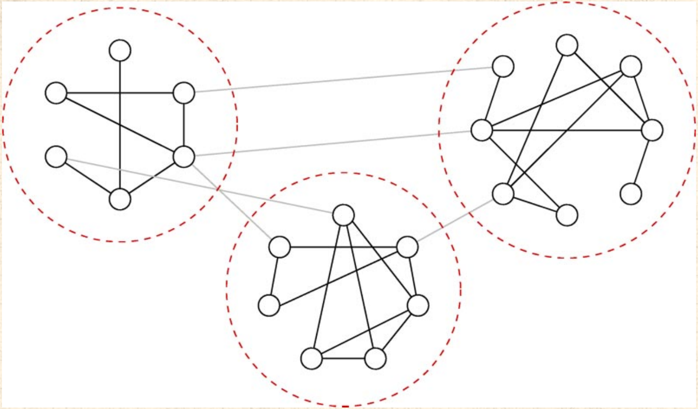
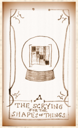
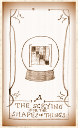




Idea: Edges that connect communities have higher betweenness than edges within communities.

Hierarchy by division



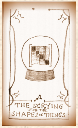
One class of structure-detection algorithms:

1. Compute edge betweenness for whole network.

Hierarchy by division

One class of structure-detection algorithms:

1. Compute edge betweenness for whole network.
2. Remove edge with highest betweenness.



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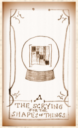
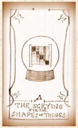
3. Recompute edge betweenness



Hierarchy by division

One class of structure-detection algorithms:

1. Compute edge betweenness for whole network.
2. Remove edge with highest betweenness.
3. Recompute edge betweenness
4. Repeat steps 2 and 3 until all edges are removed.



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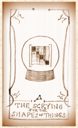

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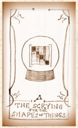

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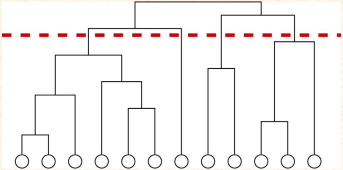

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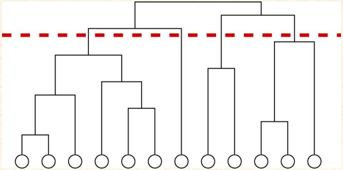
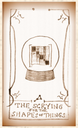
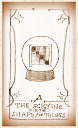


Red line indicates appearance of four (4) components at a certain level.



Key element for division approach:

- Recomputing betweenness.



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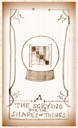

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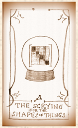

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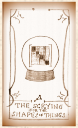

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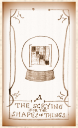

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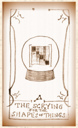
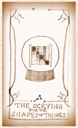
$$Q = \sum_i [e_{ii} - a_i^2]$$

where e_{ij} is the fraction of (undirected) edges travelling between identified communities i and j, and $a_i = \sum_j e_{ij}$ is the fraction of edges with at least one end in community i. ☐



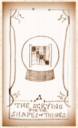
Measuring modularity:



Hierarchy by division

Test case:

- Generate random community-based networks.



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- Generate random community-based networks.
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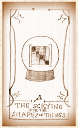

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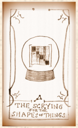

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- Generate random community-based networks.
- $N = 128$ with four communities of size 32.
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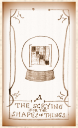
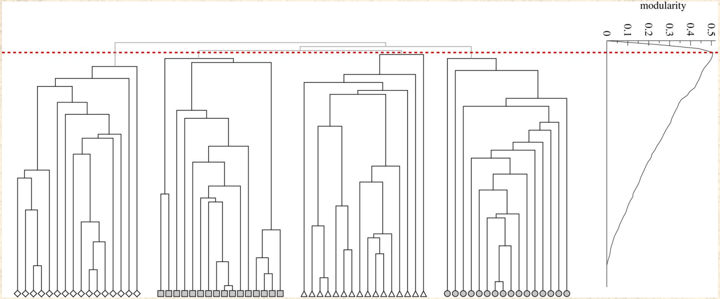


Test case:

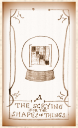
- Generate random community-based networks.
- $N = 128$ with four communities of size 32.
- Add edges randomly within and across communities.
- Example:

$$\langle k \rangle_{\text{in}} = 6 \text{ and } \langle k \rangle_{\text{out}} = 2.$$

Hierarchy by division



- Maximum modularity $Q \simeq 0.5$ obtained when four communities are uncovered.



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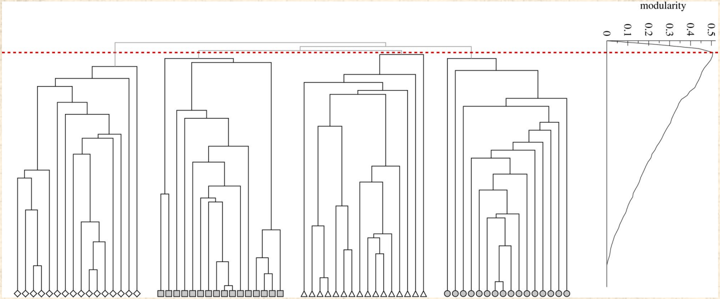
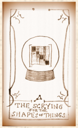


- Maximum modularity $Q \simeq 0.5$ obtained when four communities are uncovered.
- Further 'discovery' of internal structure is somewhat meaningless, as any communities arise accidentally.



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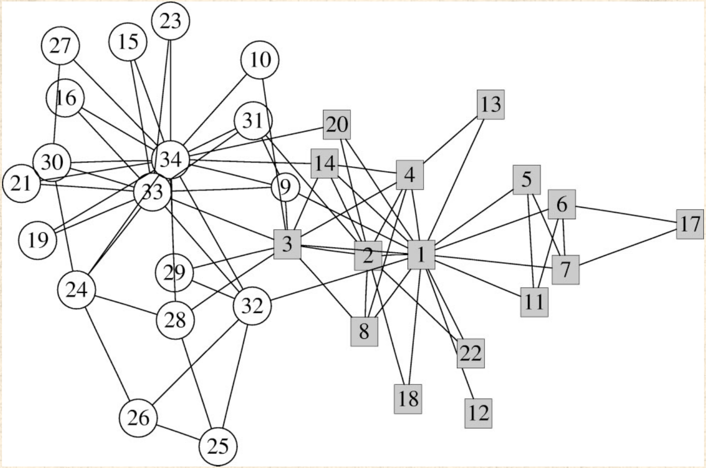
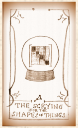
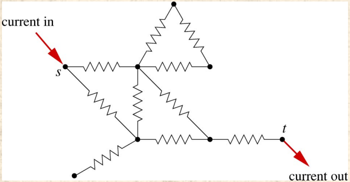


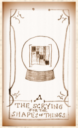
- Factions in Zachary's karate club network. [19]



Betweenness for electrons:



- Unit resistors on each edge.



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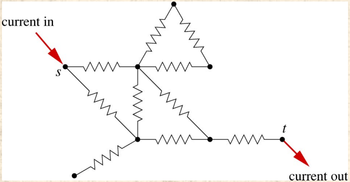

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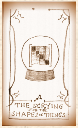

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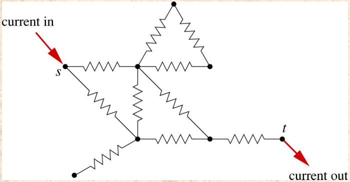

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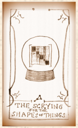

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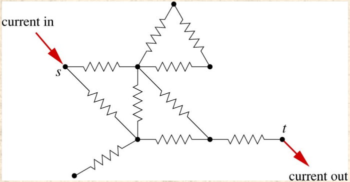

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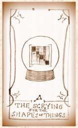

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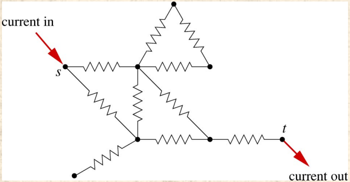

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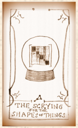

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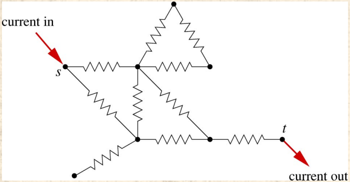

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- (Equivalent to random walk betweenness.)
- Contributing electronic betweenness for edge between nodes i and j:

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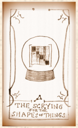

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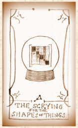

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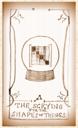

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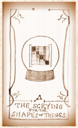

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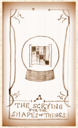

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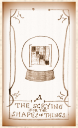

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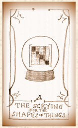

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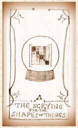

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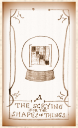

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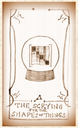

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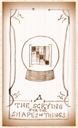

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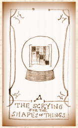
- Write right hand side as $[I^{\text{ext}}]_{i,st} = \delta_{is} - \delta_{it}$, where I^{ext}_{st} holds external source and sink currents.



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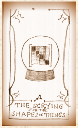

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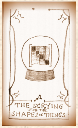

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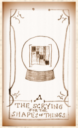

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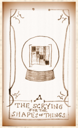
- Do not compute an inverse!



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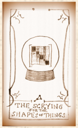
- Solve for voltage vector $\vec{V}$ by **LU** decomposition (Gaussian elimination).
- Do not compute an inverse!
- Note: voltage offset is arbitrary so no unique solution.



Electronic betweenness

- Write right hand side as $[I^{\text{ext}}]_{i,st} = \delta_{is} - \delta_{it}$, where I^{ext}_{st} holds external source and sink currents.
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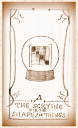

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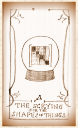
- In fact, $\mathcal{N}(\mathbf{K} - \mathbf{A}) = \{c\vec{1}, c \in R\}$ since $(\mathbf{K} - \mathbf{A})\vec{1} = \vec{0}$.



Alternate betweenness measures:

Random walk betweenness:

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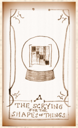

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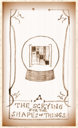

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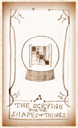

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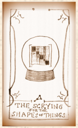

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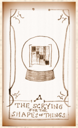

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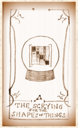

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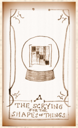
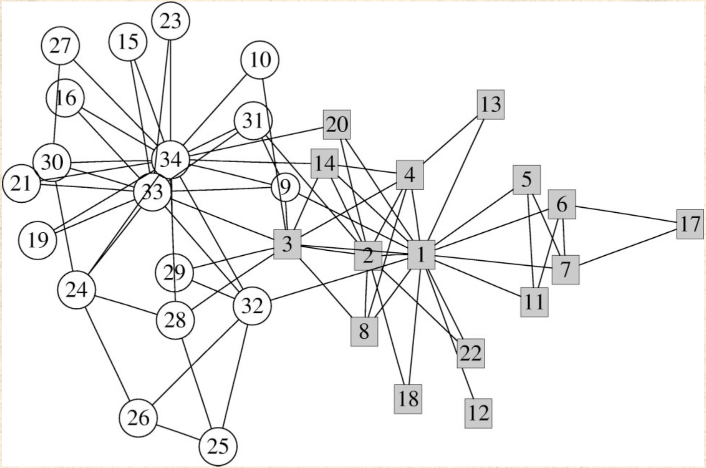
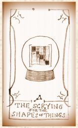
- Equivalent to electronic betweenness (see also diffusion).



Hierarchy by division



Factions in Zachary's karate club network. [19]



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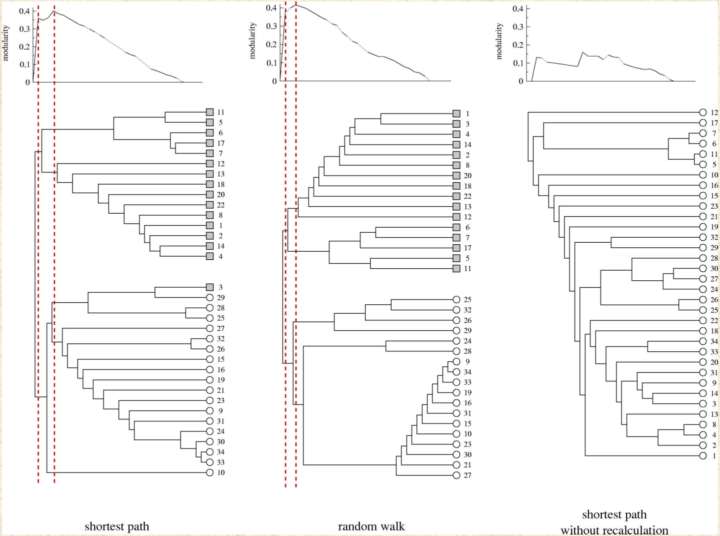
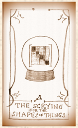
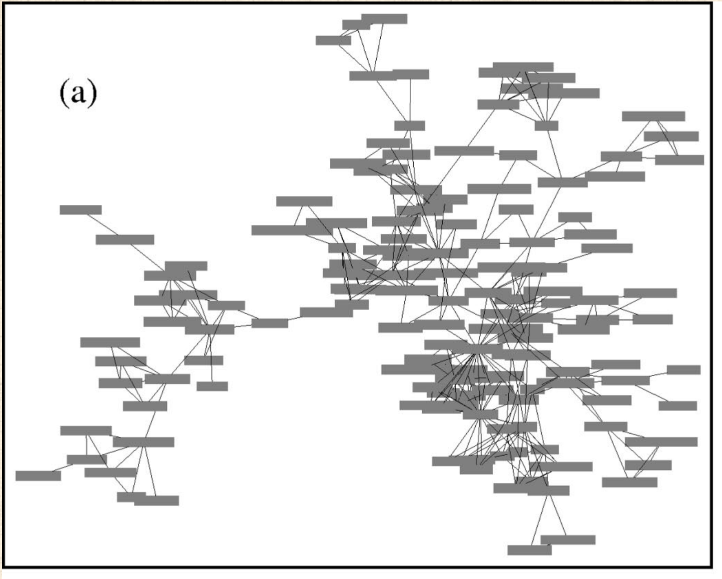
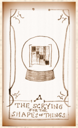


- Third column shows what happens if we don't recompute betweenness after each edge removal.



Scientists working on networks (2004)





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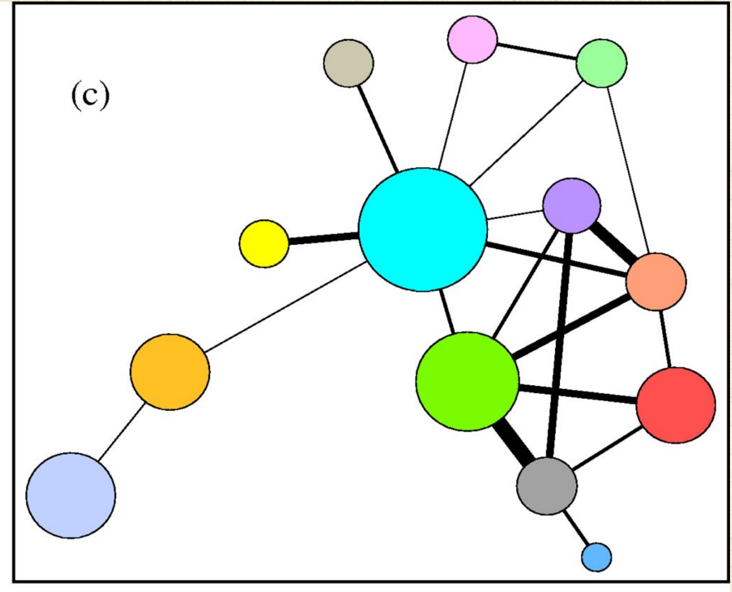

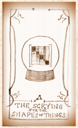

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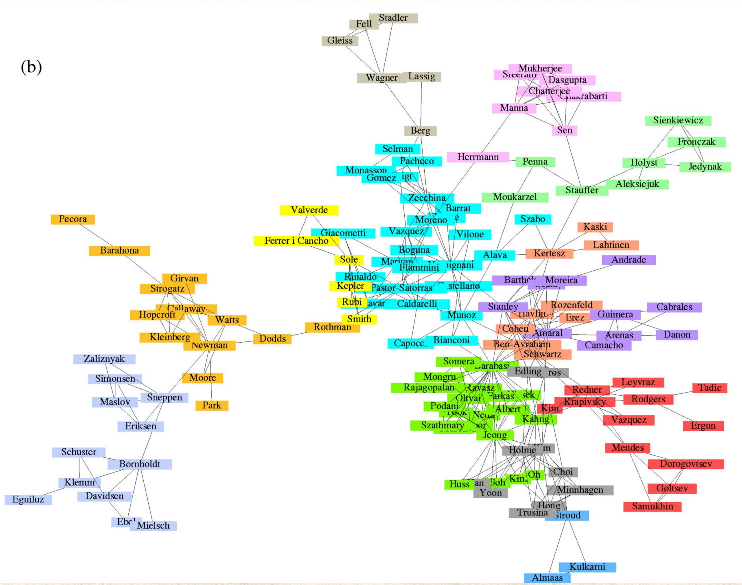
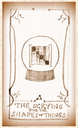
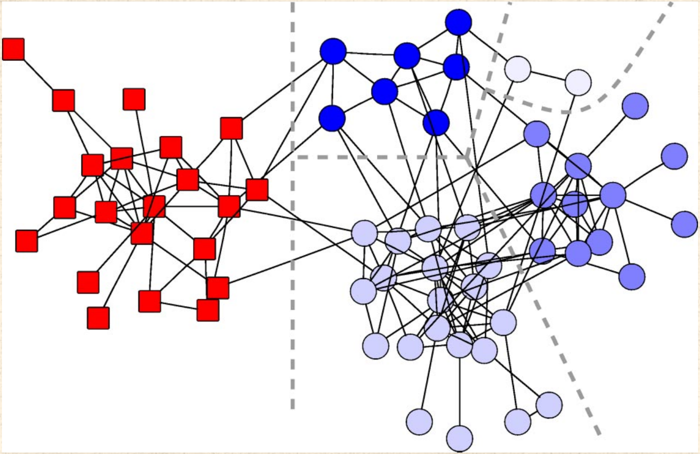
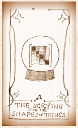


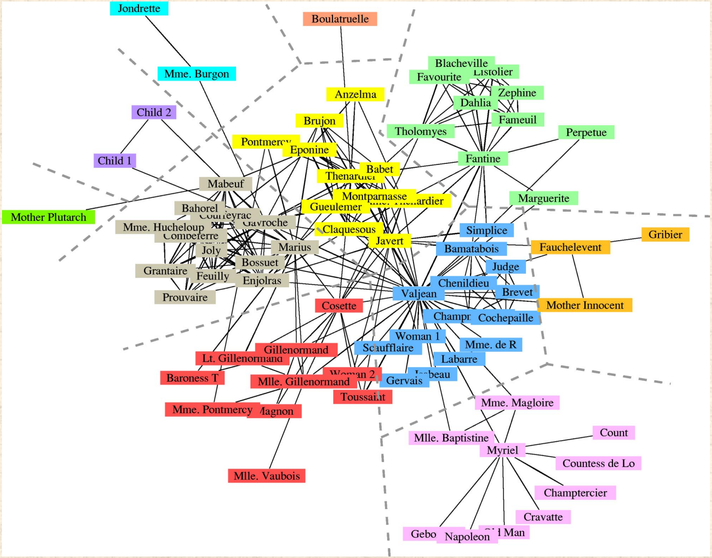


Dolphins!

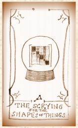




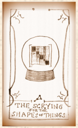
Les Miserables



More network analyses for Les Miserables here and here.

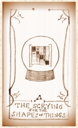


Outline



Shuffling for structure

- "Extracting the hierarchical organization of complex systems"
  Sales-Pardo *et al.*, PNAS (2007) [14, 15]



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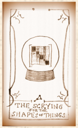
- Consider all partitions of networks into m groups



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- "Extracting the hierarchical organization of complex systems"
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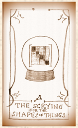
- Consider all partitions of networks into m groups
- As for Newman and Girvan approach, aim is to find partitions with maximum modularity:

$$Q = \sum_i [e_{ii} - (\sum_j e_{ij})^2] = \mathrm{Tr}\mathbf{E} - ||\mathbf{E}^2||_1.$$



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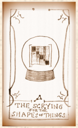

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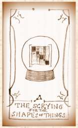
- Defn: Two partitions are connected if they differ only by the reassignment of a single node.



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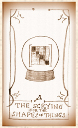

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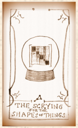

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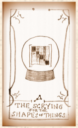
- M_{ij}^{aff} = **Pr** random walker on modularity network ends up at a partition with i and j in the same group.
- C.f. topological overlap between i and j = # matching neighbors for i and j divided by maximum of k_i and k_j.



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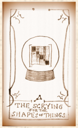
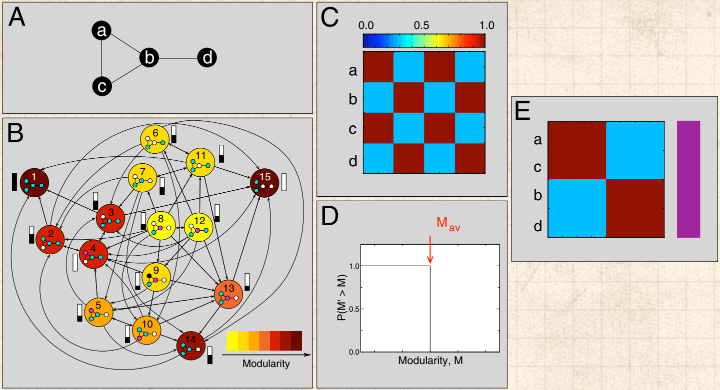
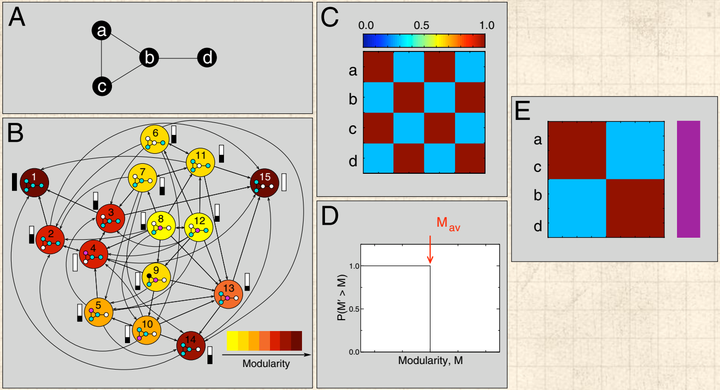


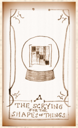


A: Base network; **B:** Partition network; **C:** Coclassification matrix; **D:** Comparison to random networks (all the same!); **E:** Ordered coclassification matrix;

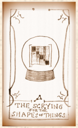
Shuffling for structure



- **A:** Base network; **B:** Partition network; **C:** Coclassification matrix; **D:** Comparison to random networks (all the same!); **E:** Ordered coclassification matrix; Conclusion: no structure...



- Method obtains a distribution of classification hierarchies.



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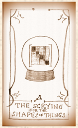

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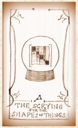

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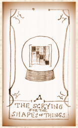

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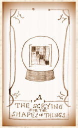
- Idea: permute nodes to minimize following cost

$$C = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij}^{\text{aff}} |i - j|.$$



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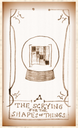

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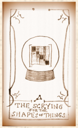
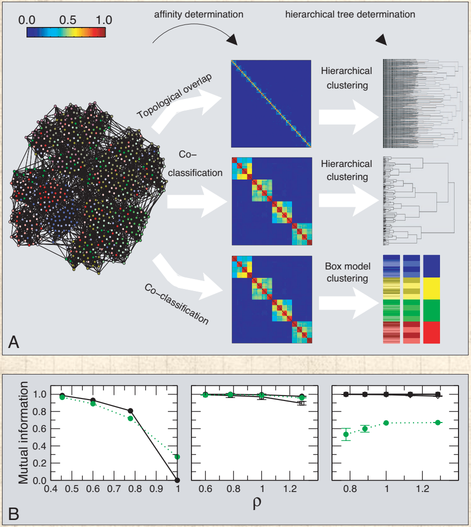
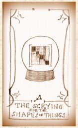
- Use simulated annealing (slow).
- Observation: should achieve same results for more general cost function: $C = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij}^{\text{aff}} f(|i - j|)$ where f is a strictly monotonically increasing function of 0, 1, 2, ...



Shuffling for structure

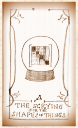


- $N = 640$,
- $\langle k \rangle = 16$,
- 3 tiered hierarchy.



Shuffling for structure

- Define cost matrix as **T** with entries $T_{ij} = f(|i - j|)$.



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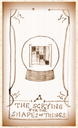
- Weird observation: if $T_{ij} = (i - j)^2$ then **T** is of rank 3, independent of N.



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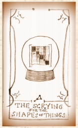

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- Discovered by numerical inspection ...
- The eigenvalues are

$$\lambda_1 = -\frac{1}{6}n(n^2-1),$$

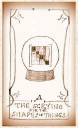
$$\lambda_2 = +\sqrt{nS_{n,4}} + S_{n,2}, \text{ and}$$

$$\lambda_3 = -\sqrt{nS_{n,4}} + S_{n,2}.$$

where

$$S_{n,2} = \frac{1}{12}n(n^2-1), \text{ and}$$

$$S_{n,4} = \frac{1}{240}n(n^2-1)(3n^2-7).$$



Shuffling for structure

- Eigenvectors

$$(\vec{v}_1)_i = \left(i - \frac{n+1}{2}\right),$$

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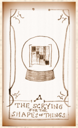

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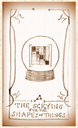
- Remarkably,

$$T = \lambda_1 \hat{v}_1 \hat{v}_1^\mathsf{T} + \lambda_2 \hat{v}_2 \hat{v}_2^\mathsf{T} + \lambda_3 \hat{v}_3 \hat{v}_3^\mathsf{T}.$$



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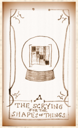
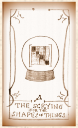
- The next step: figure out how to capitalize on this...



Shuffling for structure

Table 1. Top-level structure of real-world networks

Network	Nodes	Edges	Modules	Main modules
Air transportation	3,618	28,284	57	8
E-mail	1,133	10,902	41	8
Electronic circuit	516	686	18	11
Escherichia coli KEGG	739	1,369	39	13
E. coli UCSD	507	947	28	17



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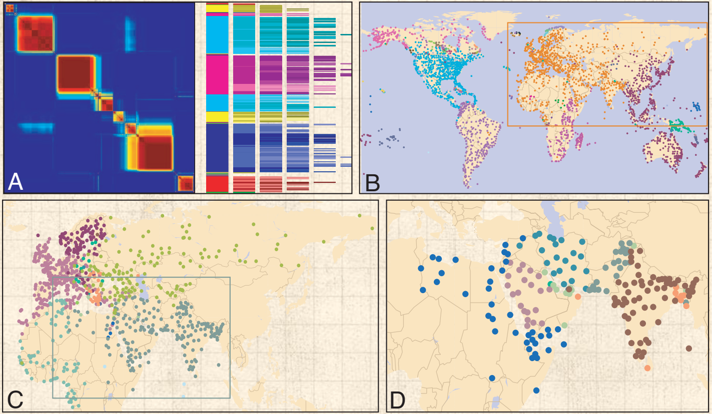
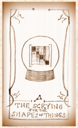
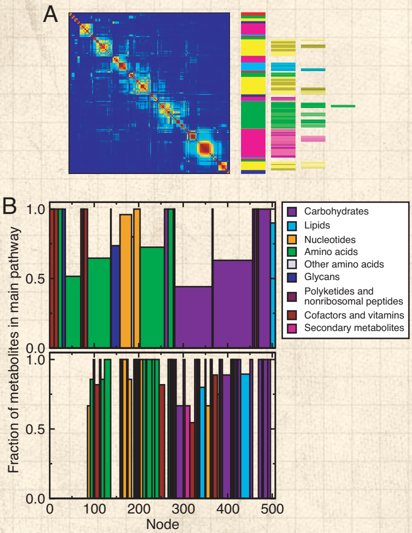


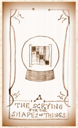


- Modules found match up with geopolitical units.

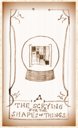
Shuffling for structure



Modularity structure for metabolic network of E. coli (UCSD reconstruction).

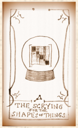


Outline



General structure detection

- “Detecting communities in large networks”
  Capocci *et al.* (2005) [4]



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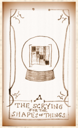

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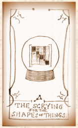

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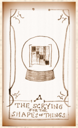
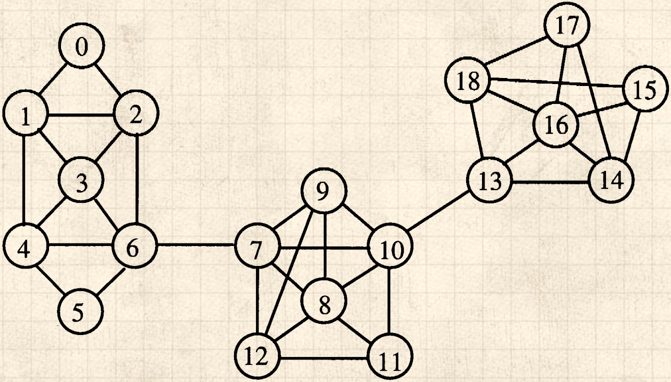
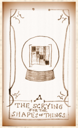
- Builds on Kleinberg's HITS algorithm.



General structure detection

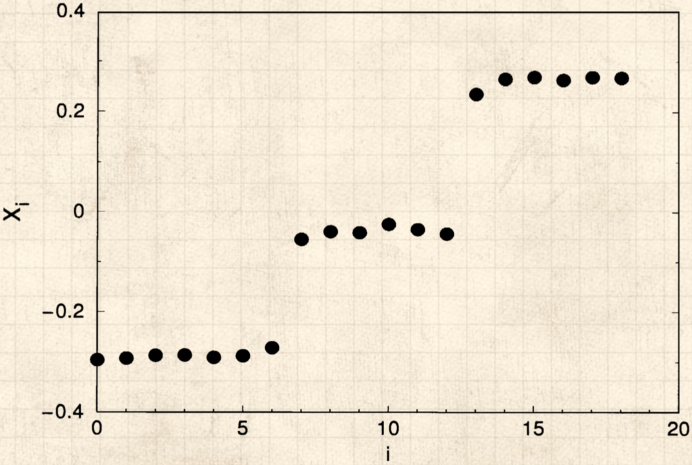
- Example network:

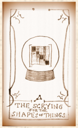




General structure detection

- Second eigenvector's components:





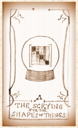
General structure detection

- Network of word associations for 10616 words.
- Average in-degree of 7.
- Using 2nd to 11th evectors of a modified version of $\mathbf{AA}^{\mathsf{T}}$:

Table 1
Words most correlated to science, literature and piano in the eigenvectors of $Q^{-1}WW^{\mathrm{T}}$

Science	1	Literature	1	Piano	1
Scientific	0.994	Dictionary	0.994	Cello	0.993
Chemistry	0.990	Editorial	0.990	Fiddle	0.992
Physics	0.988	Synopsis	0.988	Viola	0.990
Concentrate	0.973	Words	0.987	Banjo	0.988
Thinking	0.973	Grammar	0.986	Saxophone	0.985
Test	0.973	Adjective	0.983	Director	0.984
Lab	0.969	Chapter	0.982	Violin	0.983
Brain	0.965	Prose	0.979	Clarinet	0.983
Equation	0.963	Topic	0.976	Oboe	0.983
Examine	0.962	English	0.975	Theater	0.982

Values indicate the correlation.



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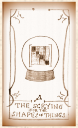
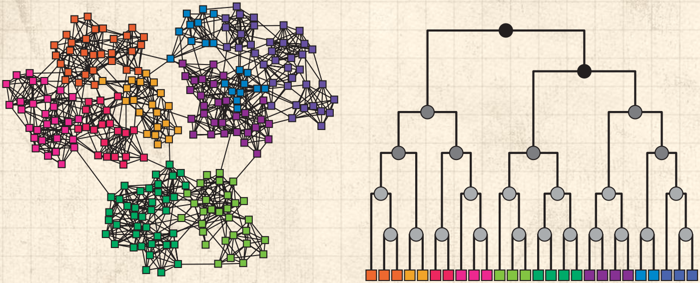
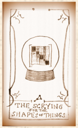


Hierarchies and missing links

Clauset *et al.*, Nature (2008) [5]



- Idea: Shades indicate probability that nodes in left and right subtrees of dendogram are connected.



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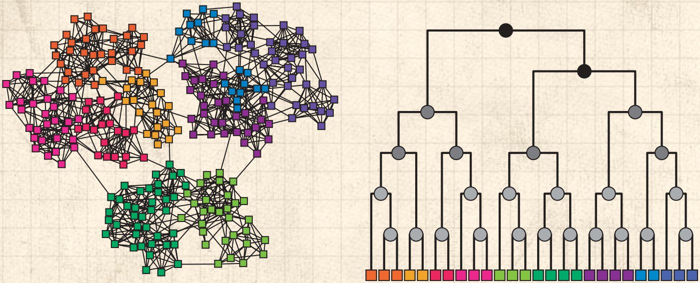

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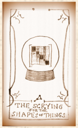

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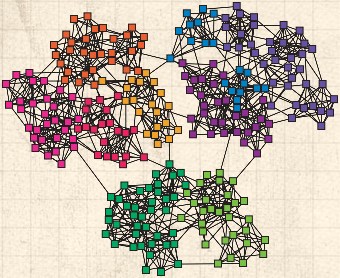

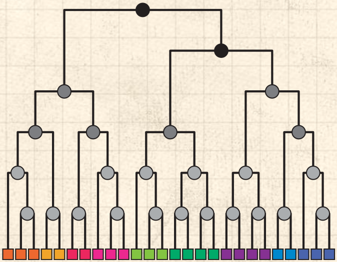

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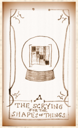

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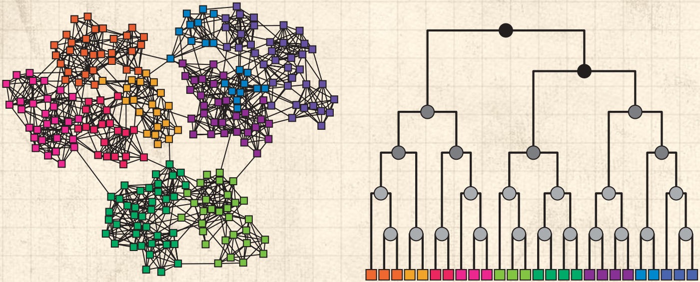

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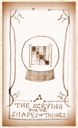
- Obtain probability that links are missing (big problem...).



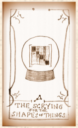
Hierarchies and missing links

- Model also predicts reasonably well
  1. average degree,
  2. clustering,
  3. and average shortest path length.

Table 1 | Comparison of original and resampled networks

Network	$\langle k \rangle_{real}$	$\langle k \rangle_{samp}$	C_{real}	C_{samp}	d_{real}	d_{samp}
T. pallidum	4.8	3.7(1)	0.0625	0.0444(2)	3.690	3.940(6)
Terrorists	4.9	5.1(2)	0.361	0.352(1)	2.575	2.794(7)
Grassland	3.0	2.9(1)	0.174	0.168(1)	3.29	3.69(2)

Statistics are shown for the three example networks studied and for new networks generated by resampling from our hierarchical model. The generated networks closely match the average degree $\langle k \rangle$, clustering coefficient C and average vertex–vertex distance d in each case, suggesting that they capture much of the structure of the real networks. Parenthetical values indicate standard errors on the final digits.



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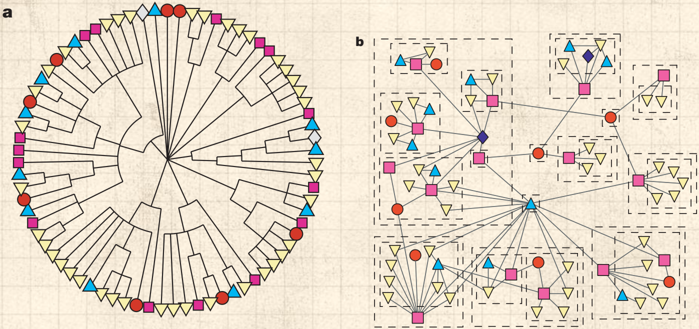
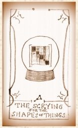


- Consensus dendogram for grassland species.



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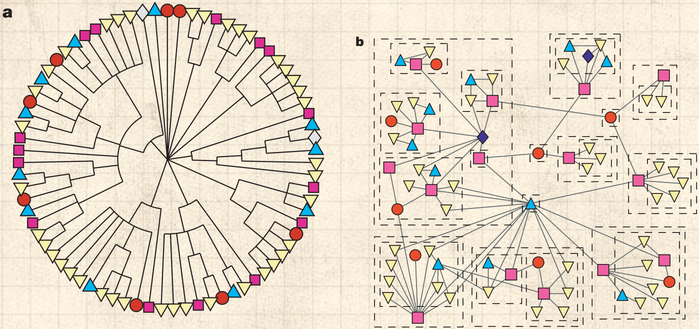
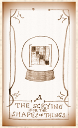


- Consensus dendogram for grassland species.
- Copes with disassortative and assortative communities.



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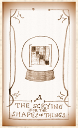
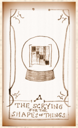


From PoCS:
Small-worldness and social searchability

Social networks and identity:



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Social networks and identity:

Identity is formed from attributes such as:

- Geographic location
- Type of employment
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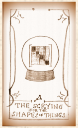

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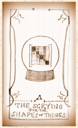

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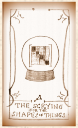
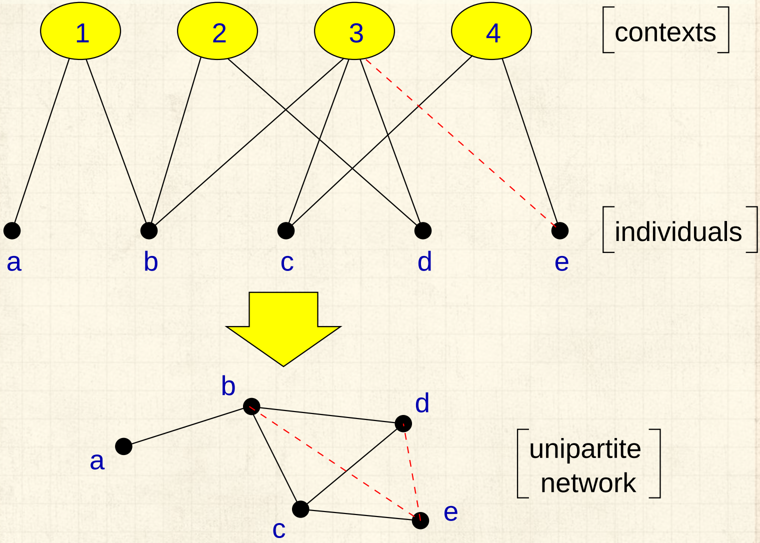
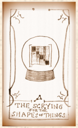
Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks.

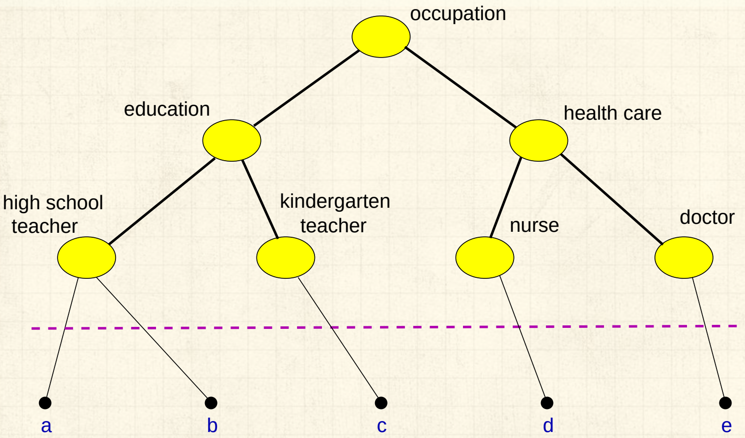


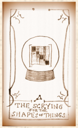
Social distance—Bipartite affiliation networks



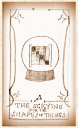


Social distance—Context distance

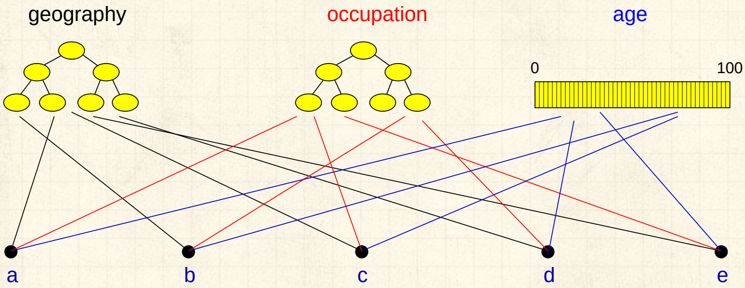




Models



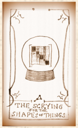
Generalized affiliation networks



Blau & Schwartz [2], Simmel [16], Breiger [3], Watts *et al.* [18]; see also Google+ Circles.

Dealing with community overlap:

- Earlier structure detection algorithms, agglomerative or divisive, force communities to be purely distinct.



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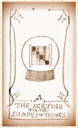

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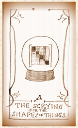

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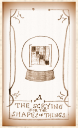

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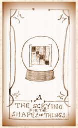

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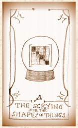
- Four new quantities:
  - m, number of a communities a node belongs to.



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- Earlier structure detection algorithms, agglomerative or divisive, force communities to be purely distinct.
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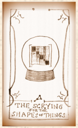

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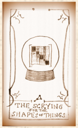

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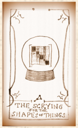

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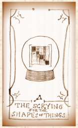
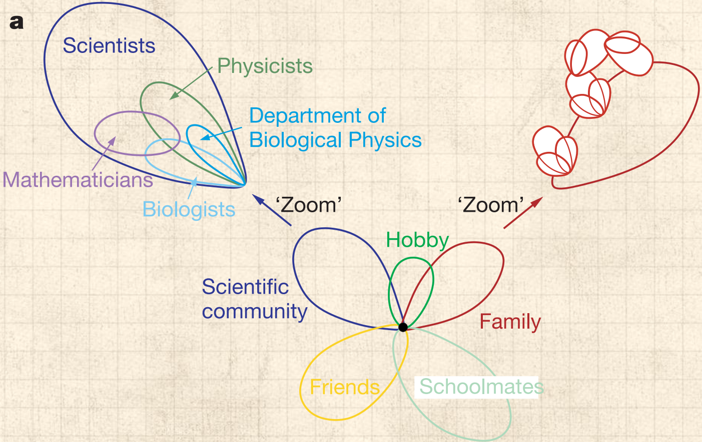
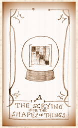
- Associated distributions: $P_{>}(m)$, $P_{>}(s^{\text{ov}}_{\alpha,\beta})$, $P_{>}(d^{\text{com}}_{\alpha})$, and $P_{>}(s^{\text{com}}_{\alpha})$.



"Uncovering the overlapping community structure of complex networks in nature and society"

Palla et al.,
Nature, **435**, 814–818, 2005. [13]





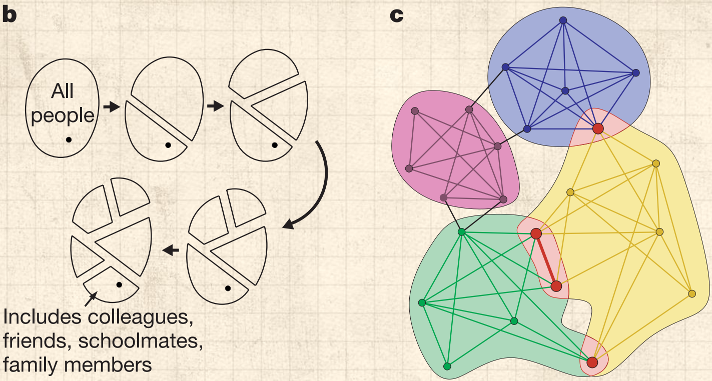
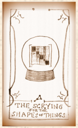


Figure 1 | **Illustration of the concept of overlapping communities.** **a**, The black dot in the middle represents either of the authors of this paper, with several of his communities around. Zooming in on the scientific community demonstrates the nested and overlapping structure of the communities, and depicting the cascades of communities starting from some members exemplifies the interwoven structure of the network of communities. **b**, Divisive and agglomerative methods grossly fail to identify the communities when overlaps are significant. **c**, An example of overlapping k-clique communities at $k = 4$. The yellow community overlaps the blue one in a single node, whereas it shares two nodes and a link with the green one. These overlapping regions are emphasized in red. Notice that any k-clique (complete subgraph of size k) can be reached only from the k-cliques of the same community through a series of adjacent k-cliques. Two k-cliques are adjacent if they share $k - 1$ nodes.



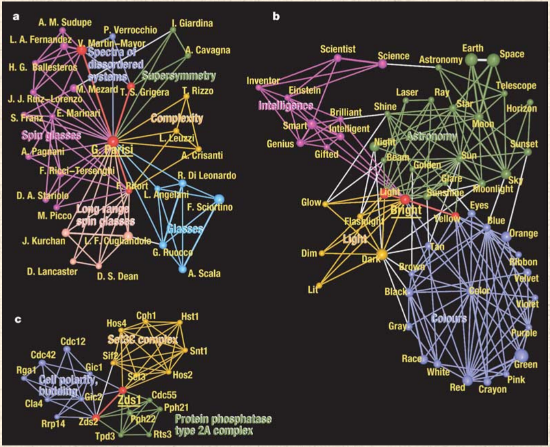
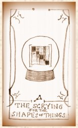
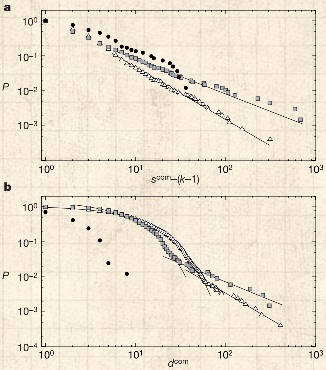


Figure 2 | The community structure around a particular node in three different networks. The communities are colour coded, the overlapping nodes and links between them are emphasized in red, and the volume of the balls and the width of the links are proportional to the total number of communities they belong to. For each network the value of k has been set to 4. **a**, The communities of G. Parisi in the co-authorship network of the Los Alamos Condensed Matter archive (for threshold weight $w^* = 0.75$) can be associated with his fields of interest. **b**, The communities of the word 'bright' in the South Florida Free Association norms list (for $w^* = 0.025$) represent the different meanings of this word. **c**, The communities of the protein Zds1 in the DIP core list of the protein–protein interactions of *S. cerevisiae* can be associated with either protein complexes or certain functions.



- Two tunable parameters: w^*, the link weight threshold, and k, the clique size.



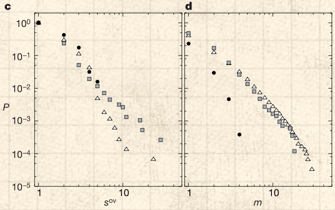
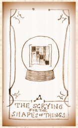
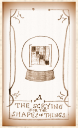
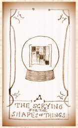


Figure 4 | Statistics of the *k*-clique communities for three large networks. The networks are the co-authorship network of the Los Alamos Condensed Matter archive (triangles, $k = 6$, $f^* = 0.93$), the word-association network of the South Florida Free Association norms (squares, $k = 4$, $f^* = 0.67$), and the protein interaction network of the yeast *S. cerevisiae* from the DIP database (circles, $k = 4$). **a**, The cumulative distribution function of the community size follows a power law with exponents between -1 (upper line) and -1.6 (lower line). **b**, The cumulative distribution of the community degree starts exponentially and then crosses over to a power law (with the same exponent as for the community size distribution). **c**, The cumulative distribution of the overlap size. **d**, The cumulative distribution of the membership number.



Outline





A link-based approach:

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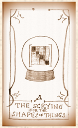

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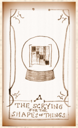

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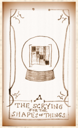

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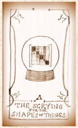
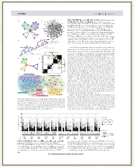
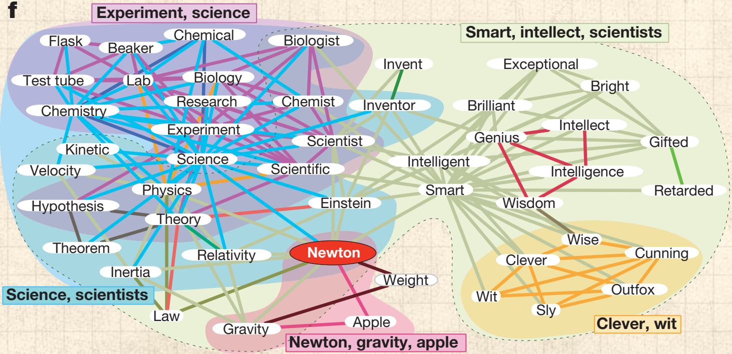
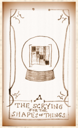


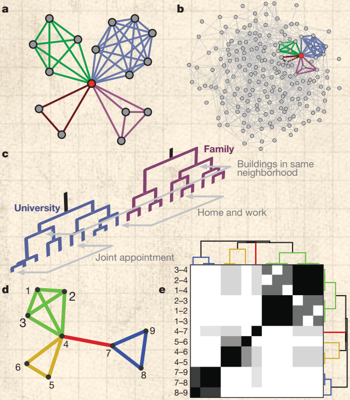


“Link communities reveal multiscale complexity in networks”

Ahn, Bagrow, and Lehmann, Nature, **466**, 761–764, 2010. [1]







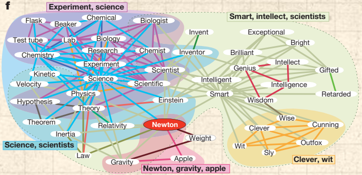


Figure 1 | Overlapping communities lead to dense networks and prevent the discovery of a single node hierarchy. **a**, Local structure in many networks is simple: an individual node sees the communities it belongs to. **b**, Complex global structure emerges when every node is in the situation displayed in **a**. **c**, Pervasive overlap hinders the discovery of hierarchical organization because nodes cannot occupy multiple leaves of a node dendrogram, preventing a single tree from encoding the full hierarchy. **d**, **e**, An example showing link communities (colours in **d**), the link similarity matrix (**e**; darker entries show more similar pairs of links) and the link dendrogram (**e**). **f**, Link communities from the full word association network around the word 'Newton'. Link colours represent communities and filled regions provide a guide for the eye. Link communities capture concepts related to science and allow substantial overlap. Note that the words were produced by experiment participants during free word associations.

Note: See details of paper on how to choose link communities well based on partition density D.

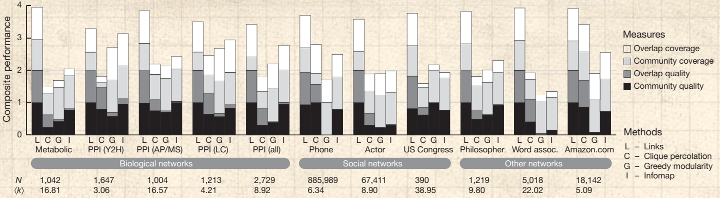
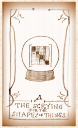


Figure 2 | Assessing the relevance of link communities using real-world networks. Composite performance (Methods and Supplementary Information) is a data-driven measure of the quality (relevance of discovered memberships) and coverage (fraction of network classified) of community and overlap. Tested algorithms are link clustering, introduced here; clique percolation[9]; greedy modularity optimization[26]; and Infomap[21]. Test networks were chosen for their varied sizes and topologies and to represent the different domains where network analysis is used. Shown for each are the number of nodes, N, and the average number of neighbours per node, $\langle k \rangle$. Link clustering finds the most relevant community structure in real-world networks. AP/MS, affinity-purification/mass spectrometry; LC, literature curated; PPI, protein–protein interaction; Y2H, yeast two-hybrid.

- Comparison of structure detection algorithms using four measures over many networks.
- Revealed communities are matched against 'known' communities recorded in network metadata.
- Link approach particularly good for dense, overlapful networks.



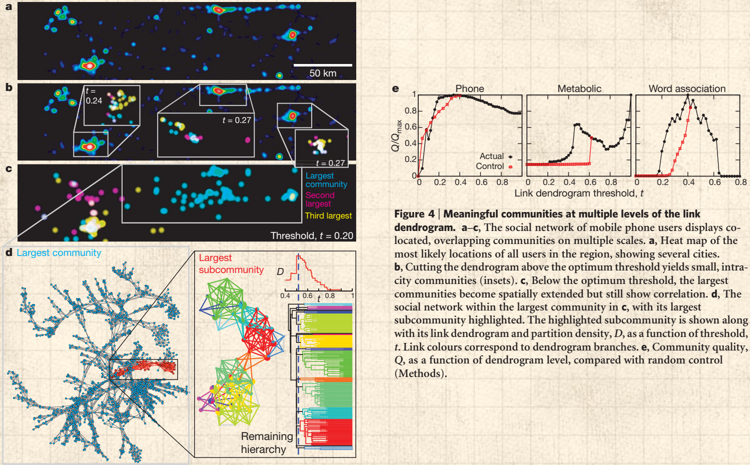


Figure 4 | Meaningful communities at multiple levels of the link dendrogram. **a**–**c**, The social network of mobile phone users displays co-located, overlapping communities on multiple scales. **a**, Heat map of the most likely locations of all users in the region, showing several cities. **b**, Cutting the dendrogram above the optimum threshold yields small, intra-city communities (insets). **c**, Below the optimum threshold, the largest communities become spatially extended but still show correlation. **d**, The social network within the largest community in **c**, with its largest subcommunity highlighted. The highlighted subcommunity is shown along with its link dendrogram and partition density, D, as a function of threshold, t. Link colours correspond to dendrogram branches. **e**, Community quality, Q, as a function of dendrogram level, compared with random control (Methods).

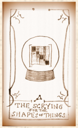
Outline

Overview

Methods

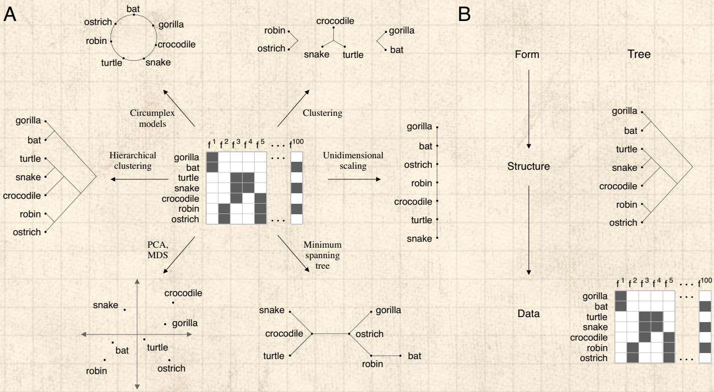
- Hierarchy by aggregation
- Hierarchy by division
- Hierarchy by shuffling
- Spectral methods
- Hierarchies & Missing Links
- Overlapping communities
- Link-based methods
- General structure detection

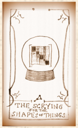
References



General structure detection

- “The discovery of structural form”
  Kemp and Tenenbaum, PNAS (2008) [8]





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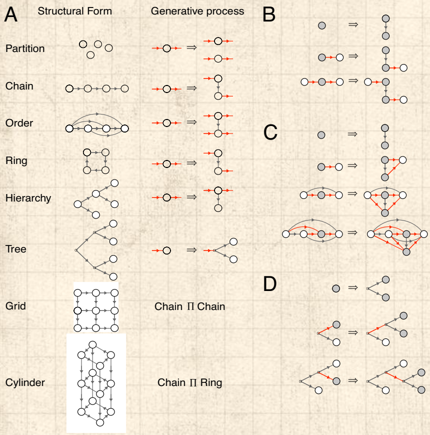
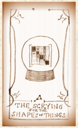


Top down description of form.



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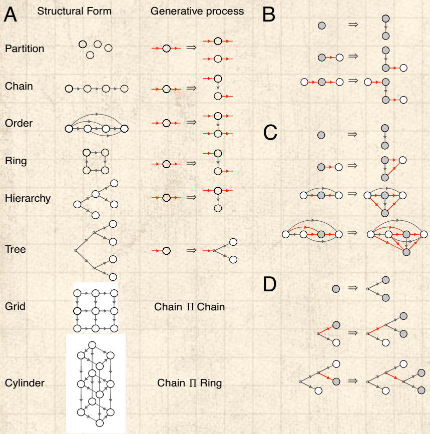

- Top down description of form.
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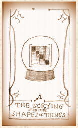

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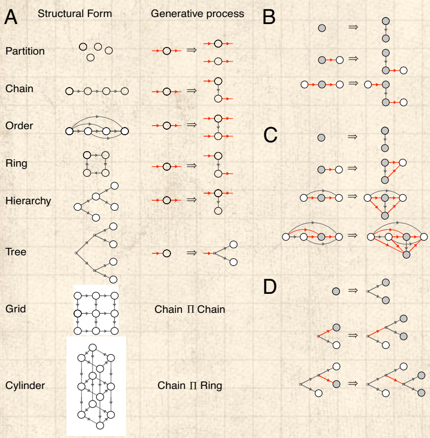
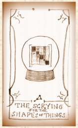
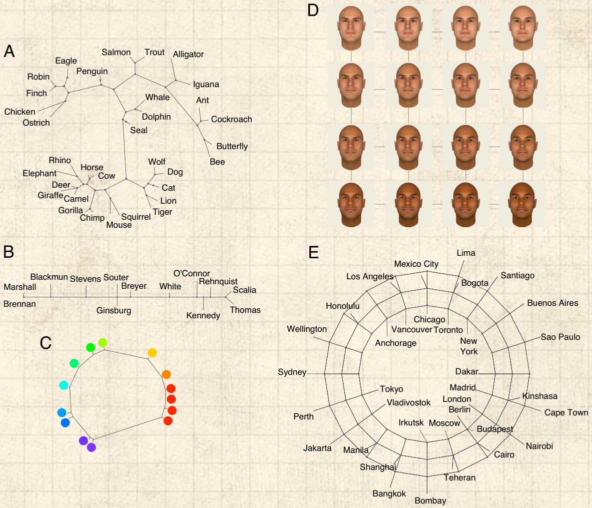


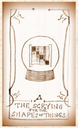
- Top down description of form.
- Node replacement graph grammar: parent node becomes two child nodes.
- B-D: Growing chains, orders, and trees.



Example learned structures:



Biological features; Supreme Court votes; perceived color differences; face differences; & distances between cities.



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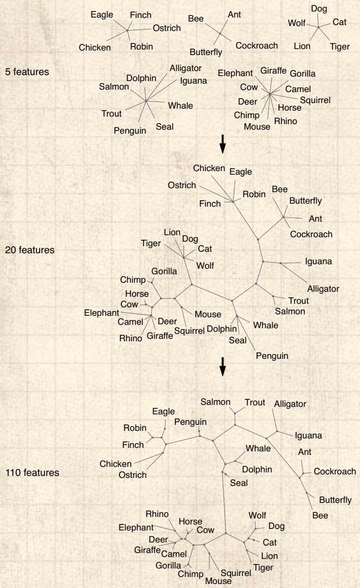
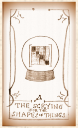


Effect of adding features on detected form.



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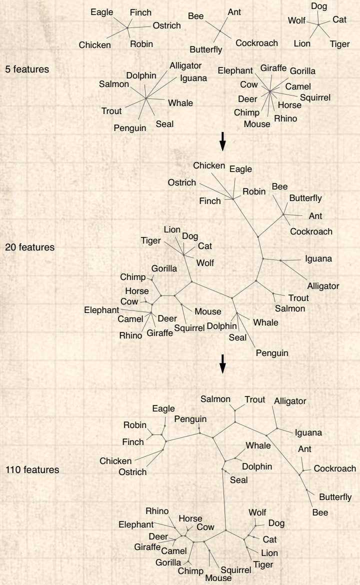


Effect of adding features on detected form.

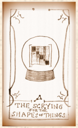
Straight partition

⇓

simple tree

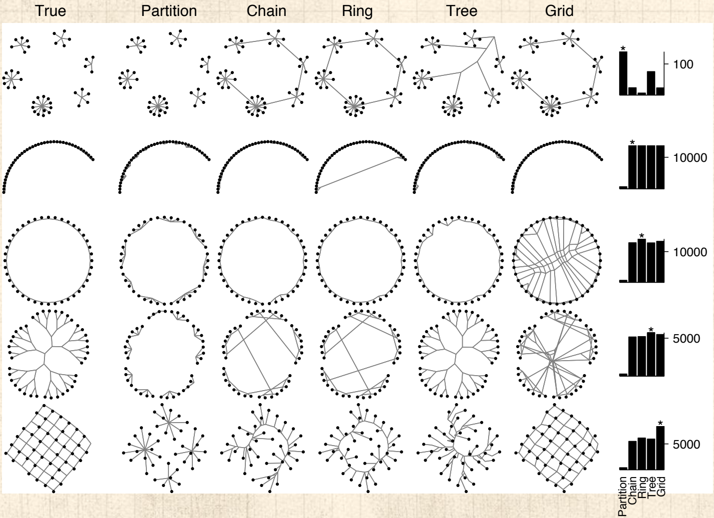
⇓

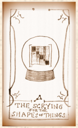
complex tree



General structure detection

- Performance for test networks.





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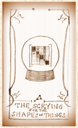

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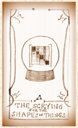

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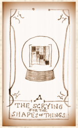

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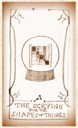

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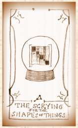

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