Random Networks

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Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

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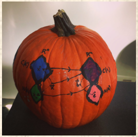
Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



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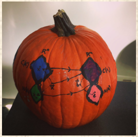

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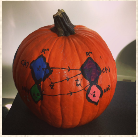
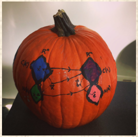
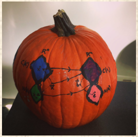


Outline

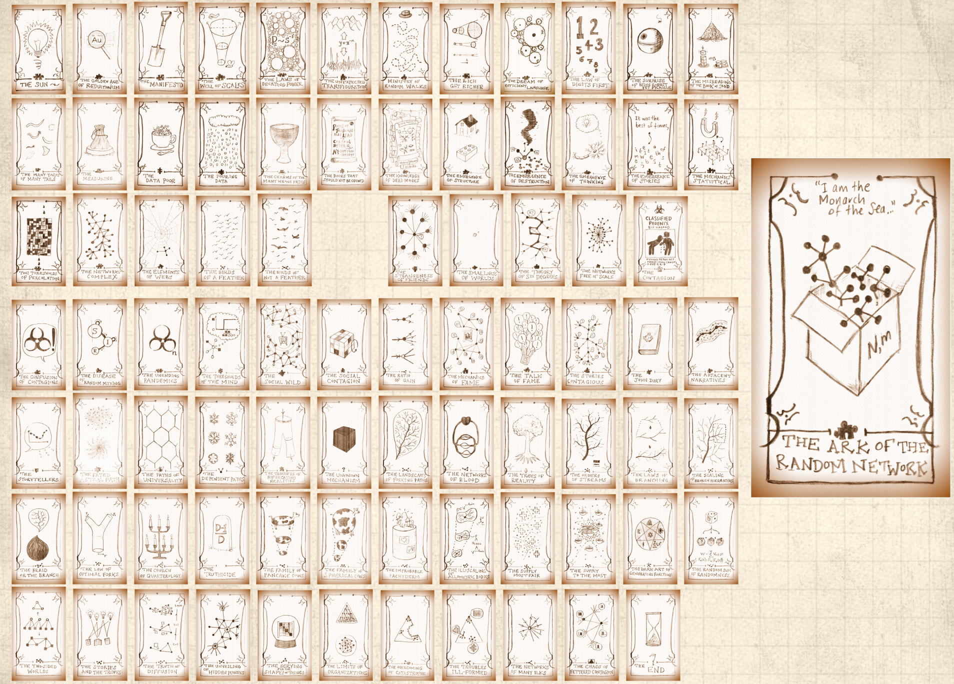


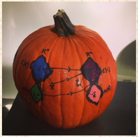
Models



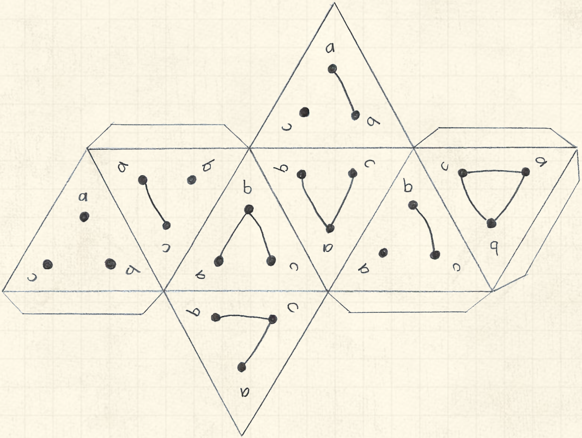
Some important models:

1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models (p^*).


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
PATHS OF THINKING THAT LEAD TO CERTAIN DESPAIR
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END
"I am the Monarch of the Sea..."
N,m
THE ARK OF THE RANDOM NETWORK

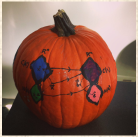


Random network generator for $N = 3$:

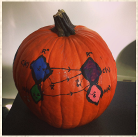


- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

Outline

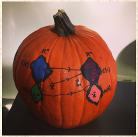


Random networks



Pure, abstract random networks:

Random networks



Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.

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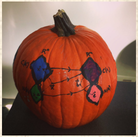

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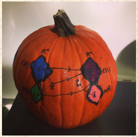

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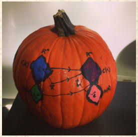

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Pure, abstract random networks:

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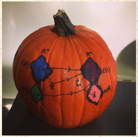
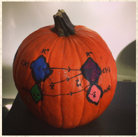
- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.



Random networks—basic features:

- Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$



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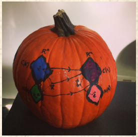

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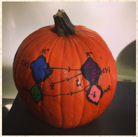

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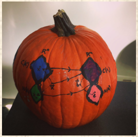
$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N(N-1)}.$$



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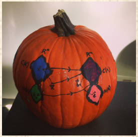

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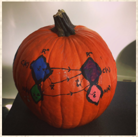
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.



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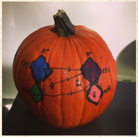
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world: links are usually costly so real networks are almost always sparse.



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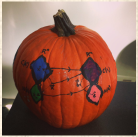
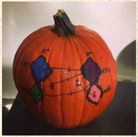


Random networks

How to build standard random networks:

- Given N and m.



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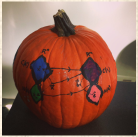

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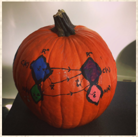

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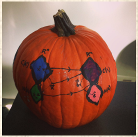
1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.



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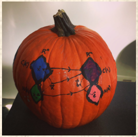
2. Take N nodes and add exactly m links by selecting edges without replacement.



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   - Useful for theoretical work.
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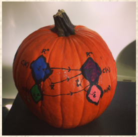

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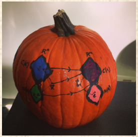

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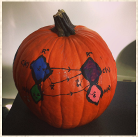

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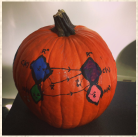
   - Best for adding relatively small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

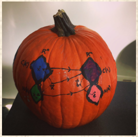


Random networks

A few more things:

- For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2}$$



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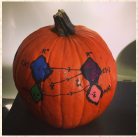
- For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2} = p\frac{1}{2}N(N-1)$$



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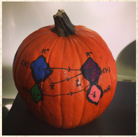
- So the expected or average degree is

$$\langle k \rangle = \frac{2\langle m \rangle}{N}$$



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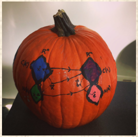

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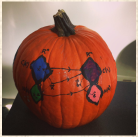

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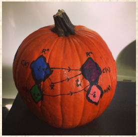

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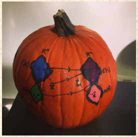
- Which is what it should be...



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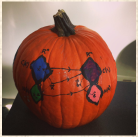
$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{\cancel{2}}{\cancel{N}} p \frac{1}{\cancel{2}} \cancel{N}(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \rightarrow 0$ as $N \rightarrow \infty$.



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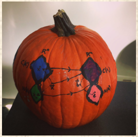
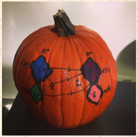


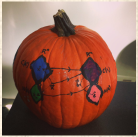
Random networks: examples



Next slides:

Example realizations of random networks

Random networks: examples



Next slides:

Example realizations of random networks

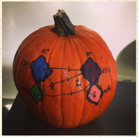
- $N = 500$

Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.

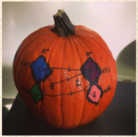


Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.

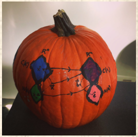


Random networks: examples

Next slides:

Example realizations of random networks

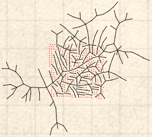
- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.



Random networks: examples for N=500

m = 100
$\langle k \rangle$ = 0.4

m = 200
$\langle k \rangle$ = 0.8



m = 230
$\langle k \rangle$ = 0.92

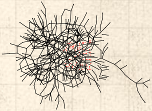
m = 240
$\langle k \rangle$ = 0.96

m = 250
$\langle k \rangle$ = 1

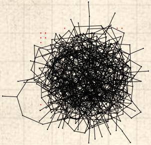
m = 260
$\langle k \rangle$ = 1.04

m = 280
$\langle k \rangle$ = 1.12

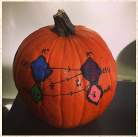
m = 300
$\langle k \rangle$ = 1.2



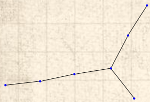
m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4



Random networks: largest components



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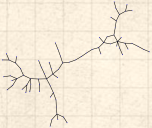
$\langle k \rangle$ = 0.8



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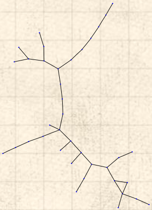
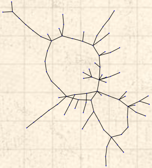
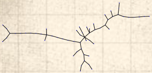


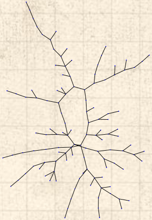
m = 240
$\langle k \rangle$ = 0.96



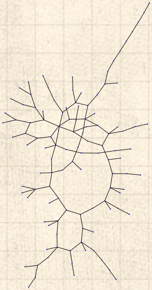
m = 250
$\langle k \rangle$ = 1



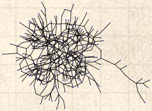
m = 260
$\langle k \rangle$ = 1.04



m = 280
$\langle k \rangle$ = 1.12



m = 300
$\langle k \rangle$ = 1.2



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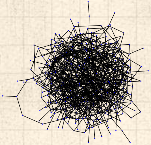
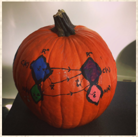


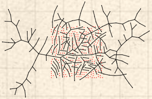
m = 1000
$\langle k \rangle$ = 4



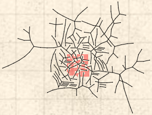
Random networks: examples for N=500

m = 250
$\langle k \rangle$ = 1

m = 250
$\langle k \rangle$ = 1



m = 250
$\langle k \rangle$ = 1



m = 250
$\langle k \rangle$ = 1

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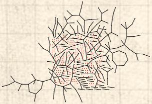
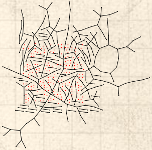


m = 250
$\langle k \rangle$ = 1



m = 250
$\langle k \rangle$ = 1

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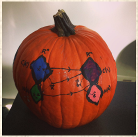
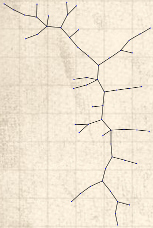


Random networks: largest components



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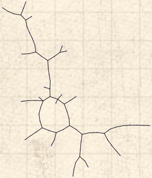
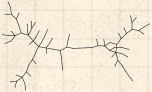


m = 250
$\langle k \rangle$ = 1



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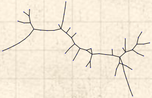
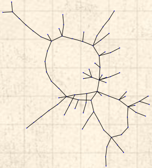
$\langle k \rangle$ = 1



m = 250
$\langle k \rangle$ = 1



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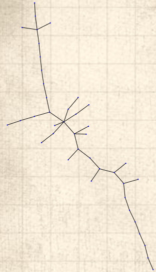

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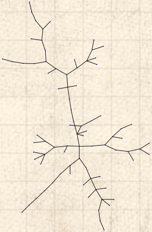

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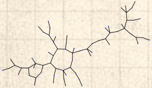

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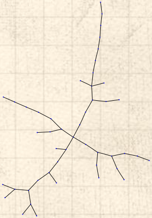

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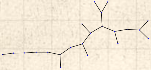

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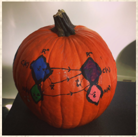
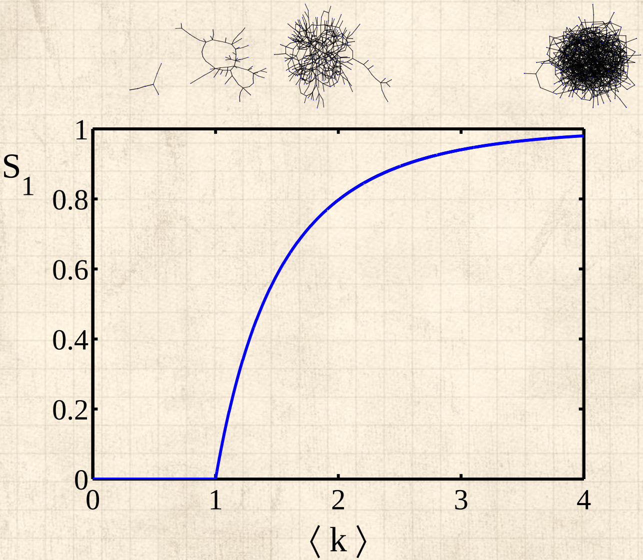
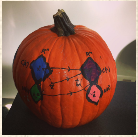
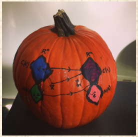


Giant component



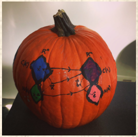


Outline



Clustering in random networks:

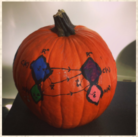
- For construction method 1, what is the clustering coefficient for a finite network?



Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient: [7]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$



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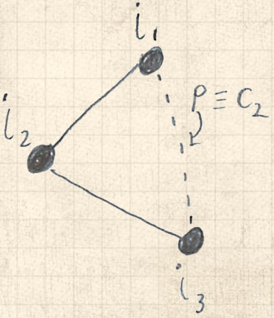
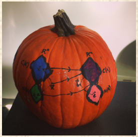


- Recall: C_2 = probability that two friends of a node are also friends.



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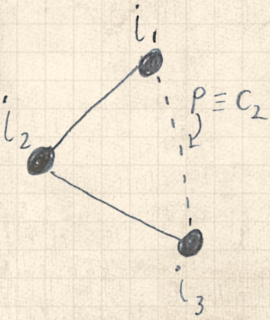
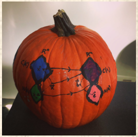


- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.



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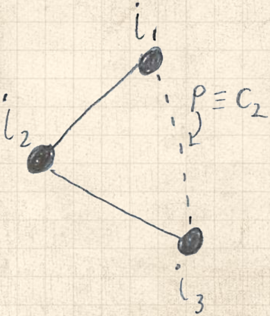
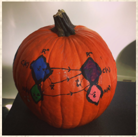


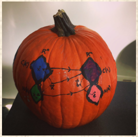
- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that

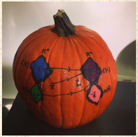
$$C_2 = p.$$



Clustering in random networks:

- So for large random networks ($N \to \infty$), clustering drops to zero.





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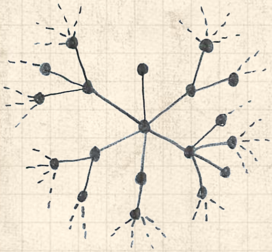
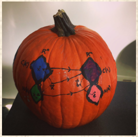
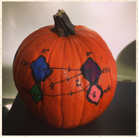


- So for large random networks ($N \to \infty$), clustering drops to zero.
- Key structural feature of random networks is that they locally look like pure branching networks
- No small loops.

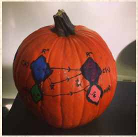


Outline



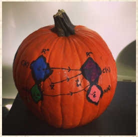
Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.



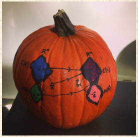
Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.



Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.



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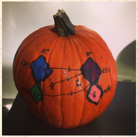

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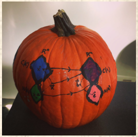
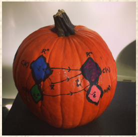
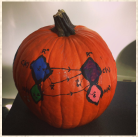
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$



Limiting form of $P(k;p,N)$:



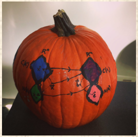


Limiting form of $P(k;p,N)$:

- Our degree distribution:
$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.

Limiting form of $P(k;p,N)$:

- Our degree distribution: $P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \to \infty$?



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- Our degree distribution: $P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
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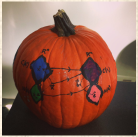

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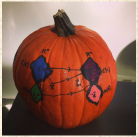

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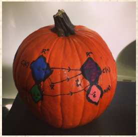

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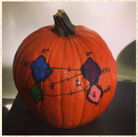

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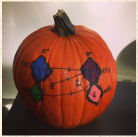
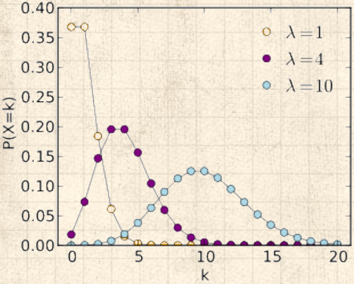
- This is a Poisson distribution with mean $\langle k \rangle$.

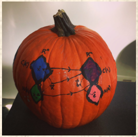


Poisson basics:

$$\boxed{P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- ‘Law of small numbers’



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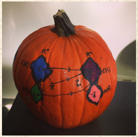

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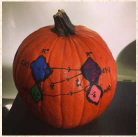

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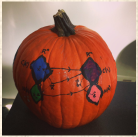

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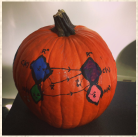

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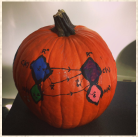

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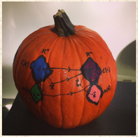

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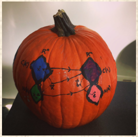

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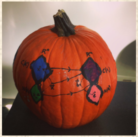

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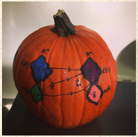

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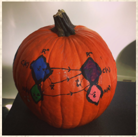

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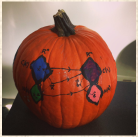

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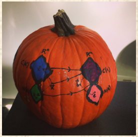

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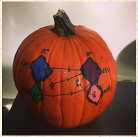

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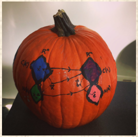

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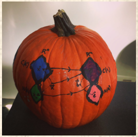

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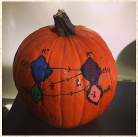

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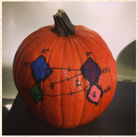

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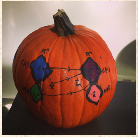

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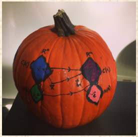

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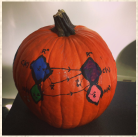

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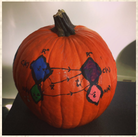
- Note: This is a special property of Poisson distribution and can trip us up...

Neural reboot (NR):

Unrelated: Feline elevation



Outline

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Coming up:

Example realizations of random networks with power law degree distributions:

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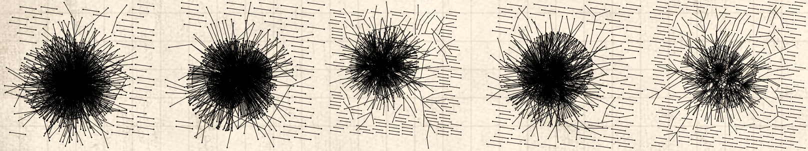
- Apart from degree distribution, wiring is random.

Random networks: examples for N=1000



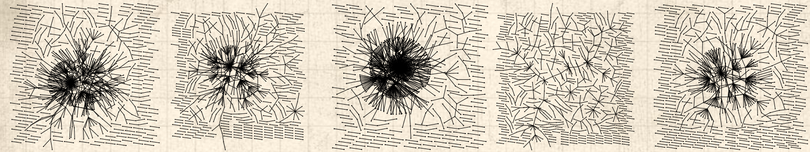
γ = 2.1 $\langle k \rangle$ = 3.448

γ = 2.19 $\langle k \rangle$ = 2.986

γ = 2.28 $\langle k \rangle$ = 2.306

γ = 2.37 $\langle k \rangle$ = 2.504

γ = 2.46 $\langle k \rangle$ = 1.856



γ = 2.55 $\langle k \rangle$ = 1.712

γ = 2.64 $\langle k \rangle$ = 1.6

γ = 2.73 $\langle k \rangle$ = 1.862

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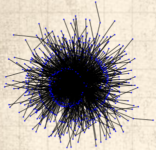
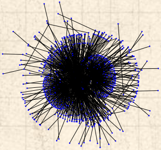
γ = 2.91 $\langle k \rangle$ = 1.49

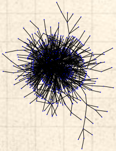
Random networks: largest components



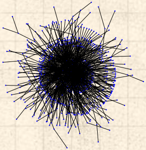
γ = 2.1
$\langle k \rangle$ = 3.448



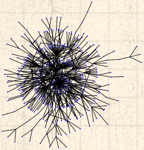
γ = 2.19
$\langle k \rangle$ = 2.986



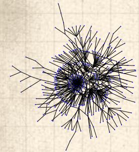
γ = 2.28
$\langle k \rangle$ = 2.306



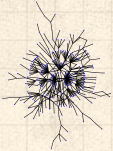
γ = 2.37
$\langle k \rangle$ = 2.504



γ = 2.46
$\langle k \rangle$ = 1.856



γ = 2.55
$\langle k \rangle$ = 1.712



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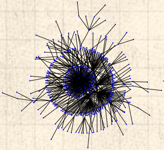

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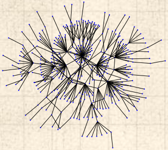

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Outline

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions
Generalized Random Networks
Configuration model
How to build in practice
Motifs
Random friends are strange
Largest component
References

Models

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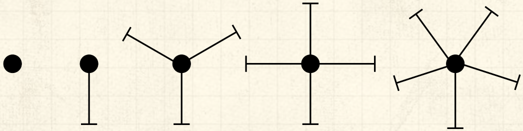
Building random networks: Stubs

Phase 1:

- Idea: start with a soup of unconnected nodes with stubs (half-edges):



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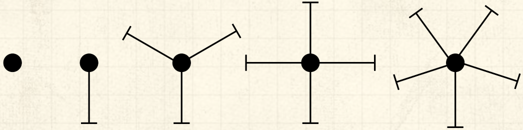

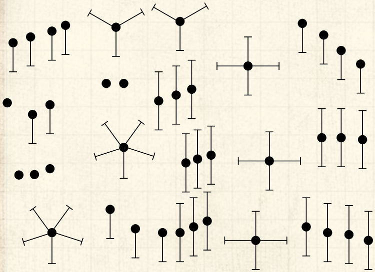

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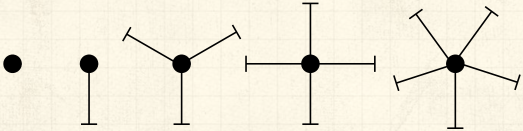

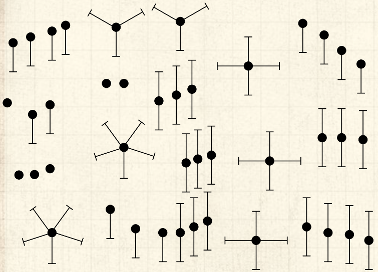

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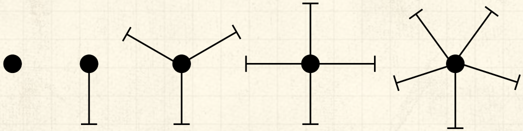

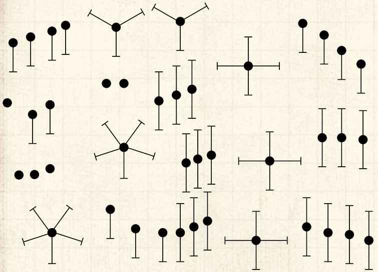

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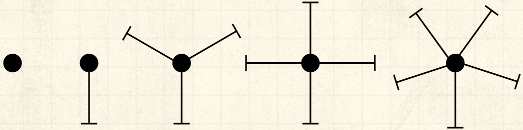
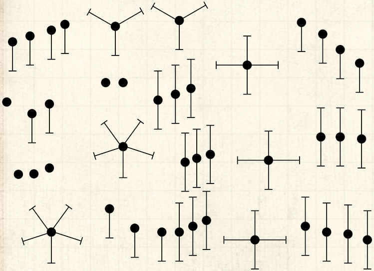




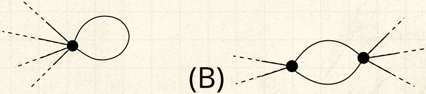
- Randomly select stubs (not nodes!) and connect them.
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.

(A) (B)



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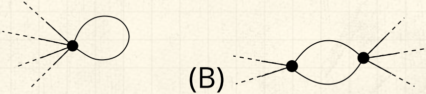

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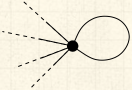
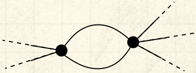
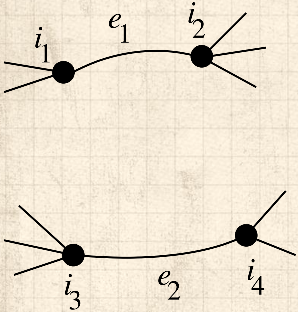
(A)  (B) 

- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

General random rewiring algorithm



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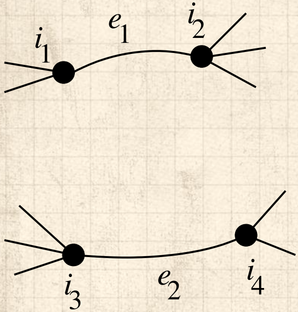

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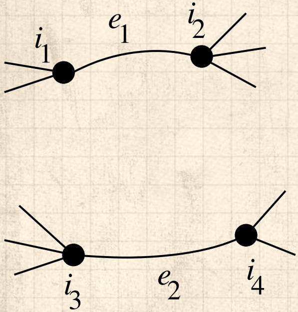

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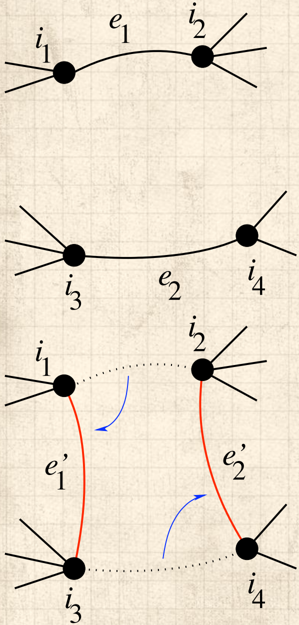

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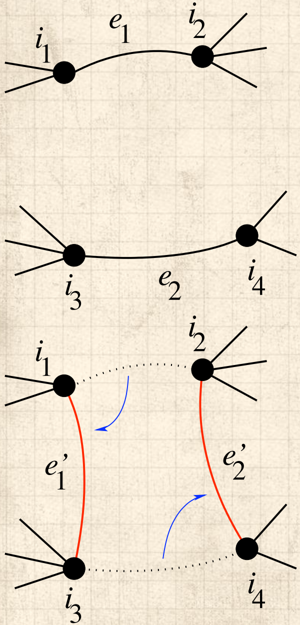

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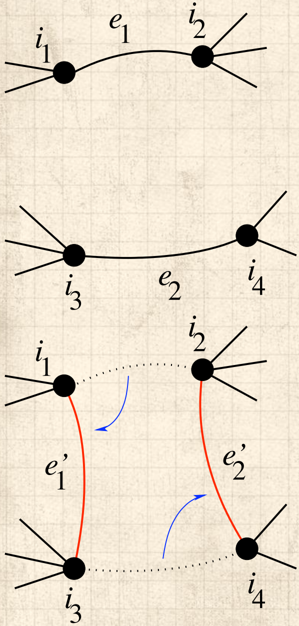

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- Same as finding on/off/on/off 4-cycles. and rotating them.

Sampling random networks

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- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [5].

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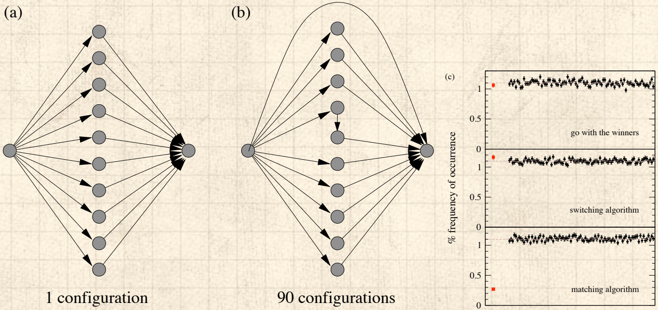
- Example from Milo et al. (2003) [5]:



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- Note: not all P_k will always give nodes that can be wired together.

Outline

Network motifs

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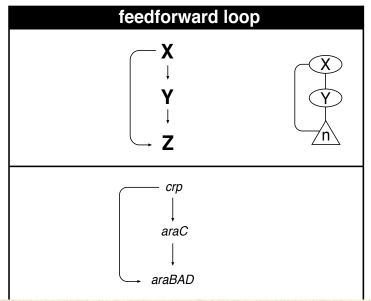
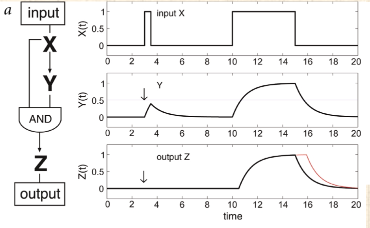
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k.
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

Network motifs





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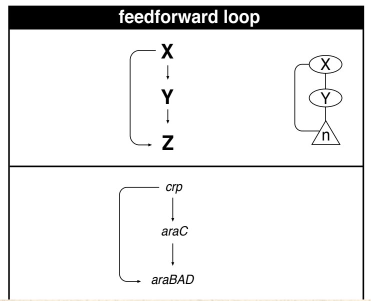

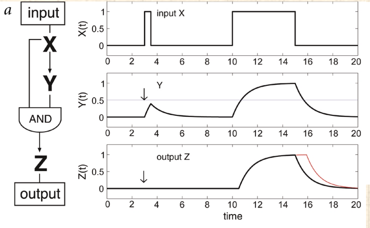

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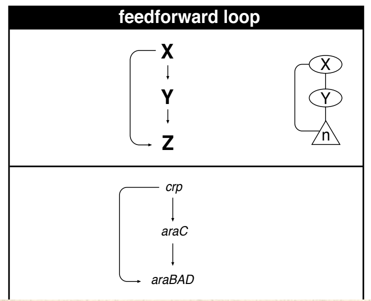

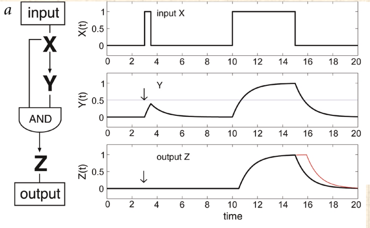

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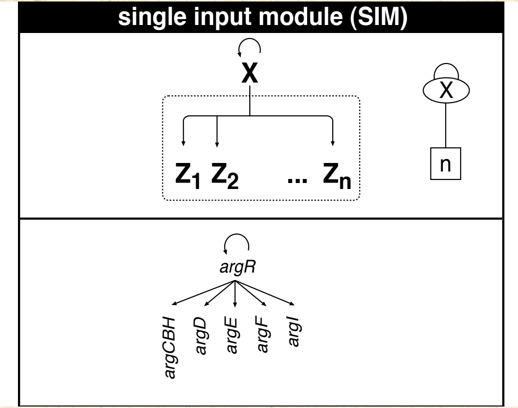
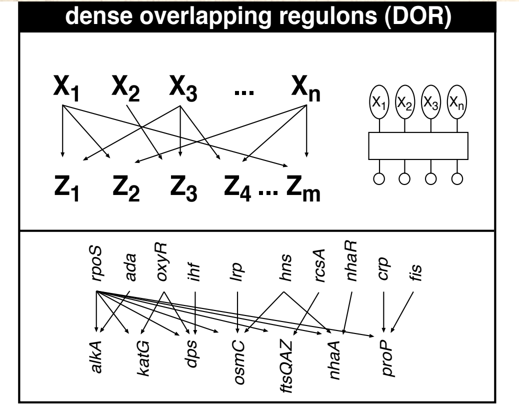
- Analogy to elevator doors.

Network motifs



- Master switch.

Network motifs



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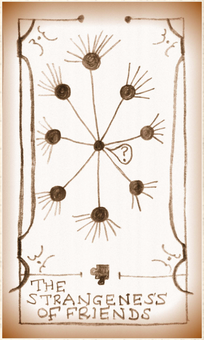
- For more, see work carried out by Wiggins *et al.* at Columbia.

Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATION
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE STRANGENESS OF FRIENDS

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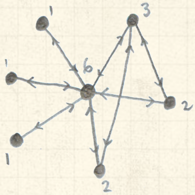
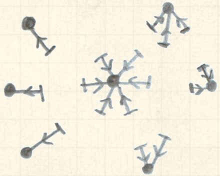
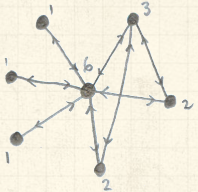
- Big deal: Rich-get-richer mechanism is built into this selection process.

Probability of randomly selecting a node of degree k by choosing from nodes:
$P_1 = 3/7$, $P_2 = 2/7$, $P_3 = 1/7$, $P_6 = 1/7$.







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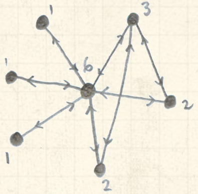

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- Natural question: what's the expected number of other friends that one friend has?

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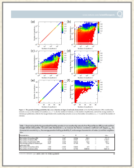
- Intuition: for networks, the more connected a node, the more likely it is to be chosen as a friend.



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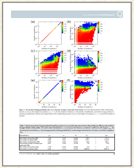

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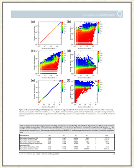

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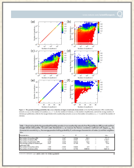

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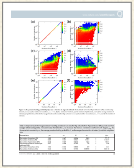

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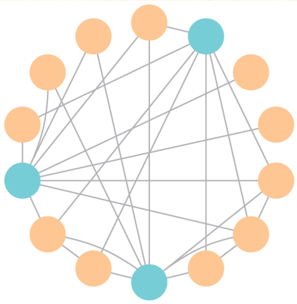

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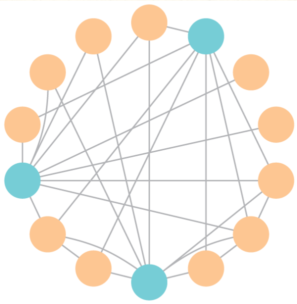

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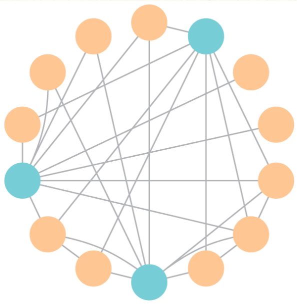
Related disappointment:



- Nodes see their friends' color choices.
- Which color is more popular?[1]
- Again: thinking in edge space changes everything.

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(Big) Reason #2:

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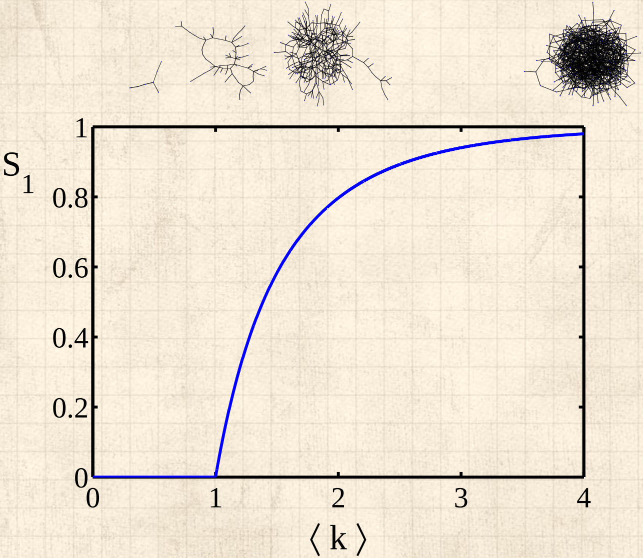
- Note: Component = Cluster

Outline

Giant component



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- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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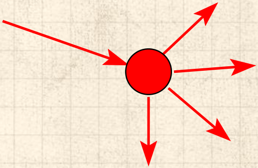

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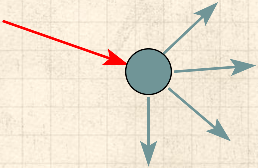

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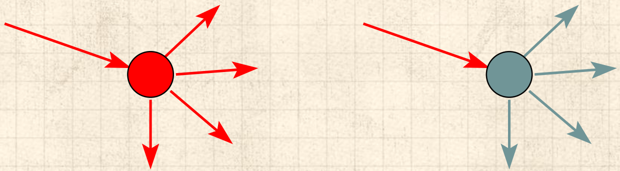

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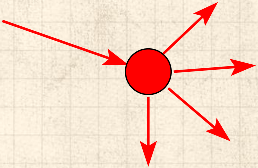

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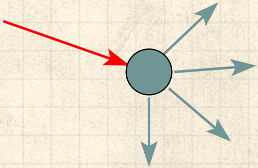

- Focus on binary case with edges and nodes either infected or not.
- First big question: for a given network and contagion process, can global spreading from a single seed occur?

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- Good: This is just our giant component condition again.

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- Resulting degree distribution $\tilde{P}_k$:

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- We say $\langle k \rangle = 1$ marks the critical point of the system.

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- How about $P_k = \delta_{k k_0}$?

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- Substitute in Poisson distribution...

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- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.

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- Really a transcritical bifurcation. [9]

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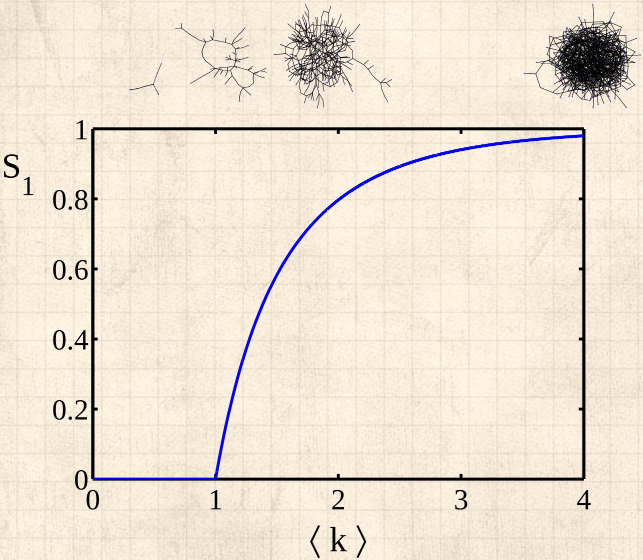

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- More detailed investigations will profit from a spot of Generatingfunctionology. [10]

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- We can sort many things out with sensible probabilistic arguments...
- CocoNuTs: We figure out the final size and complete dynamics.

Neural reboot (NR):

Falling maple leaf

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