Random Networks

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Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

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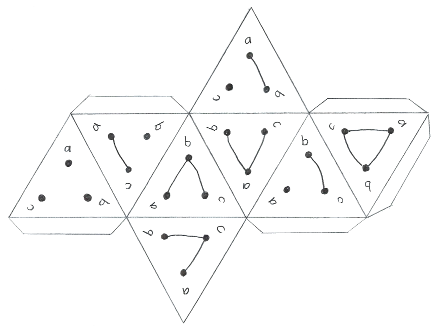
Outline

Models

Some important models:

1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models (p^*).

Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

Random networks

Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

Random networks—basic features:

- Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$

- Limit of $m = 0$: empty graph.
- Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N(N-1)}.$$

- Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world: links are usually costly so real networks are almost always sparse.

Random networks

How to build standard random networks:

- Given N and m.
- Two probablistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.
   - Useful for theoretical work.
2. Take N nodes and add exactly m links by selecting edges without replacement.
   - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
   - Best for adding relatively small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

Random networks

A few more things:

- For method 1, # links is probablistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

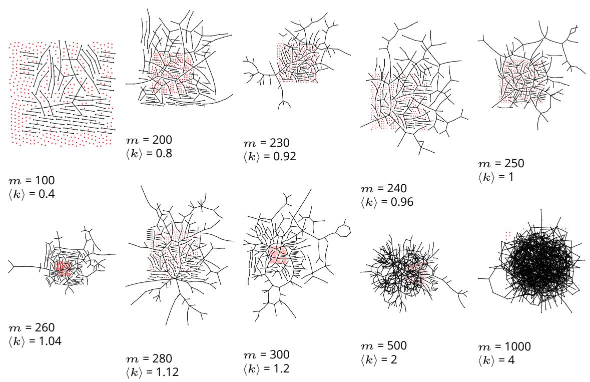
- So the expected or average degree is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

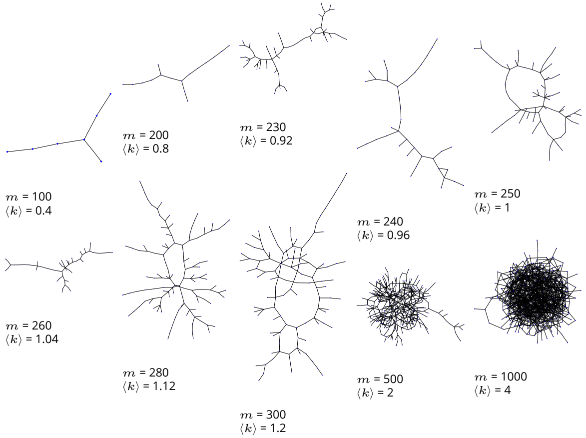
$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{\not 2}{\not N} p \frac{1}{\not 2} \not N(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

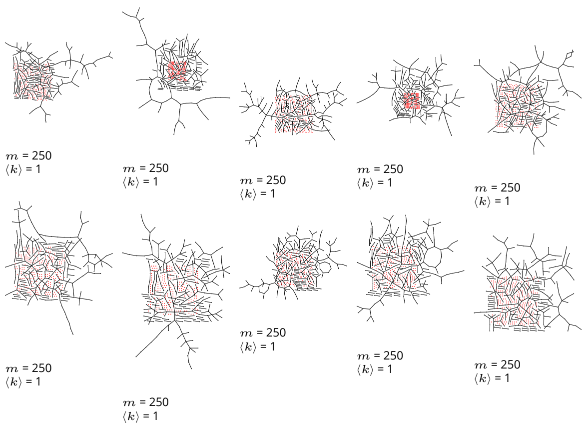
Random networks: examples for N=500



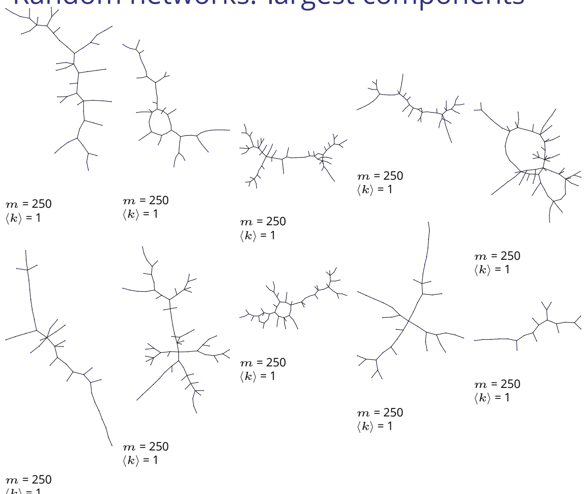
Random networks: largest components



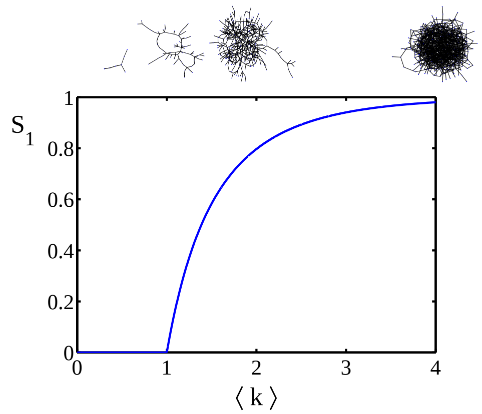
Random networks: examples for N=500



Random networks: largest components



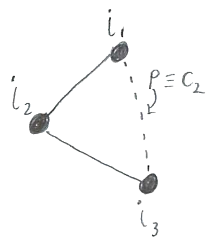
Giant component



Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient: [7]

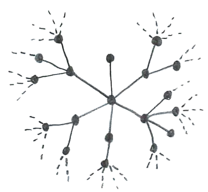
$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$



- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that

$$C_2 = p.$$

Clustering in random networks:



- So for large random networks ($N \to \infty$), clustering drops to zero.
- Key structural feature of random networks is that they locally look like pure branching networks
- No small loops.

Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

Limiting form of $P(k; p, N)$:

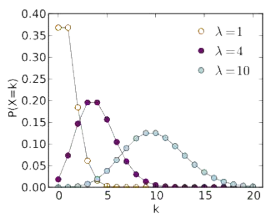
- Our degree distribution: $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \to \infty$?
- We must end up with the normal distribution right?
- If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \to \infty$.
- But we want to keep $\langle k \rangle$ fixed...
- So examine limit of $P(k; p, N)$ when $p \to 0$ and $N \to \infty$ with $\langle k \rangle = p(N-1)$ = constant.

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \to \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

- This is a Poisson distribution with mean $\langle k \rangle$.

Poisson basics:

$$P(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- 'Law of small numbers'

Poisson basics:

- Normalization: we must have

$$\sum_{k=0}^{\infty} P(k;\langle k\rangle) = 1$$

- Checking:

$$\sum_{k=0}^{\infty} P(k;\langle k\rangle) = \sum_{k=0}^{\infty} \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle}$$

$$= e^{-\langle k\rangle} \sum_{k=0}^{\infty} \frac{\langle k\rangle^k}{k!}$$

$$= e^{-\langle k\rangle} e^{\langle k\rangle} = 1$$

Poisson basics:

- Mean degree: we must have

$$\langle k\rangle = \sum_{k=0}^{\infty} k P(k;\langle k\rangle).$$

- Checking:

$$\sum_{k=0}^{\infty} k P(k;\langle k\rangle) = \sum_{k=0}^{\infty} k \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle}$$

$$= e^{-\langle k\rangle} \sum_{k=1}^{\infty} \frac{\langle k\rangle^k}{(k-1)!}$$

$$= \langle k\rangle e^{-\langle k\rangle} \sum_{k=1}^{\infty} \frac{\langle k\rangle^{k-1}}{(k-1)!}$$

$$= \langle k\rangle e^{-\langle k\rangle} \sum_{i=0}^{\infty} \frac{\langle k\rangle^i}{i!} = \langle k\rangle e^{-\langle k\rangle} e^{\langle k\rangle} = \langle k\rangle$$

- In CocoNuTs, we find a different, crazier way of doing this...

Pure random networks
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How to build theoretically
Some visual examples
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Degree distributions
Generalized Random Networks
Configuration model
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Motifs
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Poisson basics:

- The variance of degree distributions for random networks turns out to be very important.
- Using calculation similar to one for finding $\langle k\rangle$ we find the second moment to be:

$$\langle k^2\rangle = \langle k\rangle^2 + \langle k\rangle.$$

- Variance is then

$$\sigma^2 = \langle k^2\rangle - \langle k\rangle^2 = \langle k\rangle^2 + \langle k\rangle - \langle k\rangle^2 = \langle k\rangle.$$

- So standard deviation σ is equal to $\sqrt{\langle k\rangle}$.
- Note: This is a special property of Poisson distribution and can trip us up...

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General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k.
- Also known as the configuration model.[7]
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
  2. Examining mechanisms that lead to networks with certain degree distributions.

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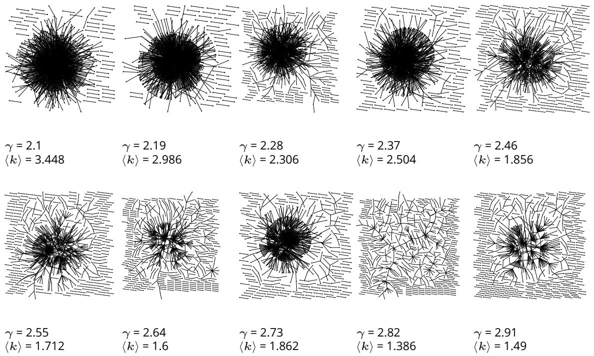
Random networks: examples for N=1000



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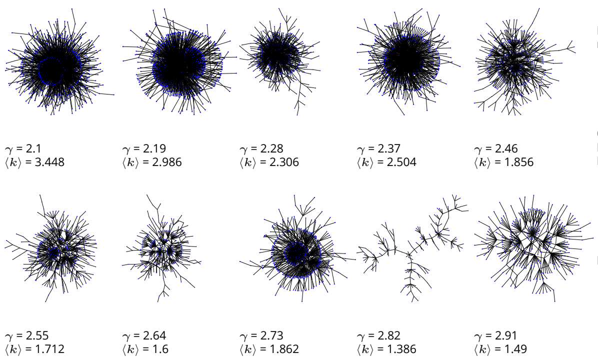
Random networks: largest components



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Models

Generalized random networks:

- Arbitrary degree distribution P_k.
- Create (unconnected) nodes with degrees sampled from P_k.
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

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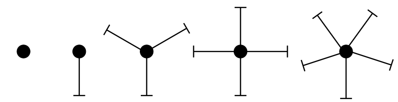
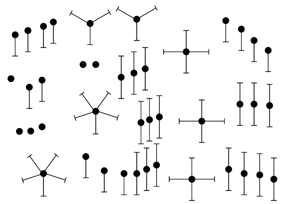
Building random networks: Stubs

Phase 1:

- Idea: start with a soup of unconnected nodes with stubs (half-edges):





- Randomly select stubs (not nodes!) and connect them.
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

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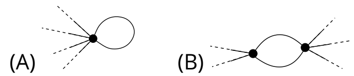
Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.



- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

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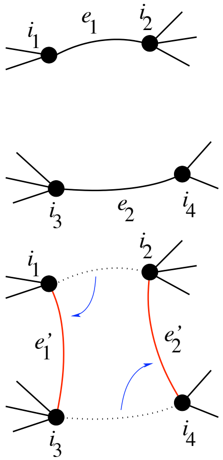
General random rewiring algorithm

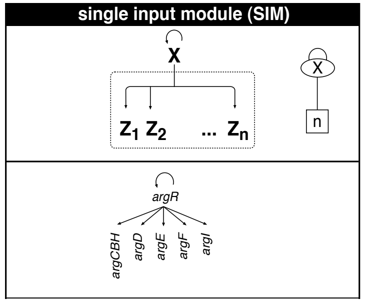


- Randomly choose two edges. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.
- Rewire one end of each edge.
- Node degrees do not change.
- Works if e_1 is a self-loop or repeated edge.
- Same as finding on/off/on/off 4-cycles. and rotating them.

Sampling random networks

- What if we have P_k instead of N_k?
- Must now create nodes before start of the construction algorithm.
- Generate N nodes by sampling from degree distribution P_k.
- Easy to do exactly numerically since k is discrete.
- Note: not all P_k will always give nodes that can be wired together.

Network motifs



- Master switch.

Sampling random networks

Phase 2:

- Use rewiring algorithm to remove all self and repeat loops.

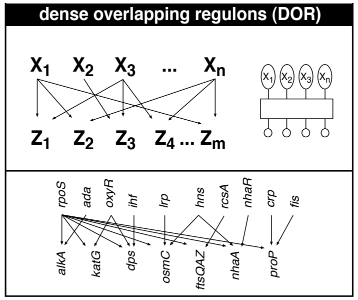
Phase 3:

- Randomize network wiring by applying rewiring algorithm liberally.
- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [5].

Network motifs

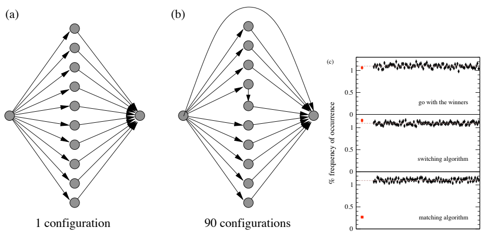
- Idea of motifs [8] introduced by Shen-Orr, Alon et al. in 2002.
- Looked at gene expression within full context of transcriptional regulation networks.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k.
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

Network motifs

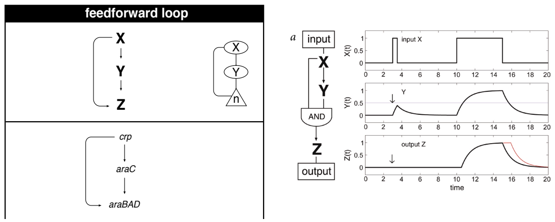


Random sampling

- Problem with only joining up stubs is failure to randomly sample from all possible networks.
- Example from Milo et al. (2003) [5]:



Network motifs



- Z only turns on in response to sustained activity in X.
- Turning off X rapidly turns off Z.
- Analogy to elevator doors.

Network motifs

- Note: selection of motifs to test is reasonable but nevertheless ad-hoc.
- For more, see work carried out by Wiggins *et al.* at Columbia.

The edge-degree distribution:

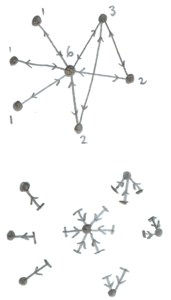
- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
- Now choosing nodes based on their degree (i.e., size):

$$\boxed{Q_k \propto kP_k}$$

- Normalized form:

$$Q_k = \frac{kP_k}{\sum_{k'=0}^{\infty} k' P_{k'}} = \frac{kP_k}{\langle k \rangle}.$$

- Big deal: Rich-get-richer mechanism is built into this selection process.



- Probability of randomly selecting a node of degree k by choosing from nodes: $P_1 = 3/7$, $P_2 = 2/7$, $P_3 = 1/7$, $P_6 = 1/7$.
- Probability of landing on a node of degree k after randomly selecting an edge and then randomly choosing one direction to travel: $Q_1 = 3/16$, $Q_2 = 4/16$, $Q_3 = 3/16$, $Q_6 = 6/16$.
- Probability of finding # outgoing edges = k after randomly selecting an edge and then randomly choosing one direction to travel: $R_0 = 3/16$ $R_1 = 4/16$, $R_2 = 3/16$, $R_5 = 6/16$.

The edge-degree distribution:

- For networks, Q_k is also the probability that a friend (neighbor) of a random node has k friends.
- Useful variant on Q_k:

  R_k = probability that a friend of a random node has k other friends.

-
$$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

- Equivalent to friend having degree $k+1$.
- Natural question: what's the expected number of other friends that one friend has?

The edge-degree distribution:

- Given R_k is the probability that a friend has k other friends, then the average number of friends' other friends is

$$\langle k \rangle_R = \sum_{k=0}^{\infty} kR_k = \sum_{k=0}^{\infty} k\frac{(k+1)P_{k+1}}{\langle k \rangle}$$

$$= \frac{1}{\langle k \rangle}\sum_{k=1}^{\infty} k(k+1)P_{k+1}$$

$$= \frac{1}{\langle k \rangle}\sum_{k=1}^{\infty} \left((k+1)^2 - (k+1)\right)P_{k+1}$$

(where we have sneakily matched up indices)

$$= \frac{1}{\langle k \rangle}\sum_{j=0}^{\infty}(j^2 - j)P_j \quad \text{(using j = k+1)}$$

$$= \frac{1}{\langle k \rangle}\left(\langle k^2 \rangle - \langle k \rangle\right)$$

The edge-degree distribution:

- Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle}\left(\langle k^2 \rangle - \langle k \rangle\right)$, is true for all random networks, independent of degree distribution.
- For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle}\left(\langle k \rangle^2 + \langle k \rangle - \langle k \rangle\right) = \langle k \rangle$$

- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...

The edge-degree distribution:

- In fact, R_k is rather special for pure random networks ...
- Substituting

$$P_k = \frac{\langle k \rangle^k}{k!}e^{-\langle k \rangle}$$

into

$$R_k = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

we have

$$R_k = \frac{(k+1)}{\langle k \rangle}\frac{\langle k \rangle^{(k+1)}}{(k+1)!}e^{-\langle k \rangle} = \frac{\cancel{(k+1)}}{\cancel{\langle k \rangle}}\frac{\langle k \rangle^{(k+\cancel{1})}}{\cancel{(k+1)}k!}e^{-\langle k \rangle}$$

$$= \frac{\langle k \rangle^k}{k!}e^{-\langle k \rangle} \equiv P_k.$$

- #samesies.

Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle}\left(\langle k^2 \rangle - \langle k \rangle\right) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the 1st and 2nd moments of P_k and not just the 1st moment.
- Three peculiarities:
  1. We might guess $\langle k_2 \rangle = \langle k \rangle(\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
  2. If P_k has a large second moment, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
  3. Your friends really are different from you... [4, 6]
  4. See also: class size paradoxes (nod to: Gelman)

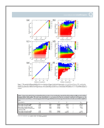
Two reasons why this matters

More on peculiarity #3:

- A node's average # of friends: $\langle k \rangle$
- Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle\left(1 + \frac{\sigma^2}{\langle k \rangle^2}\right) \geq \langle k \rangle$$

- So only if everyone has the same degree (variance= $\sigma^2 = 0$) can a node be the same as its friends.
- Intuition: for networks, the more connected a node, the more likely it is to be chosen as a friend.



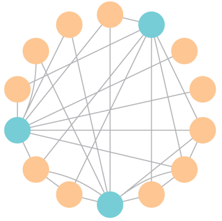
"Generalized friendship paradox in complex networks: The case of scientific collaboration"
Eom and Jo,
Nature Scientific Reports, **4**, 4603, 2014. [3]

Your friends really are ~~monsters~~ #winners:[1]

- Go on, hurt me: Friends have more coauthors, citations, and publications.
- Other horrific studies: your connections on Twitter have more followers than you, are happier than you [1], more sexual partners than you, ...
- The hope: Maybe they have more enemies and diseases too.
- Research possibility: The Frenemy Paradox.

[1] Some press here [MIT Tech Review].

Related disappointment:



- Nodes see their friends' color choices.
- Which color is more popular?[1]
- Again: thinking in edge space changes everything.

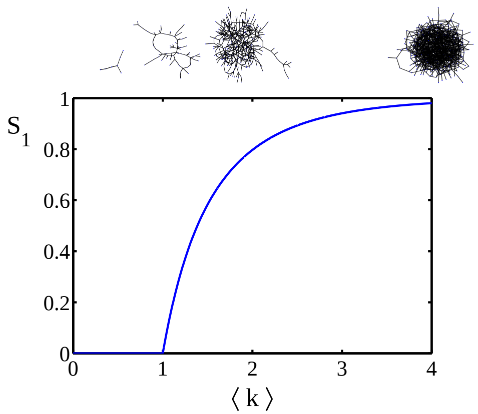
[1] https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/

Two reasons why this matters

(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that ∃ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- Defn: Giant component = component that comprises a non-zero fraction of a network as $N \to \infty$.
- Note: Component = Cluster

Giant component



Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition (or percolation condition):

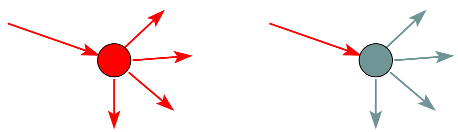
$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.
- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

Spreading on Random Networks

- For random networks, we know local structure is pure branching.
- Successful spreading is ∴ contingent on single edges infecting nodes.

Success Failure:



- Focus on binary case with edges and nodes either infected or not.
- First big question: for a given network and contagion process, can global spreading from a single seed occur?

Global spreading condition

- We need to find: [2]
  R = the average # of infected edges that one random infected edge brings about.
- Call **R** the gain ratio.
- Define B_{k1} as the probability that a node of degree k is infected by a single infected edge.
-

$$\mathbf{R} = \sum_{k=0}^{\infty} \underbrace{\frac{kP_k}{\langle k \rangle}}_{\text{prob. of connecting to a degree } k \text{ node}} \bullet \underbrace{(k-1)}_{\text{\# outgoing infected edges}} \bullet \underbrace{B_{k1}}_{\text{Prob. of infection}}$$

$$+ \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet \underbrace{0}_{\text{\# outgoing infected edges}} \bullet \underbrace{(1-B_{k1})}_{\text{Prob. of no infection}}$$

Global spreading condition

- Our global spreading condition is then:

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet B_{k1} > 1.$$

- Case 1–Rampant spreading: If $B_{k1} = 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) = \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- Good: This is just our giant component condition again.

Global spreading condition

- Case 2—Simple disease-like: If $B_{k1} = \beta < 1$ then

$$\mathbf{R} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet (k-1) \bullet \beta > 1.$$

- A fraction (1-β) of edges do not transmit infection.
- Analogous phase transition to giant component case but critical value of $\langle k \rangle$ is increased.
- Aka bond percolation.
- Resulting degree distribution $\tilde{P}_k$:

$$\tilde{P}_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$

Giant component for standard random networks:

- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Determine condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
- When $\langle k \rangle < 1$, all components are finite.
- Fine example of a continuous phase transition.
- We say $\langle k \rangle = 1$ marks the critical point of the system.

Random networks with skewed P_k:

- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \ge 1$, then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=1}^{\infty} x^{2-\gamma} \mathrm{d}x$$

$$\propto x^{3-\gamma}\Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- So giant component always exists for these kinds of networks.
- Cutoff scaling is k^{-3}: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.
- How about $P_k = \delta_{kk_0}$?

Giant component

And how big is the largest component?

- Define S_1 as the size of the largest component.
- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let's find S_1 with a back-of-the-envelope argument.
- Define δ as the probability that a randomly chosen node does not belong to the largest component.
- Simple connection: $\delta = 1 - S_1$.
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.
- So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

- Substitute in Poisson distribution...

Giant component

- Carrying on:

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k$$

$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.$$

- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

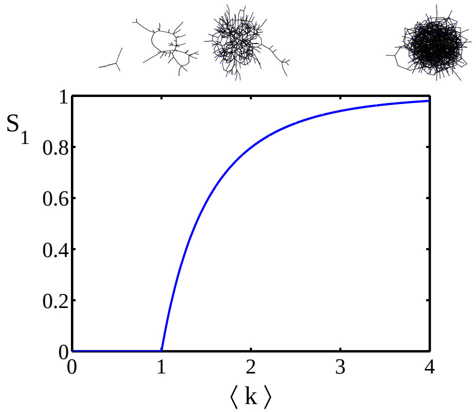
Giant component

- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1:

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation. [9]

Giant component



Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
- We need a separate probability δ' for the chance that an edge leads to the giant (infinite) component.
- We can sort many things out with sensible probabilistic arguments...
- More detailed investigations will profit from a spot of Generatingfunctionology. [10]
- CocoNuTs: We figure out the final size and complete dynamics.

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