

Mechanisms for Generating Power-Law Size Distributions, Part 1

Last updated: 2021/10/06, 20:26:04 EDT

Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021-2022 | @pocsvox

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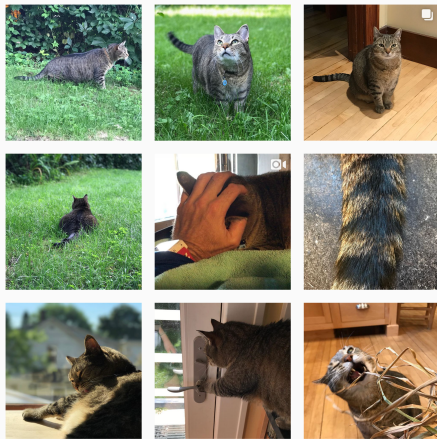
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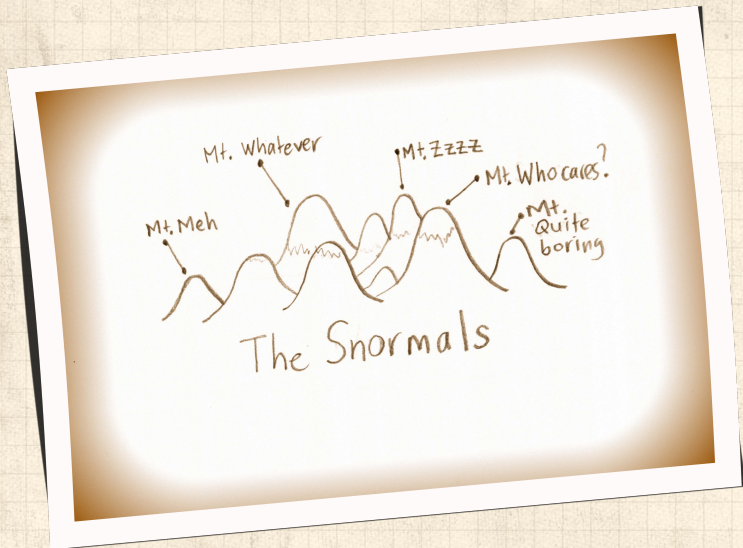
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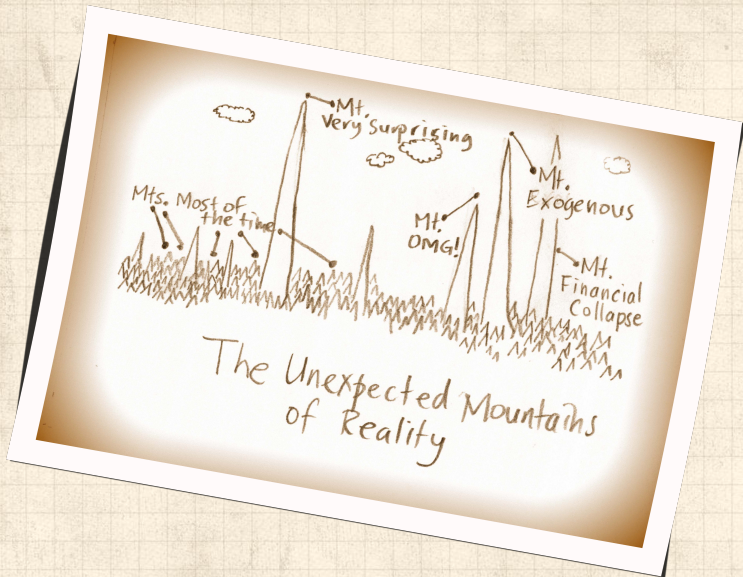
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Mechanisms:

A powerful story in the rise of complexity:

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Mechanisms:

A powerful story in the rise of complexity:

 structure arises out of randomness.

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Mechanisms:

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 Exhibit A: Random walks. 

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Mechanisms:

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 [Exhibit A: Random walks.](#) 

The essential random walk:

 One spatial dimension.

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 [Exhibit A: Random walks.](#) 

The essential random walk:

 One spatial dimension.

 Time and space are discrete

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Mechanisms:


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 **Exhibit A:** Random walks. 

The essential random walk:

 One spatial dimension.

 Time and space are discrete

 Random walker (e.g., a zombie texter ) starts at origin $x = 0$.

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Mechanisms:


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 structure arises out of randomness.


 **Exhibit A:** Random walks. 

The essential random walk:

 One spatial dimension.

 Time and space are discrete

 Random walker (e.g., a zombie texter ) starts at origin $x = 0$.

 Step at time t is ϵ_t :

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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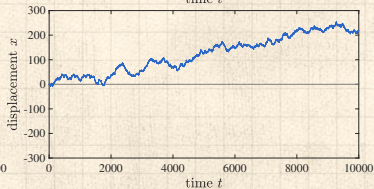
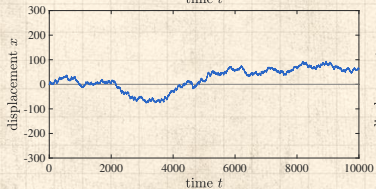
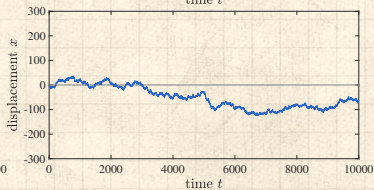
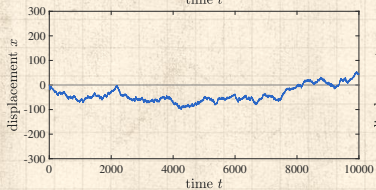
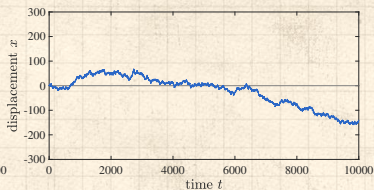
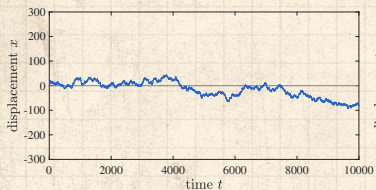
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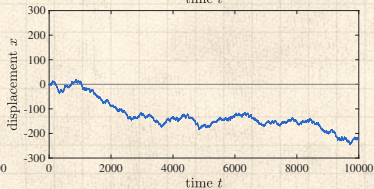
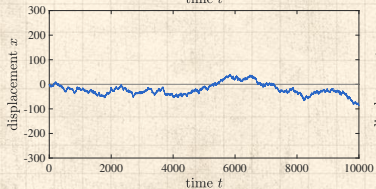
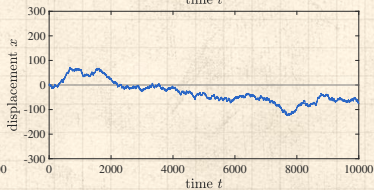
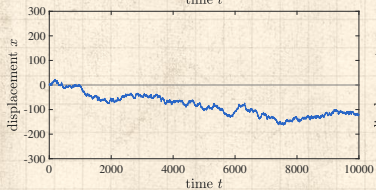
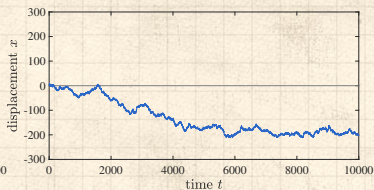
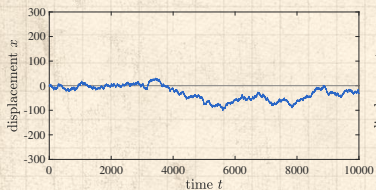
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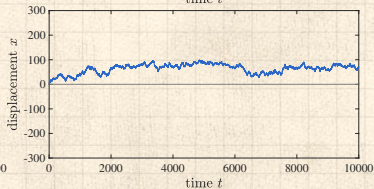
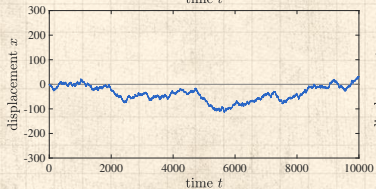
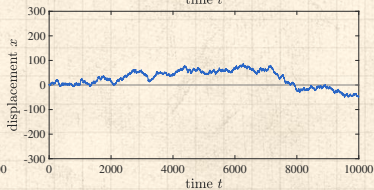
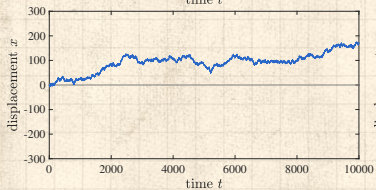
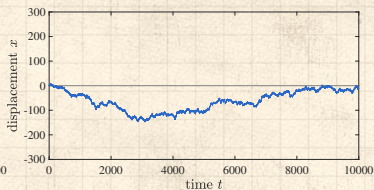
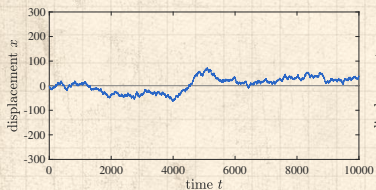
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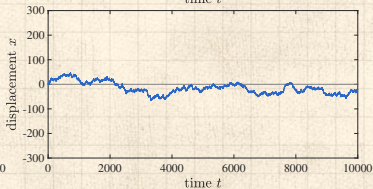
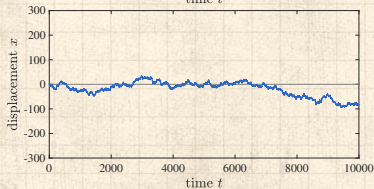
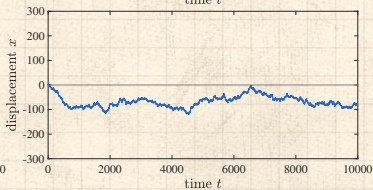
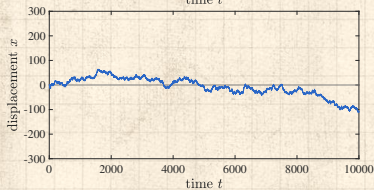
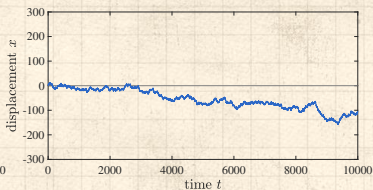
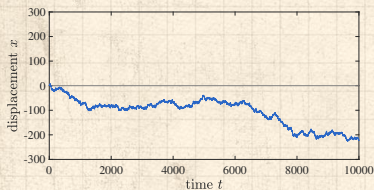
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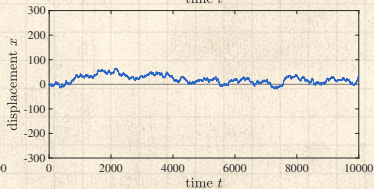
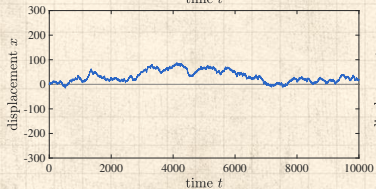
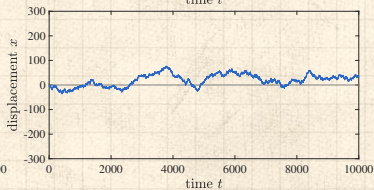
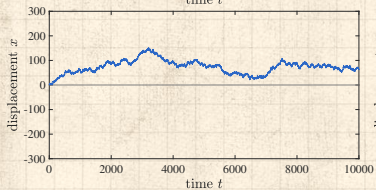
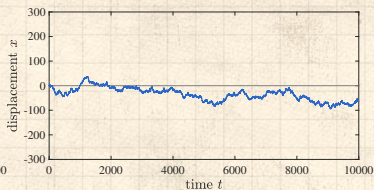
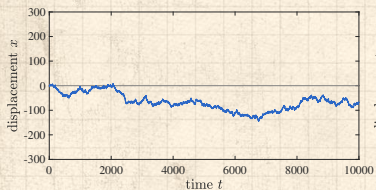
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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle$$

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Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle$$

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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$

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Random walks:

Displacement after t steps:

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Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$

 At any time step, we 'expect' our zombie texter to be back at their starting place.

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Displacement after t steps:

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Expected displacement:

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- At any time step, we 'expect' our zombie texter to be back at their starting place.
- Obviously fails for odd number of steps...

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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our zombie texter to be back at their starting place.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our texting undead friend lurching back to $x=0$ must diminish, right?

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Variations sum: *

$$\text{Var}(x_t) = \text{Var} \left(\sum_{i=1}^t \epsilon_i \right)$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

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Variances sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var} \left(\sum_{i=1}^t \epsilon_i \right) \\ &= \sum_{i=1}^t \text{Var}(\epsilon_i)\end{aligned}$$

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* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

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A non-trivial scaling law arises out of additive aggregation or accumulation.

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Great moments in Televised Random Walks:

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<http://www.youtube.com/watch?v=05gqx6eSy00?rel=0>

Plinko! from the Price is Right.

Also known as the bean machine, the quincunx (simulation), and the Galton box.



Random walk basics:

Counting random walks:

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Random walk basics:

Counting random walks:

 Each **specific** random walk of length t appears with a chance $1/2^t$.

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Random walk basics:

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- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.

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- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.

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- Random walk must displace by $+(j - i)$ after t steps.

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- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 5 

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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How does $P(x_t)$ behave for large t ?

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How does $P(x_t)$ behave for large t ?



Take time $t = 2n$ to help ourselves.

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How does $P(x_t)$ behave for large t ?

Take time $t = 2n$ to help ourselves.

$x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$

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How does $P(x_t)$ behave for large t ?

 Take time $t = 2n$ to help ourselves.

 $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$

 x_{2n} is even so set $x_{2n} = 2k$.

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How does $P(x_t)$ behave for large t ?

Take time $t = 2n$ to help ourselves.

$x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$


x_{2n} is even so set $x_{2n} = 2k$.


Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have


$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$




How does $P(x_t)$ behave for large t ?


 Take time $t = 2n$ to help ourselves.

 $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$


 x_{2n} is even so set $x_{2n} = 2k$.

 Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have

$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$


 For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:


$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$


[Insert question from assignment 5](#) 




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[Insert question from assignment 5](#) 

 The whole is different from the parts. **#nutritious**



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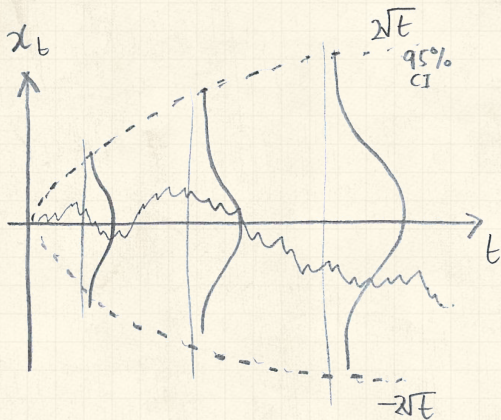
[Insert question from assignment 5](#)


The whole is different from the parts. #nutritious

See also: [Stable Distributions](#)



Universality is also not left-handed:



This is Diffusion : the most essential kind of spreading (more later).



View as Random Additive Growth Mechanism.

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So many things are connected:

Pascal's Triangle



Could have been the Pyramid of Pingala ¹ or the Triangle of Khayyam, Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.

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¹Stigler's Law of Eponymy  showing excellent form again.

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Counting encoded in algebraic forms (and much more).

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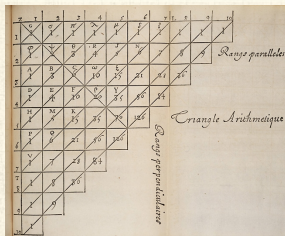
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$(h + t)^n = \sum_{k=0}^n \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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$(h + t)^3 = hhh + hht + hth + thh + htt + tht + tth + ttt$

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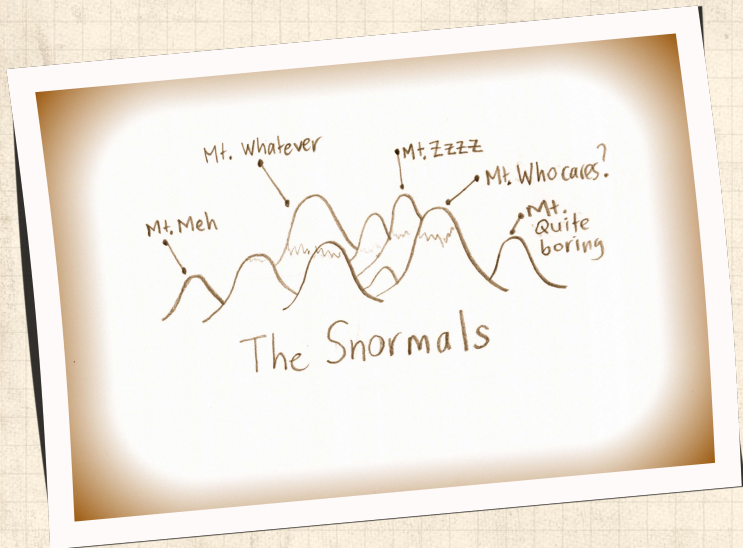
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Random walks are even weirder than you might think...

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 Think of a coin flip game with ten thousand tosses.



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☇ The most likely number of lead changes is...



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- ☰ In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$



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 Even crazier:

The expected time between tied scores = ∞



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Random walks are even weirder than you might think...

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The expected time between tied scores = ∞

See Feller, Intro to Probability Theory, Volume I [5]



Applied knot theory:



“Designing tie knots by random walks” 

Fink and Mao,
Nature, **398**, 31–32, 1999. [6]

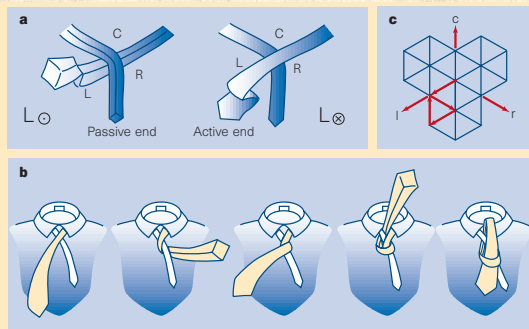


Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie.
a. The two ways of beginning a knot, L_{\ominus} and L_{\otimes} . For knots beginning with L_{\ominus} , the tie must begin inside-out. **b.** The four-in-hand, denoted by the sequence $L_{\otimes} R_{\otimes} L_{\otimes} C_{\otimes} T_{\otimes}$. **c.** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\uparrow \uparrow \uparrow \downarrow$.

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Applied knot theory:

Table 1 **Aesthetic tie knots**

h	γ	γ/h	$K(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		$L_0 R_0 C_0 T$
4	1	0.25	1	-1	1	Four-in-hand	$L_0 R_0 L_0 C_0 T$
5	2	0.40	2	-1	0	Pratt knot	$L_0 C_0 R_0 L_0 C_0 T$
6	2	0.33	4	0	0	Half-Windsor	$L_0 R_0 C_0 L_0 R_0 C_0 T$
7	2	0.29	6	-1	1		$L_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
7	3	0.43	4	0	1		$L_0 C_0 R_0 C_0 L_0 R_0 C_0 T$
8	2	0.25	8	0	2		$L_0 R_0 L_0 C_0 R_0 L_0 R_0 C_0 T$
8	3	0.38	12	-1	0	Windsor	$L_0 C_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
9	3	0.33	24	0	0		$L_0 R_0 C_0 L_0 R_0 C_0 L_0 R_0 C_0 T$
9	4	0.44	8	-1	2		$L_0 C_0 R_0 C_0 L_0 C_0 R_0 L_0 C_0 T$

Knots are characterized by half-winding number h , centre number γ , centre fraction γ/h , knots per class $K(h, \gamma)$, symmetry s , balance b , name and sequence.

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 h = number of moves

 γ = number of center moves

 $K(h, \gamma) = 2^{\gamma-1} \binom{h-\gamma-2}{\gamma-1}$

 $s = \sum_{i=1}^h x_i$ where $x = -1$ for L and $+1$ for R .

 $b = \frac{1}{2} \sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.



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The problem of first return:

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The problem of first return:

 What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?

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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
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- Some physical structures may result from random walks.



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The problem of first return:

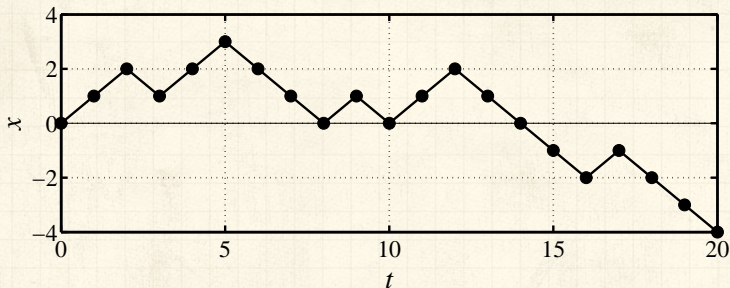
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- Will our zombie texter always return to the origin?
- What about higher dimensions?

Reasons for caring:

- We will find a power-law size distribution with an interesting exponent.
- Some physical structures may result from random walks.
- We'll start to see how different scalings relate to each other.



For random walks in 1-d:



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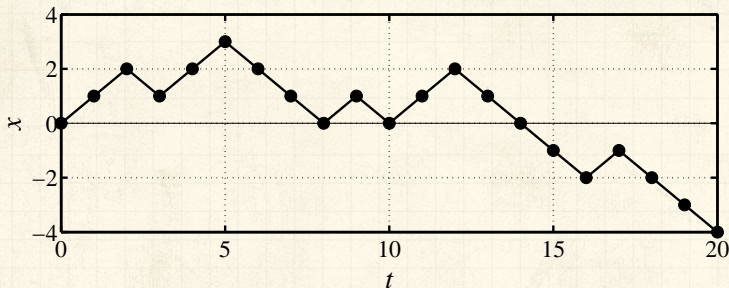
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For random walks in 1-d:



A **return** to origin can only happen when $t = 2n$.

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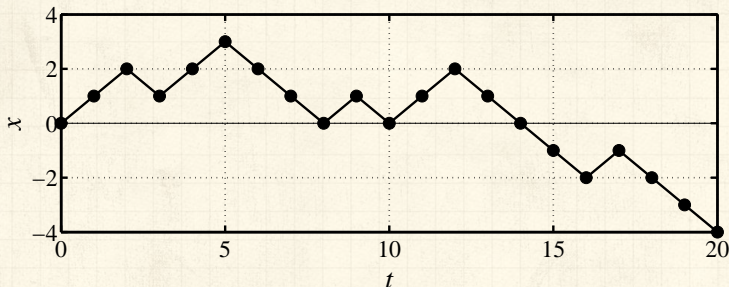
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For random walks in 1-d:



 A **return** to origin can only happen when $t = 2n$.

 In example above, returns occur at $t = 8, 10,$ and 14 .

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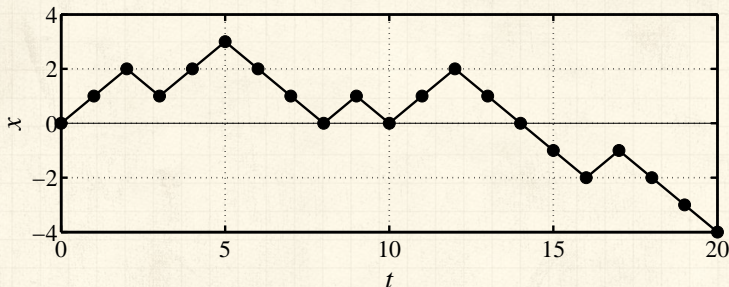
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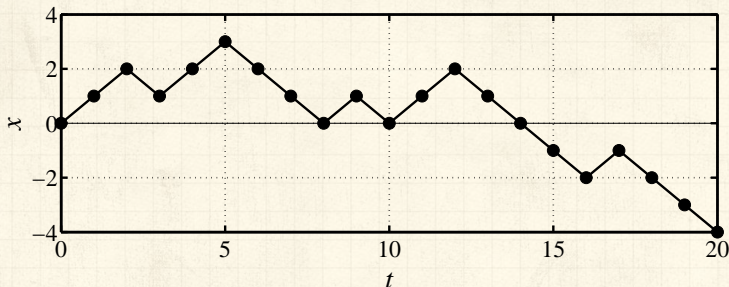
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🧱 In example above, returns occur at $t = 8, 10,$ and 14 .

🧱 Call $P_{\text{fr}(2n)}$ the probability of **first return** at $t = 2n$.



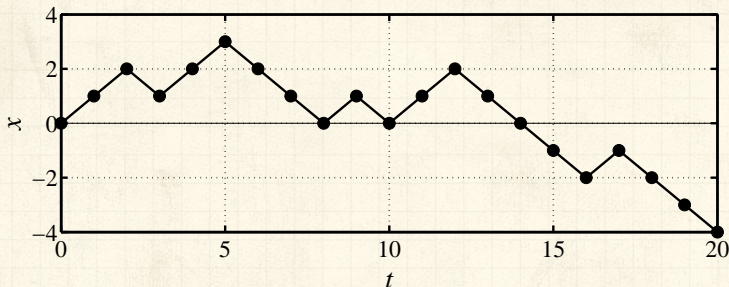
For random walks in 1-d:





-  A **return** to origin can only happen when $t = 2n$.
-  In example above, returns occur at $t = 8, 10,$ and 14 .
-  Call $P_{fr(2n)}$ the probability of **first return** at $t = 2n$.
-  Probability calculation \equiv Counting problem (combinatorics/statistical mechanics).





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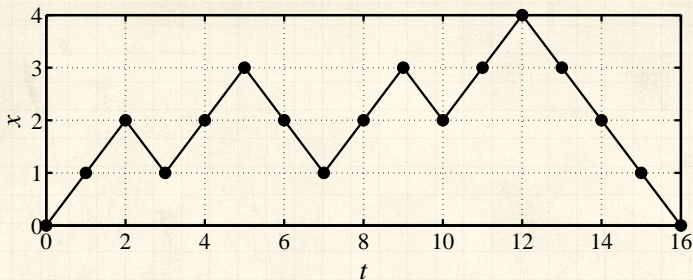
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 Call $P_{\text{fr}(2n)}$ the probability of **first return** at $t = 2n$.

 Probability calculation \equiv Counting problem (combinatorics/statistical mechanics).

 **Idea:** Transform first return problem into an easier return problem.





Can assume zombie texter first lurches to $x = 1$.

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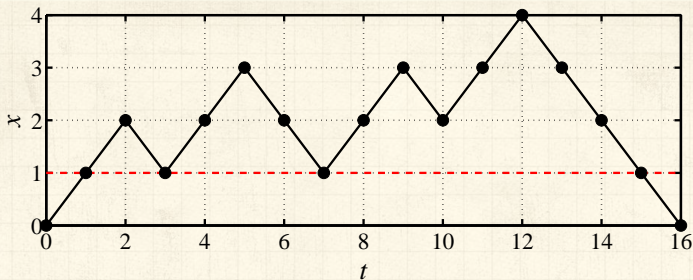
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- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).

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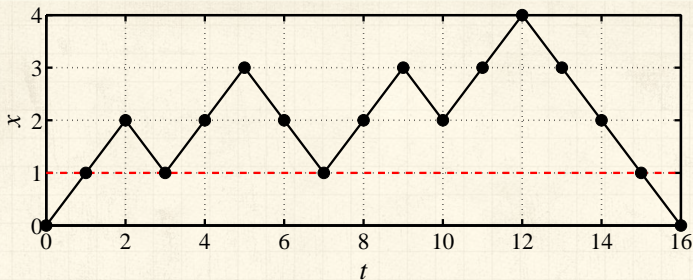
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-  Can assume zombie texter first lurches to $x = 1$.
-  Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
-  Now want walks that can return many times to $x = 1$.

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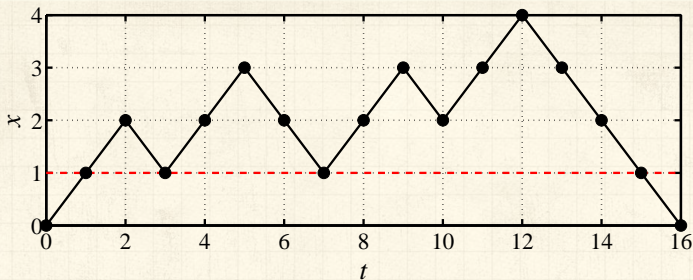
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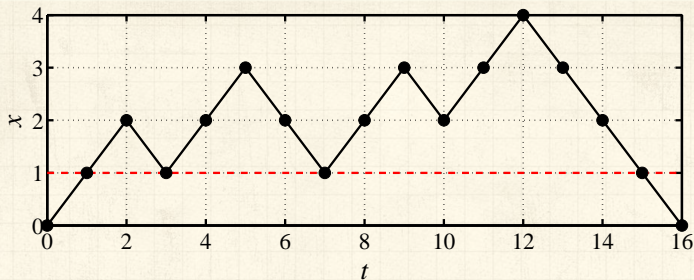
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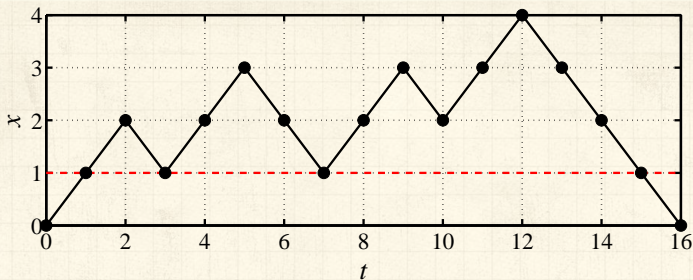
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- $P_{fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$





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- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for texters that first lurch to $x = -1$.



Counting first returns:

Approach:

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Counting first returns:

Approach:

 Move to counting numbers of walks.

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Counting first returns:

Approach:

-  Move to counting numbers of walks.
-  Return to probability at end.

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Counting first returns:

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- Move to counting numbers of walks.
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- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea:** If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.



Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
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- Call walks that drop below $x = 1$ **excluded walks**.



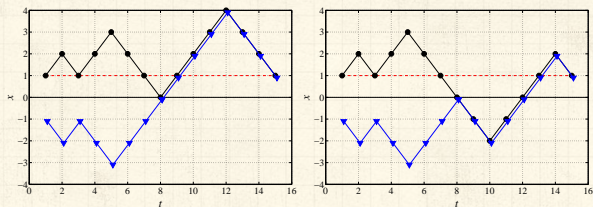
Counting first returns:

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- Call walks that drop below $x = 1$ **excluded walks**.
- We'll use a method of images to identify these excluded walks.



Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.

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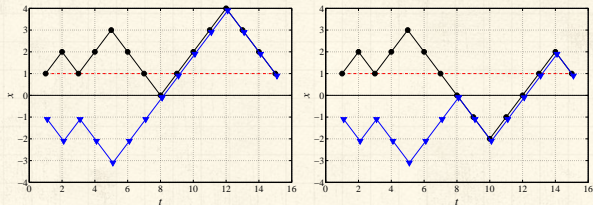
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Examples of excluded walks:



Key observation for excluded walks:

-  For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
-  Matching path first mirrors and then tracks after first reaching $x=0$.

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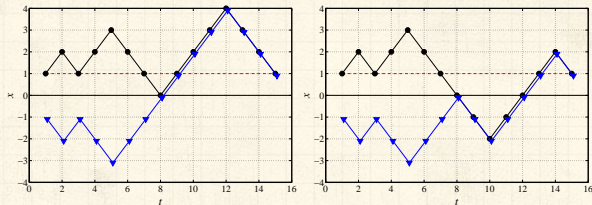
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Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0 , there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once

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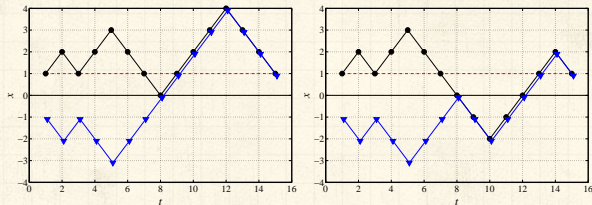
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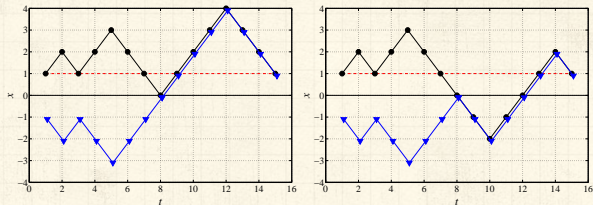
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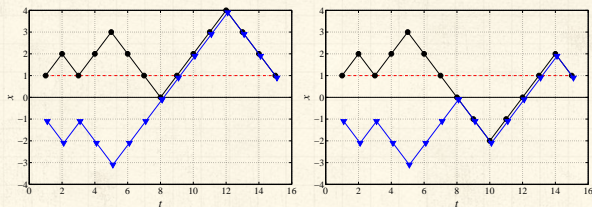
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- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once
= # of t -step paths starting at $x=-1$ and ending at $x=1$ = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$

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Probability of first return:

Insert question from assignment 5  :

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Probability of first return:

Insert question from assignment 5  :

 Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$

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Probability of first return:

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$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}$$

 Normalized number of paths gives probability.

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 Total number of possible paths = 2^{2n} .

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$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

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$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}$$

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$$\begin{aligned} P_{\text{fr}}(2n) &= \frac{1}{2^{2n}} N_{\text{fr}}(2n) \\ &\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}} \\ &= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \end{aligned}$$

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 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

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 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

 Same scaling holds for continuous space/time walks.

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 $P(t)$ is normalizable.

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🧱 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

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🧱 But mean, variance, and all higher moments are infinite. #totalmadness

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Recurrence: Random walker always returns to origin



But mean, variance, and all higher moments are infinite. **#totalmadness**



Even though walker must return, expect a long wait...



One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.



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Higher dimensions ↗:



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🧱 Walker in $d = 2$ dimensions must also return



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🧱 Walker may not return in $d \geq 3$ dimensions



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🧱 Associated human ~~genius~~: George Pólya 



Random walks

On finite spaces:

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Random walks

On finite spaces:

 In any finite homogeneous space, a random walker will visit every site with equal probability

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Random walks

On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system

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On networks:

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On networks:

-  On networks, a random walker visits each node with frequency \propto node degree **#groovy**



Random walks

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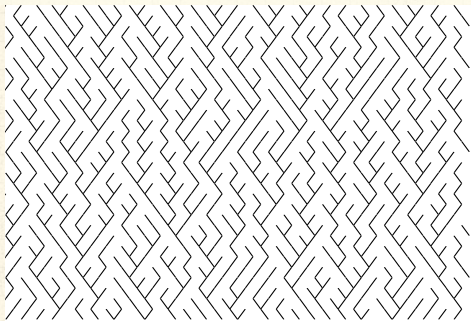
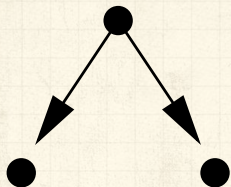
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On networks:

- 🧱 On networks, a random walker visits each node with frequency \propto node degree **#groovy**
- 🧱 Equal probability still present: walkers traverse **edges** with equal frequency. **#totallygroovy**



Scheidegger Networks [17, 4]



-  Random directed network on triangular lattice.
-  Toy model of real networks.
-  'Flow' is southeast or southwest with equal probability.

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Scheidegger networks



Creates basins with random walk boundaries.

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Scheidegger networks

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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Scheidegger networks

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- Basin termination = first return random walk problem.



Scheidegger networks


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- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$





Scheidegger networks


 Creates basins with random walk boundaries.


 **Observe** that subtracting one random walk from another gives random walk with increments:

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 Random walk with probabilistic pauses.

 Basin termination = first return random walk problem.

 Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$

 For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.



Connections between exponents:



For a basin of length l , width $\propto l^{1/2}$

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Connections between exponents:

 For a basin of length l , width $\propto l^{1/2}$

 Basin area $a \propto l \cdot l^{1/2} = l^{3/2}$

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 $dl \propto d(a^{2/3}) = 2/3a^{-1/3}da$





Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

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 Invert: $\ell \propto a^{2/3}$


 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$


 **Pr**(basin area = a) da
= **Pr**(basin length = ℓ) $d\ell$





Connections between exponents:

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 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$





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Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

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 **Pr**(basin area = a) da
= **Pr**(basin length = ℓ) $d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
= $a^{-4/3} da$
= $a^{-\tau} da$



Connections between exponents:

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Connections between exponents:



Both basin area and length obey power law distributions

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Connections between exponents:

-  Both basin area and length obey power law distributions
-  Observed for real river networks

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Connections between exponents:

-  Both basin area and length obey power law distributions
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- Plan: Redo calc with γ , τ , and h .



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$$\tau = 1 + h(\gamma - 1)$$

 Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.



Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to:^[3]

$$\tau = 2 - h$$

and

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-  Only one exponent is independent (take h).
-  Simplifies system description.
-  Expect Scaling Relations where power laws are found.
-  Need only characterize Universality  class with independent exponents.



Death ...

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Failure:


 A very simple model of failure/death

 x_t = entity's 'health' at time t

 Start with $x_0 > 0$.

 Entity fails when x hits 0.





"Explaining mortality rate plateaus" 

Weitz and Fraser,
Proc. Natl. Acad. Sci., **98**, 15383–15386,
2001. [18]





... and the NBA:


Basketball and other sports ^[2]:

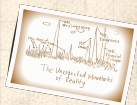
 Three arcsine laws  (Lévy ^[12]) for continuous-time random walk last time T :

$$\frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}.$$

The arcsine distribution  applies for:
(1) fraction of time positive, (2) the last time the walk changes sign,
and (3) the time the maximum is achieved.

 Well approximated by basketball score lines ^[8, 2].

 Australian Rules Football has some differences ^[11].



More than randomness



Can generalize to Fractional Random Walks ^[15, 16, 14]

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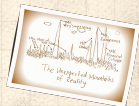
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Fractional Brownian Motion ↗, Lévy flights ↗

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"On $1/f$ noise and other distributions with long tails."

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 In 1-d, standard deviation σ scales as

$$\sigma \sim t^\alpha$$

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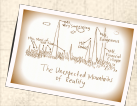
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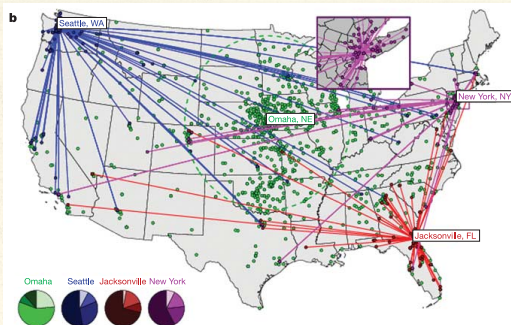
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 Extensive memory of path now matters...





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🧱 First big studies of movement and interactions of people.

🧱 Brockmann *et al.* ^[1] “Where’s George” study.

🧱 Beyond Lévy: Superdiffusive in space but with long waiting times.

🧱 Tracking movement via cell phones ^[9] and Twitter ^[7].



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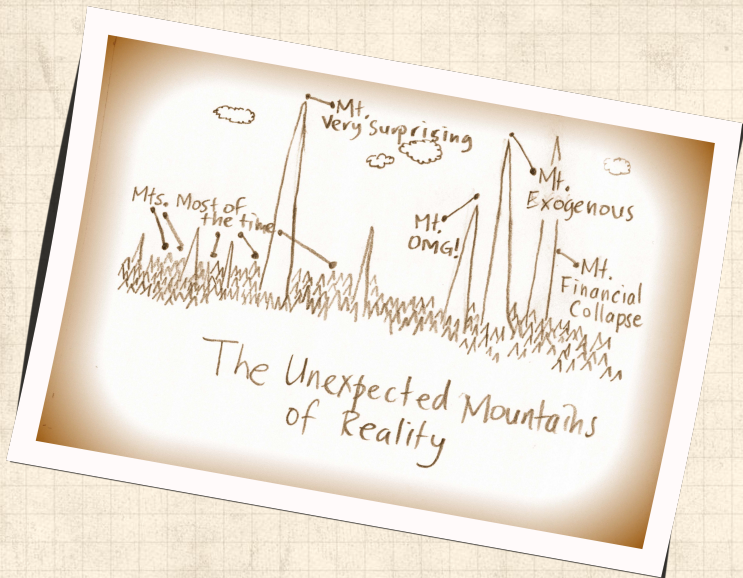
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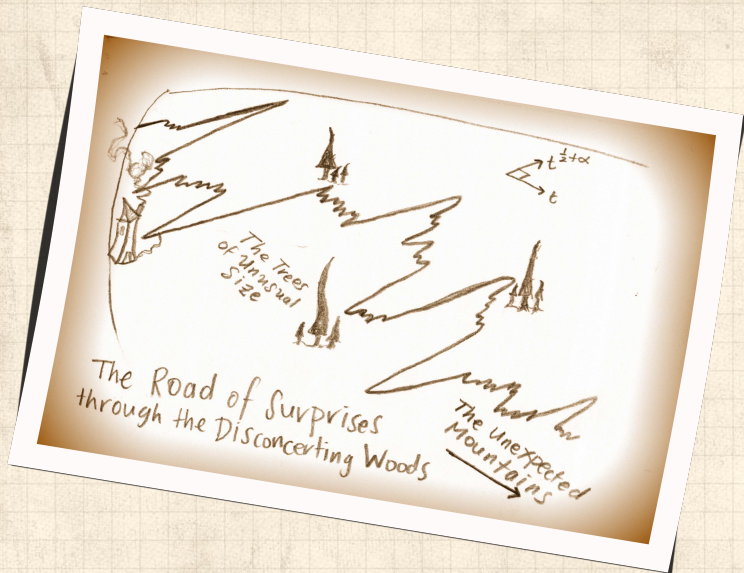
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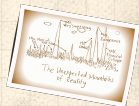
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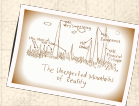
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