Random walks and diffusion on networks

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Outline

Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node j and take time to be discrete.
- Q: What's the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step t, our walker is at node i.
- We want to characterize the evolution of $\vec{p}(t)$.
- First task: connect $\vec{p}(t+1)$ to $\vec{p}(t)$.
- Let's call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is texting.

Where is Barry?

- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix A where

  $a_{ij} = 1$ if i has an edge leading to j,
  $a_{ij} = 0$ otherwise.

- Barry is at node j at time t with probability $p_j(t)$.
- In the next time step, he randomly lurches toward one of j's neighbors.
- Barry arrives at node i from node j with probability $\frac{1}{k_j}$ if an edge connects j to i.
- Equation-wise:

$$p_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} p_j(t).$$

  where k_j is j's degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$.

Inebriation and diffusion:

- Excellent observation: The same equation applies for stuff moving around a network, such that at each time step all material at node i is sent to its neighbors.
- $x_i(t)$ = amount of stuff at node i at time t.
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$$x_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} x_j(t).$$

- Random walking is equivalent to diffusion.

Where is Barry?

- Linear algebra-based excitement: $p_i(t+1) = \sum_{j=1}^{n} a_{ji} \frac{1}{k_j} p_j(t)$ is more usefully viewed as

$$\vec{p}(t+1) = A^{\mathsf{T}} K^{-1} \vec{p}(t)$$

  where $[K_{ij}] = [\delta_{ij} k_i]$ has node degrees on the main diagonal and zeros everywhere else.
- So... we need to find the dominant eigenvalue of $A^{\mathsf{T}} K^{-1}$.
- Expect this eigenvalue will be 1 (doesn't make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.

Where is Barry?

- By inspection, we see that

$$\vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}$$

  satisfies $\vec{p}(\infty) = A^{\mathsf{T}} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.
- We will find Barry at node i with probability proportional to its degree k_i.
- Beautiful implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don't see that.

Other pieces:

- Goodness: $A^{\mathsf{T}} K^{-1}$ is similar to a real symmetric matrix if $A = A^{\mathsf{T}}$.
- Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2} A^{\mathsf{T}} K^{-1} K^{1/2} = K^{-1/2} A^{\mathsf{T}} K^{-1/2}.$$

- Since $A^{\mathsf{T}} = A$, we have

$$(K^{-1/2} A K^{-1/2})^{\mathsf{T}} = K^{-1/2} A K^{-1/2}.$$

- Upshot: $A^{\mathsf{T}} K^{-1} = A K^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.