Power-Law Mechanisms: Variable Transformation

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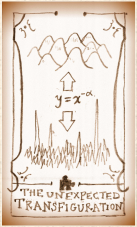
Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number, 2025–2026

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



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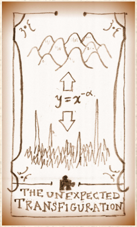

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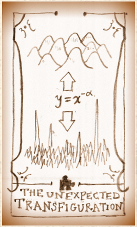
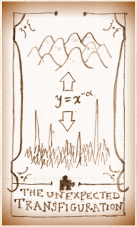
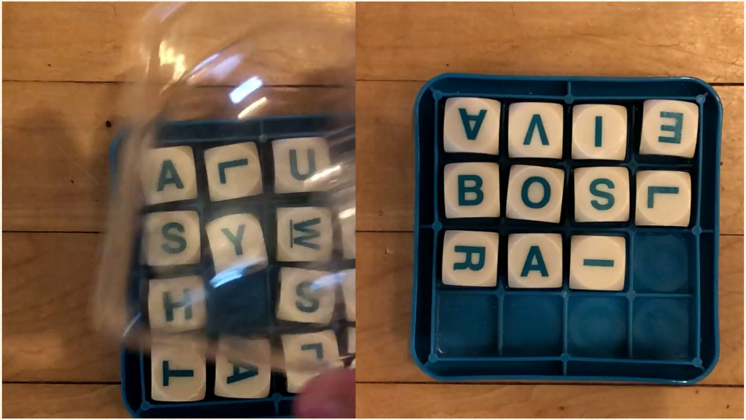
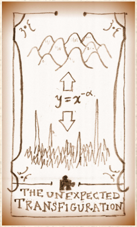


Outline

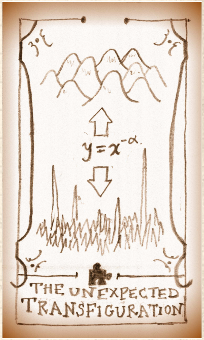


The Boggoracle Speaks:

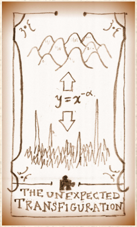


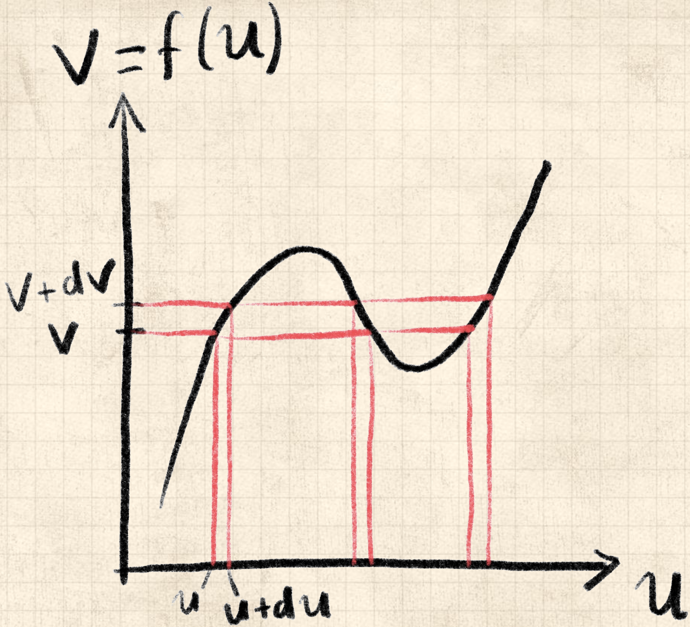


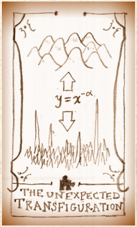

THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS


$y = x^{-\alpha}$
THE UNEXPECTED TRANSFIGURATION

Outline




v = f(u)
v+dv
v
u
u+du
u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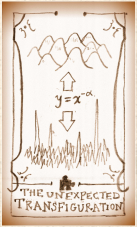
Variable Transformation

Understand power laws as arising from:



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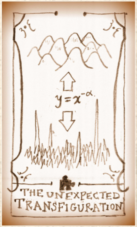

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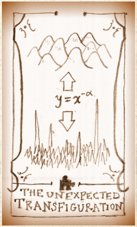

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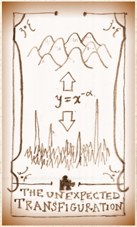

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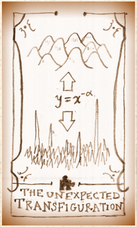

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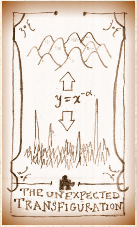
- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$



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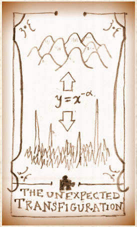
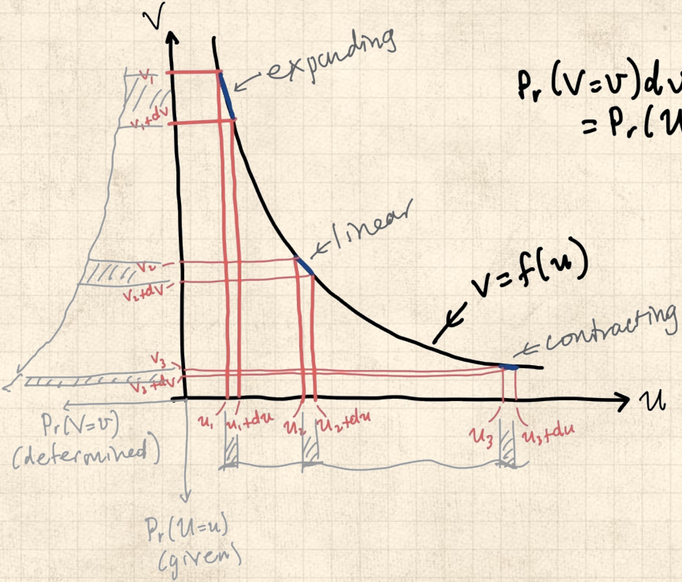
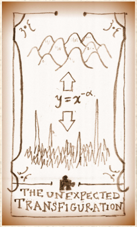
- Often easier to do by hand...

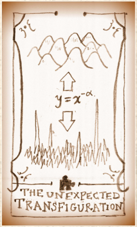


$$\Pr(V=v)\,dv = \Pr(U=u)\,du$$



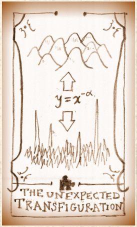


Example



Example

- Assume relationship between x and y is 1-1.



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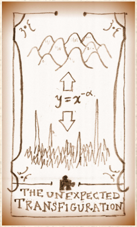
- Power-law relationship between variables: $y = cx^{-\alpha}, \alpha > 0$



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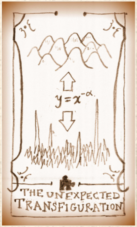

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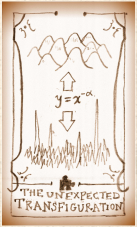

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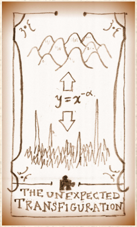
$$
\begin{aligned}
\mathrm{d}y &= \mathrm{d}\left(cx^{-\alpha}\right) \\
&= c(-\alpha)x^{-\alpha-1}\mathrm{d}x
\end{aligned}
$$



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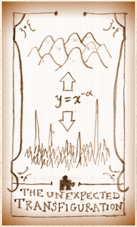

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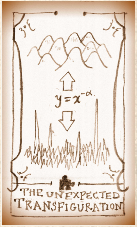

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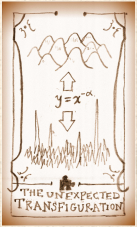
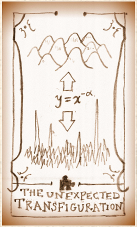


Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$



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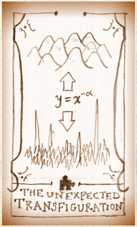

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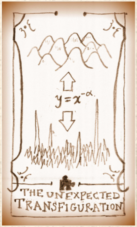
- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$



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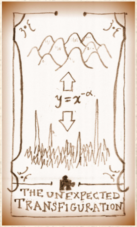
- If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

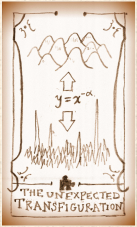


Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$



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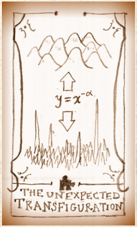

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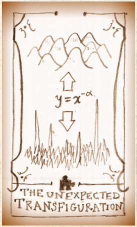
- More later when we cover robustness.



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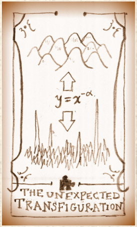

Gravity

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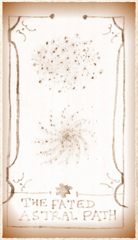

[1] Stigler's Law of Eponymy.

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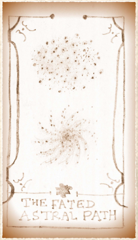
- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.



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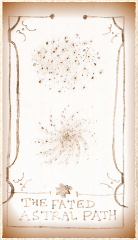

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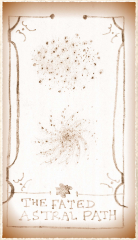

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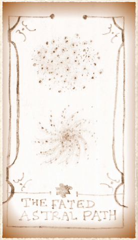
- Again, the humans naming things after humans, poorly.[1]



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- Law of gravity:

$$F \propto r^{-2}$$

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- Assume stars are distributed randomly in space (oops?)
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Matter is concentrated in stars: [2]

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- Connect differentials: $\mathrm{d}r \propto \mathrm{d}F^{-\frac{1}{2}} \propto F^{-\frac{3}{2}}\mathrm{d}F$

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- Mean is finite.
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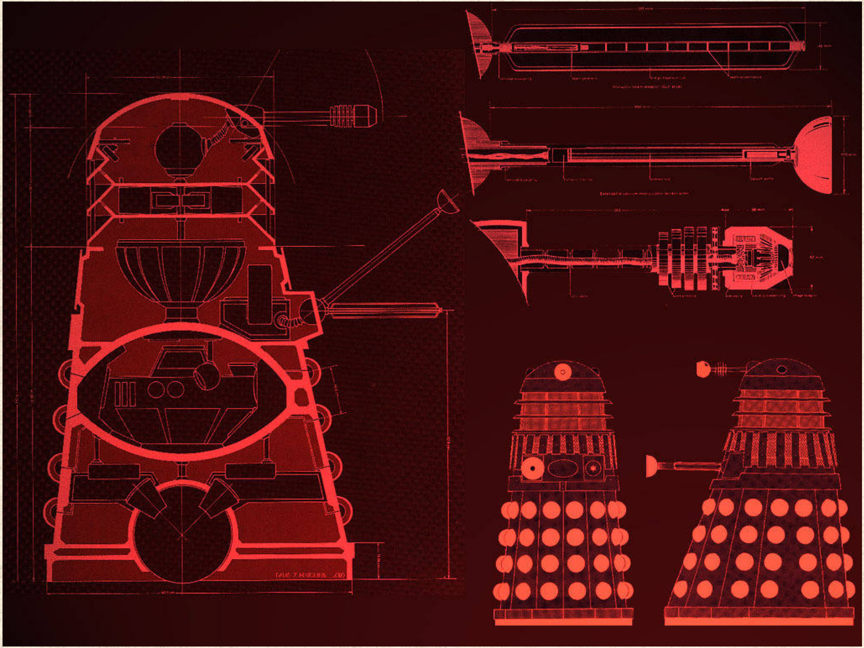
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

☐ Todo: Build Dalek army.



Outline

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- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

References I

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