

Power-Law Mechanisms: Variable Transformation

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Principles of Complex Systems, Vols. 1, 2, & 3D
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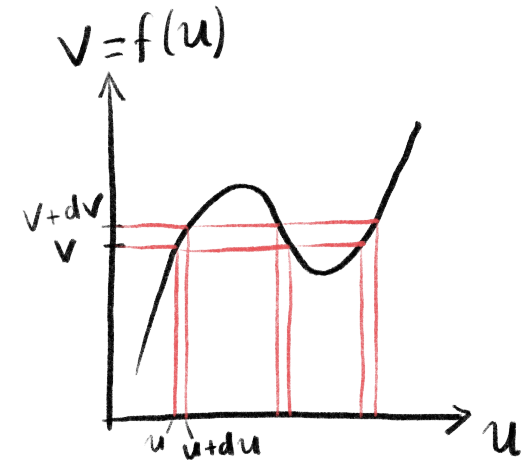
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Outline

Variable transformation
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Variable Transformation

Understand power laws as arising from:

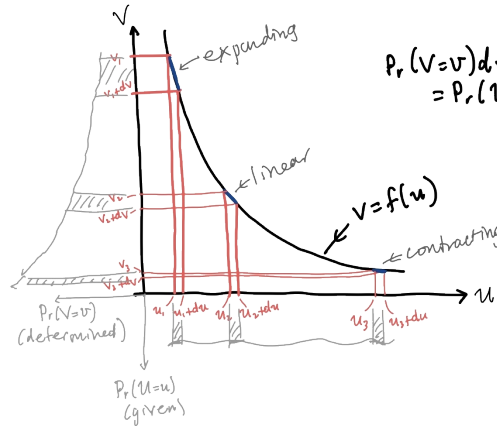
1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

$$\begin{aligned} P_Y(y)dy &= \sum_{x|f(x)=y} P_X(x)dx \\ &= \sum_{y|f(x)=y} P_X(f^{-1}(y)) \left| \frac{dy}{f'(f^{-1}(y))} \right| \end{aligned}$$

Often easier to do by hand...

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Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables:
 $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha} x^{\alpha+1} dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

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Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\left(\frac{y}{c} \right)^{-1/\alpha} \right) \frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

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Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

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Gravity

- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtmark who was thinking about electrostatics and plasma ^[1].
- Again, the humans naming things after humans, poorly.¹



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¹Stigler's Law of Eponymy 

Matter is concentrated in stars: ^[2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

- Connect differentials: $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$

Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2} dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2}) F^{-3/2} dF$$



$$\propto (F^{-1/2})^2 F^{-3/2} dF$$



$$= F^{-1-3/2} dF$$



$$= F^{-5/2} dF.$$

Gravity:

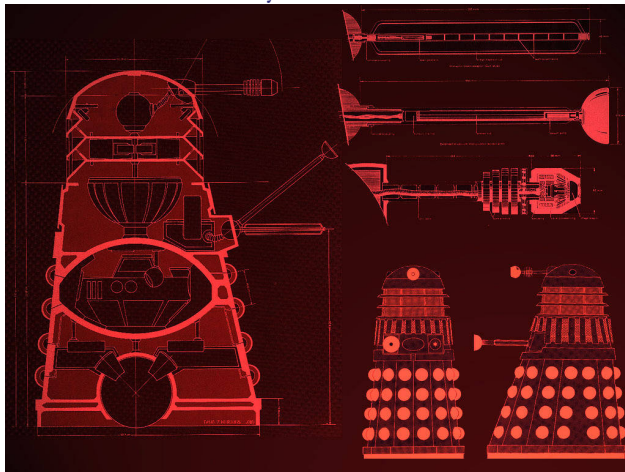


$$P_F(F) = F^{-5/2} dF$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance = ∞ .
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

□ Todo: Build Dalek army.



Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument [↗](#)...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

References I

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Über die verbreiterung von spektrallinien.
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- D. Sornette.
[Critical Phenomena in Natural Sciences.](#)
Springer-Verlag, Berlin, 1st edition, 2003.

