

Scale-free networks

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number,
2023–2024 | @pocsvox

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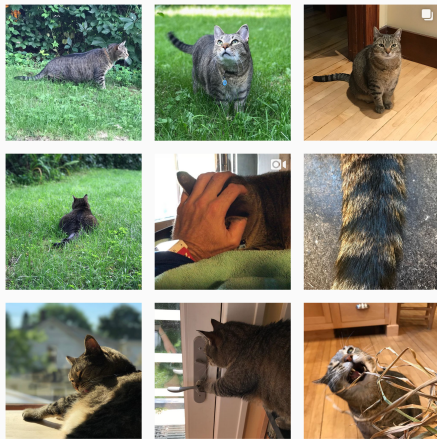
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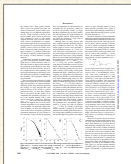
Scale-free networks


 Real networks with power-law degree distributions became known as **scale-free** networks.

 Scale-free refers specifically to the **degree distribution** having a **power-law decay** in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$


 One of the seminal works in complex networks:



"Emergence of scaling in random networks" 

Barabási and Albert,
Science, **286**, 509–511, 1999. [2]

Times cited: ~ 43,853  (as of May 19, 2023)

 Somewhat misleading nomenclature ...

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Scale-free networks

- Scale-free networks are **not fractal** in any sense.
- Usually talking about networks whose links are **abstract, relational, informational, ...**(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

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Some real data (we are feeling brave):

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From Barabási and Albert's original paper [2]:

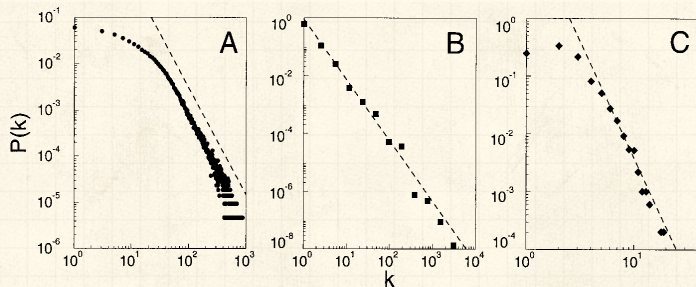


Fig. 1. The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes **(A)** $\gamma_{\text{actor}} = 2.3$, **(B)** $\gamma_{\text{www}} = 2.1$ and **(C)** $\gamma_{\text{power}} = 4$.



Random networks: largest components

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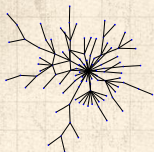
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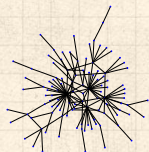
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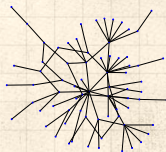
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$$\gamma = 2.5$$
$$\langle k \rangle = 1.8$$



$$\gamma = 2.5$$
$$\langle k \rangle = 2.05333$$



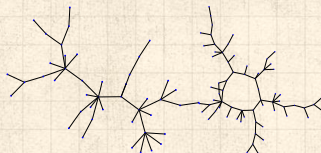
$$\gamma = 2.5$$
$$\langle k \rangle = 1.66667$$



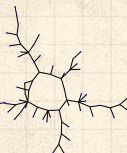
$$\gamma = 2.5$$
$$\langle k \rangle = 1.92$$



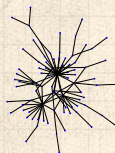
$$\gamma = 2.5$$
$$\langle k \rangle = 1.6$$



$$\gamma = 2.5$$
$$\langle k \rangle = 1.50667$$



$$\gamma = 2.5$$
$$\langle k \rangle = 1.62667$$



$$\gamma = 2.5$$
$$\langle k \rangle = 1.8$$



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The big deal:

 We move beyond describing networks to finding **mechanisms** for why certain networks are the way they are.


A big deal for scale-free networks:


 How does the exponent γ depend on the mechanism?

 Do the mechanism details matter?





BA model

 Barabási-Albert model = BA model.


 Key ingredients:


Growth and **Preferential Attachment (PA)**.

 **Step 1:** start with m_0 disconnected nodes.

 **Step 2:**

1. **Growth**—a new node appears at each time step $t = 0, 1, 2, \dots$
2. Each new node makes m links to nodes already present.
3. **Preferential attachment**—Probability of connecting to i th node is $\propto k_i$.

 In essence, we have a **rich-gets-richer** scheme.

 Yes, we've seen this all before in Simon's model.

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BA model

 **Definition:** A_k is the attachment kernel for a node with degree k .

 For the original model:

$$A_k = k$$

 **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.


 For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} k N_k(t)}$$




where $N(t) = m_0 + t$ is # nodes at time t
and $N_k(t)$ is # degree k nodes at time t .



Approximate analysis

-  When $(N + 1)$ th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

-  Assumes probability of being connected to is **small**.
-  Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
-  Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

$$\frac{d}{dt}k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.



Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

Rearrange and solve:

$$\frac{dk_i(t)}{k_i(t)} = \frac{dt}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}$$

Next find c_i ...



Know i th node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$


All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which **flattens out** growth curve.

First-mover advantage: Early nodes do **best**.


Clearly, a Ponzi scheme .



We are already at the Zipf distribution:


 Degree of node i is the size of the i th ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$


 From before:

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

so $t_{i,\text{start}} \sim i$ which is the rank.

 We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}.$$

 Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\gamma = 1 + 1/(1/2) = 3.$$

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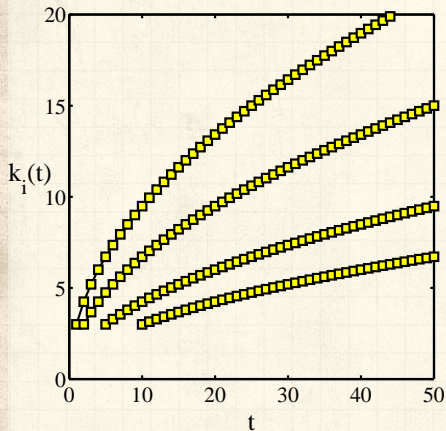
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 $m = 3$

 $t_{i,\text{start}} =$
1, 2, 5, and 10.



Degree distribution

So what's the **degree distribution** at time t ?

Use fact that birth time for added nodes is distributed uniformly between time 0 and t :

$$\Pr(t_{i,\text{start}})dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t}$$

Also use

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—Jacobian:

$$\frac{dt_{i,\text{start}}}{dk_i} = -2 \frac{m^2 t}{k_i(t)^3}.$$



Degree distribution



$$\Pr(k_i)dk_i = \Pr(t_{i,\text{start}})dt_{i,\text{start}}$$



$$= \Pr(t_{i,\text{start}})dk_i \left| \frac{dt_{i,\text{start}}}{dk_i} \right|$$



$$= \frac{1}{t} dk_i 2 \frac{m^2 t}{k_i(t)^3}$$









$$= 2 \frac{m^2}{k_i(t)^3} dk_i$$



$$\propto k_i^{-3} dk_i.$$



Degree distribution

-  We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
-  Typical for real networks: $2 < \gamma < 3$.
-  Range true more generally for events with size distributions that have power-law tails.
-  $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
-  In practice, $\gamma < 3$ means variance is governed by upper cutoff.
-  $\gamma > 3$: finite mean and variance (mild)



Back to that real data:

From Barabási and Albert's original paper [2]:

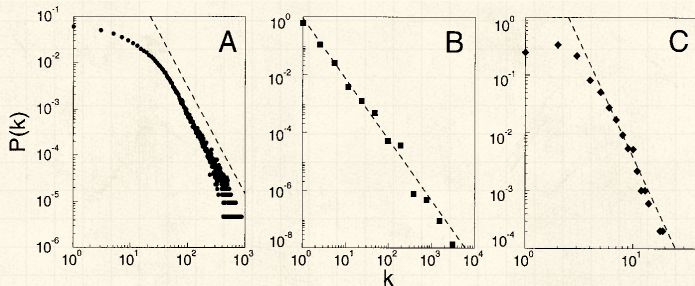


Fig. 1. The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes **(A)** $\gamma_{\text{actor}} = 2.3$, **(B)** $\gamma_{\text{www}} = 2.1$ and **(C)** $\gamma_{\text{power}} = 4$.



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Web	$\gamma \simeq 2.1$ for in-degree
Web	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

The Internet*s* is a different business...



Things to do and questions

- ☰ Vary attachment kernel.
- ☰ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion
 3. Add edge rewiring
- ☰ Deal with directed versus undirected networks.
- ☰ **Important Q.:** Are there distinct universality classes for these networks?
- ☰ **Q.:** How does changing the model affect γ ?
- ☰ **Q.:** Do we need preferential attachment and growth?
- ☰ **Q.:** Do model details matter? Maybe ...

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Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.
- For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is \therefore an **outrageous** assumption of node capability.
- But a **very simple mechanism** saves the day...

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Preferential attachment through randomness


- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an **extra step**: new nodes then connect to some of their friends' friends.
- Can also do this **at random**.
- Assuming the existing network is random, we know probability of a **random friend** having degree k is


$$Q_k \propto kP_k$$

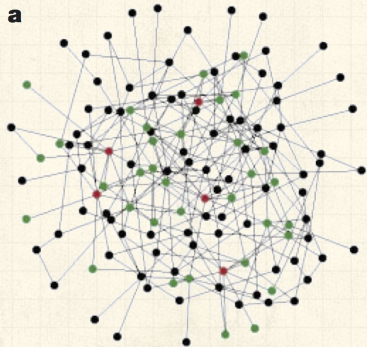
- So **rich-gets-richer** scheme can now be seen to work in a natural way.



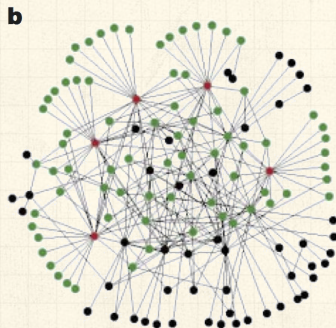
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 Albert et al., Nature, 2000:
"Error and attack tolerance of complex networks" [1]

 Standard random networks (Erdős-Rényi)
versus Scale-free networks:



Exponential



Scale-free

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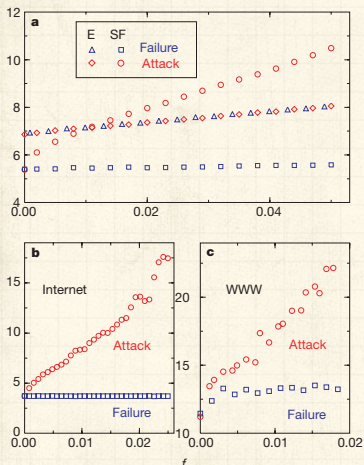
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Robustness



Plots of network diameter as a function of fraction of nodes removed



Erdős-Rényi versus scale-free networks



blue symbols = random removal



red symbols = targeted removal (most connected first)

from Albert et al., 2000



Robustness

- Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- All very reasonable: **Hubs** are a big deal.
- But:** next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 1. Physically larger nodes that may be harder to 'target'
 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

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Not a robust paper:



"The "Robust yet Fragile" nature of the Internet" [↗](#)

Doyle et al.,
Proc. Natl. Acad. Sci., **2005**, 14497–14502,
2005. [3]

-  HOT networks versus scale-free networks
-  Same degree distributions, different arrangements.
-  Doyle *et al.* take a look at the actual Internet.

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Fooling with the mechanism:

 2001: Krapivsky & Redner (KR) ^[4] explored the **general attachment kernel**:



$$\Pr(\text{attach to node } i) \propto A_k = k_i^\nu$$


where A_k is the attachment kernel and $\nu > 0$.

 KR also looked at changing the details of the attachment kernel.



Generalized model

 We'll follow KR's approach using rate equations .

 Here's the set up:

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1}N_{k-1} - A_kN_k] + \delta_{k1}$$

where N_k is the number of nodes of degree k .

1. One node with one link is added per unit time.
2. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
3. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
4. A is the correct normalization (coming up).
5. Seed with some initial network (e.g., a connected pair)
6. Detail: $A_0 = 0$

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Generalized model

 In general, probability of attaching to a **specific node** of degree k at time t is

$$\Pr(\text{attach to node } i) = \frac{A_k}{A(t)}$$


where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

 E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.

 For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

 Detail: we are ignoring initial seed network's edges.



Generalized model

So now

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1}N_{k-1} - A_k N_k] + \delta_{k1}$$

becomes

$$\frac{dN_k}{dt} = \frac{1}{2t} [(k-1)N_{k-1} - kN_k] + \delta_{k1}$$

As for BA method, look for steady-state growing solution: $N_k = n_k t$.

We replace dN_k/dt with $dn_k t/dt = n_k$.

We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_k t] + \delta_{k1}$$

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As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

Now: what happens if we start playing around with the attachment kernel A_k ?

Again, we're asking if the result $\gamma = 3$ universal ↗?

KR's natural modification: $A_k = k^\nu$ with $\nu \neq 1$.

But we'll first explore a more subtle modification of A_k made by Krapivsky/Redner [4]

Keep A_k **linear** in k but tweak details.

Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \rightarrow \infty$.

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 Recall we used the normalization:


$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$


 We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

 We assume that $A = \mu t$

 We'll find μ later and make sure that our assumption is consistent.

 As before, also assume $N_k(t) = n_k t$.

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Universality?


 For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

 This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1}n_{k-1} - A_k n_k] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

 Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}; \quad k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}.$$



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 Time for pure excitement: Find **asymptotic behavior** of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.

 For large k , we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu-1}$$

 Since μ depends on A_k , **details matter...**



Universality?

Now we need to find μ .

Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$

Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$

Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

Closed form expression for μ .

We can solve for μ in some cases.

Our assumption that $A = \mu t$ looks to be not too horrible.

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Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.

Again, we can find $\gamma = \mu + 1$ by finding μ .

Closed form expression for μ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun



$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

Craziness...

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 Rich-get-somewhat-richer:


$$A_k \sim k^\nu \text{ with } 0 < \nu < 1.$$


 General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

 Stretched exponentials (truncated power laws).

 aka Weibull distributions.

 **Universality**: now details of kernel **do not** matter.

 Distribution of degree is universal providing $\nu < 1$.



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Details:

 For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu} \right)}$$

 For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

 And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.



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 Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

 Now a **winner-take-all** mechanism.

 One single node ends up being connected to almost all other nodes.

 For $\nu > 2$, all but a finite # of nodes connect to one node.



Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
 - Description:** Characterizing very large networks
 - Explanation:** Micro story \Rightarrow Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... **#excitement**

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