Mixed, correlated random networks

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Principles of Complex Systems, Vols. 1, 2, & 3D CSYS/MATH 6701, 6713, & a pretend number, 2023–2024| @pocsvox

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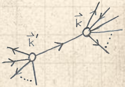

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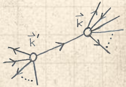
Outline




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
1 2 5 4 3 6 7 8 9
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
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THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
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THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
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THE JOHN DORY
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THE STORYTELLERS
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THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
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THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
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THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
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THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
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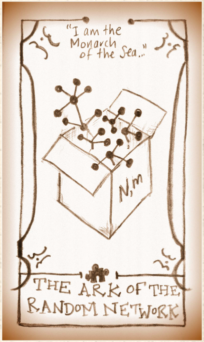
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE ORDERS THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea.."
N,m
THE ARK OF THE RANDOM NETWORK

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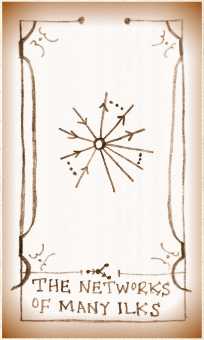


THE NETWORKS OF MANY ILKS

Random directed networks:

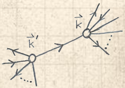
- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_{i} and k_{o} incoming and outgoing edges, otherwise random.

- Network defined by joint in- and out-degree distribution: $P_{k_{\mathrm{i}},k_{\mathrm{o}}}$
- Normalization: $\sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = 1$
- Marginal in-degree and out-degree distributions:

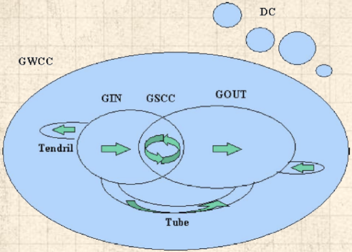
$$P_{k_{\mathrm{i}}} = \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} \text{ and } P_{k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}}$$

- Required balance:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \langle k_{\mathrm{o}} \rangle$$



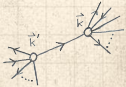
Directed network structure:



From Boguñá and Serano. [1]

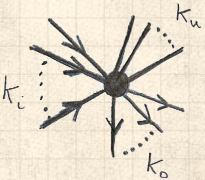
- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).

- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]



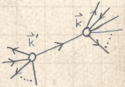
Observation:

- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.



- Consider nodes with three types of edges:
  1. k_{u} undirected edges,
  2. k_{i} incoming directed edges,
  3. k_{o} outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\ k_{\mathsf{u}} \ \ k_{\mathsf{i}} \ \ k_{\mathsf{o}}\ ]^{\mathsf{T}}.$$



- Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathsf{u}} \;\; k_{\mathsf{i}} \;\; k_{\mathsf{o}} \,]^{\mathsf{T}}.$$

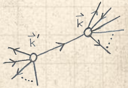
- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathsf{i}} \rangle = \sum_{k_{\mathsf{u}}=0}^{\infty} \sum_{k_{\mathsf{i}}=0}^{\infty} \sum_{k_{\mathsf{o}}=0}^{\infty} k_{\mathsf{i}} P_{\vec{k}} = \sum_{k_{\mathsf{u}}=0}^{\infty} \sum_{k_{\mathsf{i}}=0}^{\infty} \sum_{k_{\mathsf{o}}=0}^{\infty} k_{\mathsf{o}} P_{\vec{k}} = \langle k_{\mathsf{o}} \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

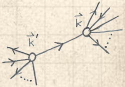
$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathsf{u}}} \delta_{k_{\mathsf{i}},0} \delta_{k_{\mathsf{o}},0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathsf{u}},0} P_{k_{\mathsf{i}},k_{\mathsf{o}}}.$$



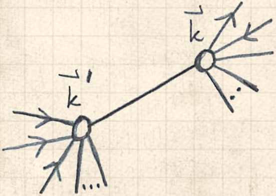
Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
  3. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an out-directed edge relative to the destination node.
- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})$.
  2. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ and $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ must be connected.



Correlations—Undirected edge balance:

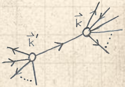
- Randomly choose an edge, and randomly choose one end.
- Say we find a degree $\vec{k}$ node at this end, and a degree $\vec{k}'$ node at the other end.
- Define probability this happens as $P^{(\mathrm{u})}(\vec{k}, \vec{k}')$.
- Observe we must have $P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k}', \vec{k})$.



- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \frac{k'_{\mathrm{u}} P(\vec{k}')}{\langle k'_{\mathrm{u}} \rangle}$$

$$P^{(\mathrm{u})}(\vec{k}', \vec{k}) = P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) \frac{k_{\mathrm{u}} P(\vec{k})}{\langle k_{\mathrm{u}} \rangle}.$$

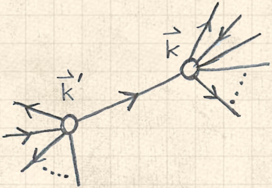


Correlations—Directed edge balance:

- The quantities

$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

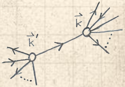


1. along an outgoing edge, or
2. against the direction of an incoming edge.

- We therefore have

$$P^{(\mathrm{dir})}(\vec{k}, \vec{k}') = P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \frac{k_{\mathrm{o}}' P(\vec{k}')}{\langle k_{\mathrm{o}}' \rangle} = P^{(\mathrm{o})}(\vec{k}' \,|\, \vec{k}) \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}.$$

- Note that $P^{(\mathrm{dir})}(\vec{k}, \vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}', \vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.



Global spreading condition: [2]

When are cascades possible?:

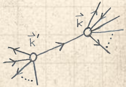
- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}=0}^{\infty} \frac{k_{\mathrm{u}} P_{k_{\mathrm{u}}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} \frac{k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}},1} > 1.$$

- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.



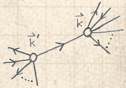
Global spreading condition:

Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

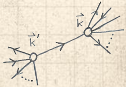
- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_\mathrm{u},1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).



Global spreading condition:

Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
- Gain ratio now more complicated:
    1. Infected directed edges can lead to infected directed or undirected edges.
    2. Infected undirected edges can lead to infected directed or undirected edges.
- Define $f^{(\mathrm{u})}(d)$ and $f^{(\mathrm{o})}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.



- Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathsf{u})}(d+1) \\ f^{(\mathsf{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathsf{u})}(d) \\ f^{(\mathsf{o})}(d) \end{bmatrix}$$

- Two separate gain equations:

$$f^{(\mathsf{u})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet (k_{\mathsf{u}} - 1) \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{u})}(d) + \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{u}} \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{o})}($$

$$f^{(\mathsf{o})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet k_{\mathsf{o}} B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{u})}(d) + \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{o}} \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{o})}(d) \right]$$

- Gain ratio matrix:

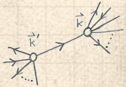
$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet (k_{\mathsf{u}} - 1) & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{u}} \\ \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet k_{\mathsf{o}} & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{o}} \end{bmatrix} \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1}$$

- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$



Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}} P^{(\mathrm{u})}(k_{\mathrm{u}} \,|\, *) \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},*}$$

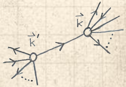
- II. Directed, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathrm{i}},k_{\mathrm{o}}} P^{(\mathrm{i})}(k_{\mathrm{i}}, k_{\mathrm{o}} \,|\, *) \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}},*}$$

- III. Mixed Directed and Undirected, Uncorrelated—

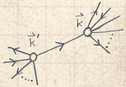
$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$



Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(\mathrm{i})}(\vec{k}\,|\,*)$ with $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ and so on.
- Edge types are now more diverse beyond directed and undirected as originating node type matters.
- Sums are now over $\vec{k}'$.



Summary of contagion conditions for correlated networks:

- IV. Undirected,
Correlated—$f_{k_\mathrm{u}}(d+1) = \sum_{k'_\mathrm{u}} R_{k_\mathrm{u} k'_\mathrm{u}} f_{k'_\mathrm{u}}(d)$

$$R_{k_\mathrm{u} k'_\mathrm{u}} = P^{(\mathrm{u})}(k_\mathrm{u} \,|\, k'_\mathrm{u}) \bullet (k_\mathrm{u} - 1) \bullet B_{k_\mathrm{u} k'_\mathrm{u}}$$

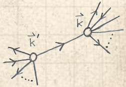
- V. Directed,
Correlated—$f_{k_\mathrm{i} k_\mathrm{o}}(d+1) = \sum_{k'_\mathrm{i}, k'_\mathrm{o}} R_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}} f_{k'_\mathrm{i} k'_\mathrm{o}}(d)$

$$R_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}} = P^{(\mathrm{i})}(k_\mathrm{i}, k_\mathrm{o} \,|\, k'_\mathrm{i}, k'_\mathrm{o}) \bullet k_\mathrm{o} \bullet B_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}}$$

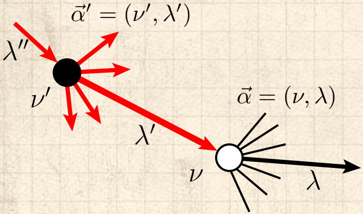
- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}}(d+1) \\ f^{(\mathrm{o})}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{k'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}'}(d) \\ f^{(\mathrm{o})}_{\vec{k}'}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k}\vec{k}'} = \begin{bmatrix} P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \bullet (k_\mathrm{u} - 1) & P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \bullet k_\mathrm{u} \\ P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \bullet k_\mathrm{o} & P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \bullet k_\mathrm{o} \end{bmatrix} \bullet B_{\vec{k}\vec{k}'}$$



Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

- $P_{\vec{\alpha}\vec{\alpha}'}$ = conditional probability that a type λ' edge emanating from a type ν' node leads to a type ν node.
- $k_{\vec{\alpha}\vec{\alpha}'}$ = potential number of newly infected edges of type λ emanating from nodes of type ν.
- $B_{\vec{\alpha}\vec{\alpha}'}$ = probability that a type ν node is eventually infected by a single infected type λ' link arriving from a neighboring node of type ν'.
- Generalized contagion condition:

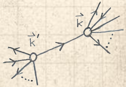
$$\max|\mu| : \mu \in \sigma\left(\mathbf{R}\right) > 1$$

- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\textrm{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\textrm{trig}} \right)^{k-1} \right],$$

$$P_{\textrm{trig}} = S_{\textrm{trig}} = \sum_k P_k \bullet \left[1 - (1 - Q_{\textrm{trig}})^k \right].$$

- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.



Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}} - 1} \right]$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k'_{\text{u}}} P(k'_{\text{u}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}} \right]$$

- II. Directed, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} | \cdot) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}} \right]$$

$$S_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P(k'_{\text{i}}, k'_{\text{o}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}} \right]$$

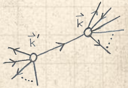
Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$



Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\textrm{trig}}(k_{\textrm{u}}) = \sum_{k'_{\textrm{u}}} P^{(\textrm{u})}(k'_{\textrm{u}} \,|\, k_{\textrm{u}}) B_{k'_{\textrm{u}}1} \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{u}}))^{k'_{\textrm{u}}-1}\right]$

$$S_{\textrm{trig}} = \sum_{k'_{\textrm{u}}} P(k'_{\textrm{u}}) \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{u}}))^{k'_{\textrm{u}}}\right]$$

- V. Directed, Correlated— $Q_{\textrm{trig}}(k_{\textrm{i}}, k_{\textrm{o}}) = \sum_{k'_{\textrm{i}}, k'_{\textrm{o}}} P^{(\textrm{u})}(k'_{\textrm{i}}, k'_{\textrm{o}} | \, k_{\textrm{i}}, k_{\textrm{o}}) B_{k'_{\textrm{i}}1} \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{i}}, k'_{\textrm{o}}))^{k'_{\textrm{o}}}\right]$

$$S_{\textrm{trig}} = \sum_{k'_{\textrm{i}}, k'_{\textrm{o}}} P(k'_{\textrm{i}}, k'_{\textrm{o}}) \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{i}}, k'_{\textrm{o}}))^{k'_{\textrm{o}}}\right]$$

Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

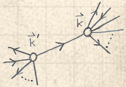
$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- More generalizations: bipartite affiliation graphs and multilayer networks.



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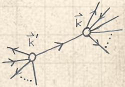

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