

Branching Networks II

Last updated: 2023/08/22, 11:48:23 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number,
2023–2024 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



The PoCSverse
Branching
Networks II
1 of 86

Horton ⇔
Tokunaga

Reducing Horton
Scaling relations

Fluctuations

Models

Nutshell

References



Licensed under the *Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License*.

These slides are brought to you by:

Sealie & Lambie
Productions



The PoCSverse
Branching
Networks II
2 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

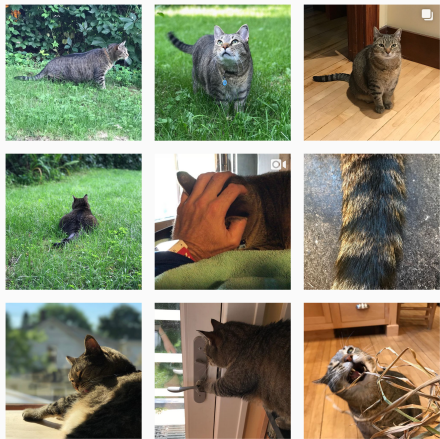
Nutshell

References



These slides are also brought to you by:

Special Guest Executive Producer



 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

The PoCSverse
Branching
Networks II
3 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References



Outline

Horton \Leftrightarrow Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References

The PoCSverse
Branching
Networks II
4 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models


Nutshell

References



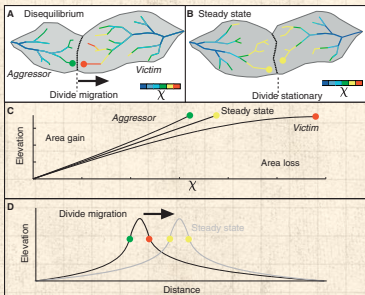
Piracy on the high χ 's:



“Dynamic Reorganization of River Basins” 

Willett et al.,

Science, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x, t)}{\partial t} = U - K A^m \left| \frac{\partial z(x, t)}{\partial x} \right|^n$$

$$z(x) = z_b + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_b}^x \left(\frac{A_0}{A(x')} \right)^{m/n} dx'$$

Piracy on the high χ 's:

The PoCSverse
Branching
Networks II
7 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations


Fluctuations

Models

Nutshell

References







https://www.youtube.com/watch?v=FnroL1_-l2c?rel=0

More: [How river networks move across a landscape](#) 
(Science Daily)



Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:





-  In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
-  Oddly, Horton's laws have **four** parameters and Tokunaga has **two** parameters.
-  $R_n, R_a, R_\ell,$ and R_s **versus** T_1 and R_T . One simple redundancy: $R_\ell = R_s$.
Insert assignment question 
-  To make a connection, clearest approach is to start with Tokunaga's law ...
-  Known result: Tokunaga \rightarrow Horton ^[18, 19, 20, 9, 2]



Let us make them happy

We need one more ingredient:

Space-fillingness


-  A network is **space-filling** if the average distance between adjacent streams is roughly constant.
-  Reasonable for river and cardiovascular networks
-  For river networks:
Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
-  In terms of basin characteristics:

$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_{\omega} \bar{s}_{\omega}}{a_{\Omega}}$$





More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

 Start looking for Horton's stream number law:

$$n_\omega / n_{\omega+1} = R_n.$$

 Estimate n_ω , the number of streams of order ω in terms of other $n_{\omega'}, \omega' > \omega$.

 Observe that each stream of order ω terminates by either:

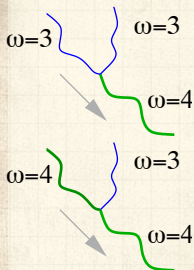
1. Running into another stream of order ω and generating a stream of order $\omega + 1$

...

▶ $2n_{\omega+1}$ streams of order ω do this

2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...

▶ $n_{\omega'} T_{\omega'-\omega}$ streams of order ω do this






More with the happy-making thing


Putting things together:



$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$

 Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain R_n .

 Insert assignment question 

 Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd} .
- Assume side streams are roughly separated by distance $1/\rho_{dd}$.
- For an order ω **stream segment**, expected length is

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + \sum_{k=1}^{\omega-1} T_k \right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^\omega$$



Horton and Tokunaga are happy

Altogether then:



$$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$



Recall $R_\ell = R_s$ so

$$R_\ell = R_s = R_T$$



And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$



Horton and Tokunaga are happy

The PoCSverse
Branching
Networks II
16 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References

Some observations:

- 🧱 R_n and R_ℓ depend on T_1 and R_T .
- 🧱 Seems that R_a must as well ...
- 🧱 Suggests Horton's laws must contain some redundancy
- 🧱 We'll in fact see that $R_a = R_n$.
- 🧱 Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]



Horton and Tokunaga are happy

The PoCSverse
Branching
Networks II
17 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations


Fluctuations

Models

Nutshell

References

The other way round


 Note: We can invert the expressions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.



$$R_T = R_\ell,$$



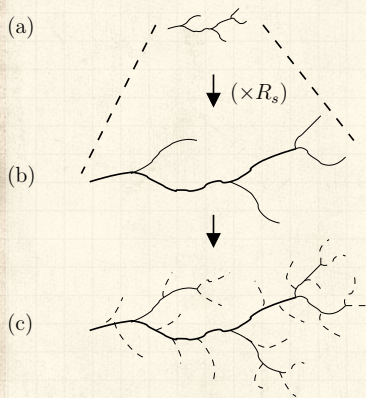
$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$

 Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...



Horton and Tokunaga are friends

From Horton to Tokunaga [2]



Assume Horton's laws hold for number and length



Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.



Scale up by a factor of R_e , orders increment to $\omega + 1$ and ω .



Maintain drainage density by adding new order $\omega - 1$ streams



Horton and Tokunaga are friends

...and in detail:

- Must retain same drainage density.
- Add an extra $(R_\ell - 1)$ first order streams for each original tributary.
- Since by definition, an order $\omega + 1$ stream segment has T_ω order 1 side streams, we have:


$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right).$$

- For large ω , Tokunaga's law is the solution—let's check ...



Horton and Tokunaga are friends

Just checking:

 Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right)$$



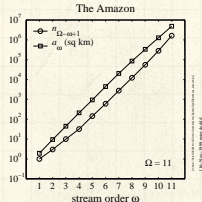
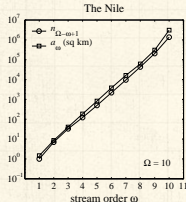
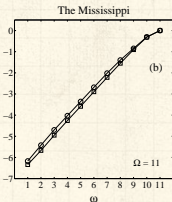
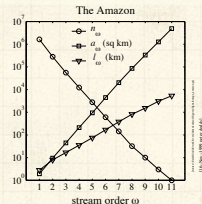
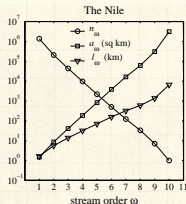
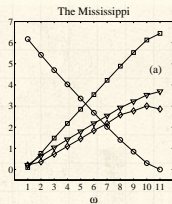
$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_1 R_\ell^{i-1} \right)$$

$$= (R_\ell - 1) \left(1 + T_1 \frac{R_\ell^{k-1} - 1}{R_\ell - 1} \right)$$

$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{k-1}}{R_\ell - 1} = T_1 R_\ell^{k-1} \quad \dots\text{yep.}$$



Horton's laws of area and number:



In bottom plots, stream number graph has been flipped vertically.



Highly suggestive that $R_n \equiv R_a \dots$



Measuring Horton ratios is tricky:

The PoCSverse
Branching
Networks II
22 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References



How robust are our estimates of ratios?



Rule of thumb: discard data for two smallest and two largest orders.



Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3, 8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7, 8]	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024




Amazon:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019



Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

 $a_\Omega \propto$ sum of all stream segment lengths in a order Ω basin (assuming uniform drainage density)

 So:

$$\begin{aligned} a_\Omega &\simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{dd} \\ &\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \hat{1}^{n_\omega}}_{n_\omega} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_\omega} \\ &= \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^\omega \end{aligned}$$




Reducing Horton's laws:

Continued ...



$$\begin{aligned} a_{\Omega} &\propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega} \\ &= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)} \\ &\sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \nearrow \end{aligned}$$





 So, a_{Ω} is growing like R_n^{Ω} and therefore:

$$R_n \equiv R_a$$



Reducing Horton's laws:


Not quite:


-  ...But this only a rough argument as Horton's laws do not imply a strict hierarchy
-  Need to account for sidebranching.
-  Insert assignment question 




Equipartitioning:


Intriguing division of area:

 Observe: Combined area of basins of order ω independent of ω .

 Not obvious: basins of low orders not necessarily contained in basin on higher orders.

 Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

 Reason:

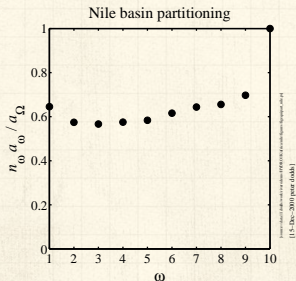
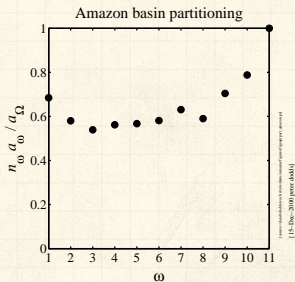
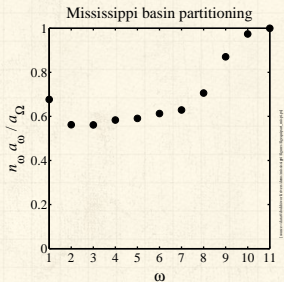
$$n_\omega \propto (R_n)^{-\omega}$$

$$\bar{a}_\omega \propto (R_a)^\omega \propto n_\omega^{-1}$$



Equipartitioning:

Some examples:



Horton \leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References



Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References







Neural Reboot: Fwoompf

<https://www.youtube.com/watch?v=5mUs70SqD4o?rel=0>



Scaling laws







The story so far:

-  Natural branching networks are **hierarchical**, **self-similar** structures
-  Hierarchy is **mixed**
-  Tokunaga's law describes detailed architecture:
$$T_k = T_1 R_T^{k-1}.$$
-  We have connected Tokunaga's and Horton's laws
-  Only two Horton laws are independent ($R_n = R_a$)
-  Only **two** parameters are **independent**:
$$(T_1, R_T) \Leftrightarrow (R_n, R_s)$$



Scaling laws

A little further ...






-  Ignore stream ordering for the moment
-  Pick a random location on a branching network p .
-  Each point p is associated with a basin and a longest stream length
-  **Q:** What is probability that the p 's drainage basin has area a ? $P(a) \propto a^{-\tau}$ for large a
-  **Q:** What is probability that the longest stream from p has length ℓ ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
-  Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$



Probability distributions with power-law decays



We see them everywhere:

-  Earthquake magnitudes (Gutenberg-Richter law)
-  City sizes (Zipf's law)
-  Word frequency (Zipf's law)^[22]
-  Wealth (maybe not—at least heavy tailed)
-  Statistical mechanics (phase transitions)^[5]



A big part of the story of complex systems



Arise from **mechanisms**: growth, randomness, optimization, ...









Our task is always to illuminate the mechanism ...



Scaling laws


Connecting exponents


-  We have the detailed picture of branching networks (Tokunaga and Horton)
-  Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story ^[17, 1, 2]
-  Let's work on $P(\ell)$...
-  Our first fudge: assume Horton's laws hold throughout a basin of order Ω .
-  (We know they deviate from strict laws for low ω and high ω but not too much.)
-  Next: place stick between teeth. Bite stick. Proceed.



Scaling laws

Finding γ :


 Often useful to work with **cumulative distributions**, especially when dealing with power-law distributions.

 The complementary cumulative distribution turns out to be most useful:

$$P_{>}(l_*) = P(l > l_*) = \int_{l=l_*}^{l_{\max}} P(l) dl$$




$$P_{>}(l_*) = 1 - P(l < l_*)$$


 Also known as the exceedance probability.



Scaling laws

Finding γ :

 The connection between $P(x)$ and $P_{>}(x)$ when $P(x)$ has a power law tail is simple:

 Given $P(l) \sim l^{-\gamma}$ large l then for large enough l_*

$$P_{>}(l_*) = \int_{l=l_*}^{l_{\max}} P(l) dl$$

$$\sim \int_{l=l_*}^{l_{\max}} l^{-\gamma} dl$$





$$= \frac{l^{-(\gamma-1)}}{-(\gamma-1)} \Big|_{l=l_*}^{l_{\max}}$$

$$\propto l_*^{-(\gamma-1)} \quad \text{for } l_{\max} \gg l_*$$




Scaling laws

Finding γ :

-  **Aim:** determine probability of randomly choosing a point on a network with main stream length $> l_*$
-  Assume some spatial sampling resolution Δ
-  Landscape is broken up into grid of $\Delta \times \Delta$ sites
-  Approximate $P_{>}(l_*)$ as

$$P_{>}(l_*) = \frac{N_{>}(l_*; \Delta)}{N_{>}(0; \Delta)}$$


where $N_{>}(l_*; \Delta)$ is the number of sites with main stream length $> l_*$.

-  Use Horton's law of stream segments:
 $\bar{s}_\omega / \bar{s}_{\omega-1} = R_s \dots$




Scaling laws


Finding γ :


 Set $l_* = \bar{l}_\omega$ for some $1 \ll \omega \ll \Omega$.



$$P_>(\bar{l}_\omega) = \frac{N_>(\bar{l}_\omega; \Delta)}{N_>(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \Delta}$$

 Δ 's cancel

 Denominator is $a_\Omega \rho_{dd}$, a constant.


 So ...using Horton's laws ...

$$P_>(\bar{l}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$




Scaling laws


Finding γ :


 We are here:

$$P_{>}(\bar{l}_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$

 Cleaning up irrelevant constants:

$$P_{>}(\bar{l}_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega'}$$

 Change summation order by substituting $\omega'' = \Omega - \omega'$.

 Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References




Scaling laws

Finding γ :



$$P_{>}(\bar{l}_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

 Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,


$$P_{>}(\bar{l}_{\omega}) \propto \left(\frac{R_n}{R_s}\right)^{\Omega-\omega} \propto \left(\frac{R_n}{R_s}\right)^{-\omega}$$

again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$





Scaling laws

Finding γ :

 Nearly there:

$$P_{>}(\bar{\ell}_\omega) \propto \left(\frac{R_n}{R_s} \right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

 Need to express right hand side in terms of $\bar{\ell}_\omega$.

 Recall that $\bar{\ell}_\omega \simeq \bar{\ell}_1 R_\ell^{\omega-1}$.




$$\bar{\ell}_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$




Scaling laws


Finding γ :

 Therefore:


$$P_{>}(\bar{l}_{\omega}) \propto e^{-\omega \ln(R_n/R_s)} = (e^{\omega \ln R_s})^{-\ln(R_n/R_s)/\ln(R_s)}$$




$$\propto \bar{l}_{\omega}^{-\ln(R_n/R_s)/\ln R_s}$$



$$= \bar{l}_{\omega}^{-(\ln R_n - \ln R_s)/\ln R_s}$$



$$= \bar{l}_{\omega}^{-\ln R_n/\ln R_s + 1}$$




$$= \bar{l}_{\omega}^{-\gamma + 1}$$




Scaling laws


Finding γ :


 And so we have:


$$\gamma = \ln R_n / \ln R_s$$

 Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert assignment question 

 Such connections between exponents are called **scaling relations**

 Let's connect to one last relationship: Hack's law



Scaling laws

Hack's law: [6]



$$l \propto a^h$$



Typically observed that $0.5 \lesssim h \lesssim 0.7$.



Use Horton laws to connect h to Horton ratios:

$$\bar{l}_\omega \propto R_s^\omega \text{ and } \bar{a}_\omega \propto R_n^\omega$$



Observe:

$$\bar{l}_\omega \propto e^{\omega \ln R_s} \propto (e^{\omega \ln R_n})^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n} \propto \bar{a}_\omega^{\ln R_s / \ln R_n} \Rightarrow h = \ln R_s / \ln R_n$$

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References



We mentioned there were a good number of 'laws': [2]

Relation: **Name or description:**

$T_k = T_1 (R_T)^{k-1}$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\ell_{\omega+1} / \ell_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law



acing Horton

ng relations

uations

els

hell

rences

Connecting exponents

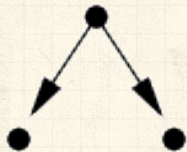
Only 3 parameters are independent:
e.g., take d , R_n , and R_s

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$ $R_T = R_s$
$n_\omega/n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



Scheidegger's model

Directed random networks ^[11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$



Functional form of all scaling laws exhibited but exponents differ from real world ^[15, 16, 14]



Useful and interesting test case



A toy model—Scheidegger's model

The PoCSverse
Branching
Networks II
48 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

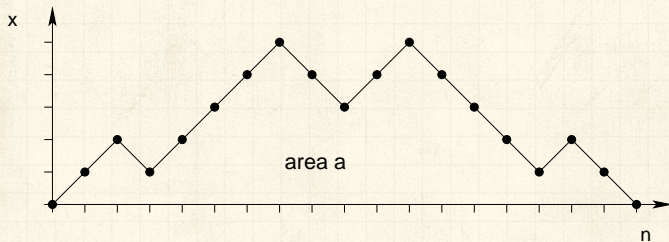
Nutshell

References

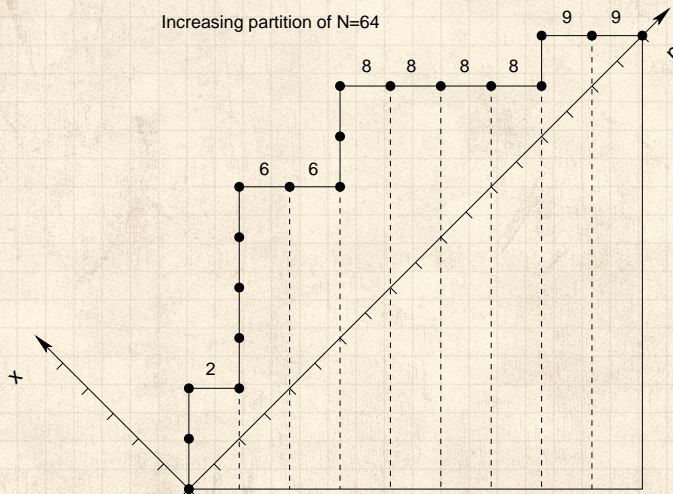
Random walk basins:



Boundaries of basins are random walks



Scheidegger's model



The PoCSverse
Branching
Networks II
49 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References



Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):



$$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

and so $P(\ell) \propto \ell^{-3/2}$.



Typical area for a walk of length n is $\propto n^{3/2}$:

$$\ell \propto a^{2/3}.$$



Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.



Note $\tau = 2 - h$ and $\gamma = 1/h$.

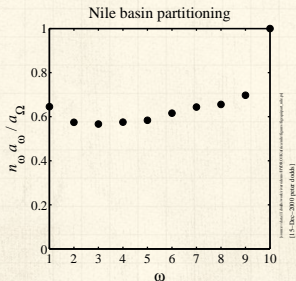
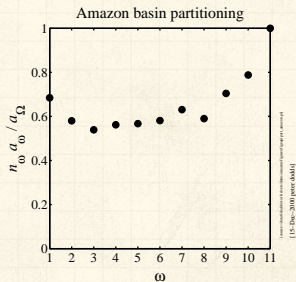
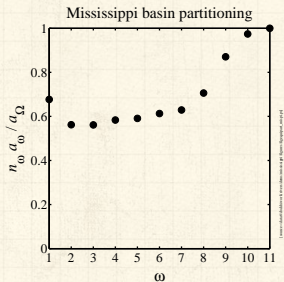


R_n and R_ℓ have not been derived analytically.



Equipartitioning reexamined:

Recall this story:



The PoCverse
Branching
Networks II
51 of 86

Horton \leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations


Models

Nutshell


References




Equipartitioning


 What about

$$P(a) \sim a^{-\tau} \quad ?$$

 Since $\tau > 1$, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

 $P(a)$ overcounts basins within basins ...

 while stream ordering separates basins ...



Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

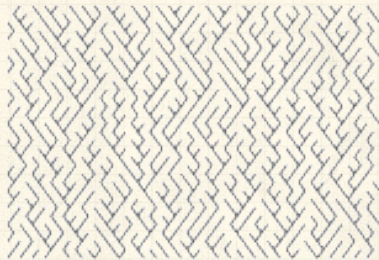
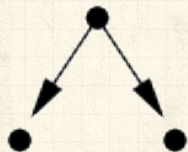
$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$

- Natural generalization to consider relationships between **probability distributions**
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



A toy model—Scheidegger's model

Directed random networks ^[11, 12]




$$P(\searrow) = P(\swarrow) = 1/2$$




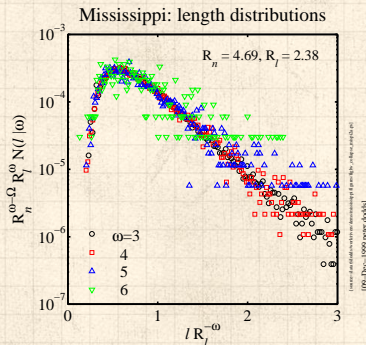
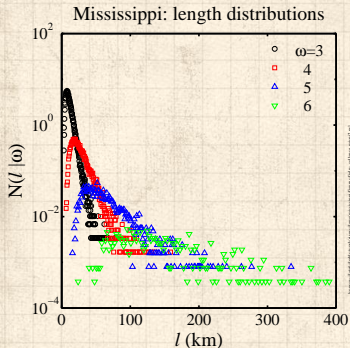
Flow is directed downwards





Generalizing Horton's laws

 $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$

 $\bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$




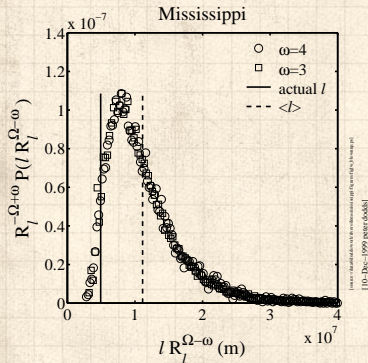
 Scaling collapse works well for intermediate orders


 All **moments** grow exponentially with order




Generalizing Horton's laws


 How well does overall basin fit internal pattern?




 Actual length = **4920 km** (at 1 km res)

 Predicted Mean length = **11100 km**

 Predicted Std dev = **5600 km**

 Actual length/Mean length = **44 %**

 Okay.



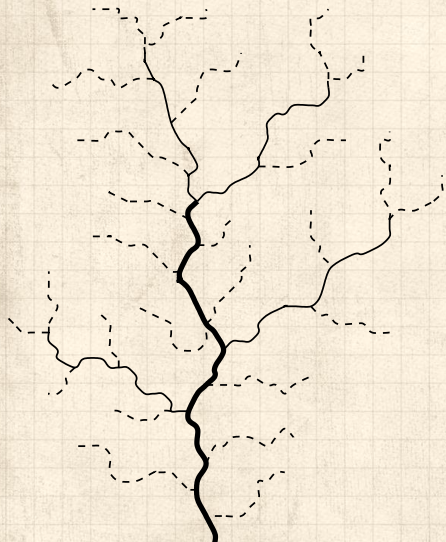
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	l_Ω	\bar{l}_Ω	σ_l	l_Ω/\bar{l}_Ω	σ_l/\bar{l}_Ω
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	a_Ω	\bar{a}_Ω	σ_a	a_Ω/\bar{a}_Ω	σ_a/\bar{a}_Ω
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



Combining stream segments distributions:



Stream segments
sum to give main
stream lengths



$$l_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$$



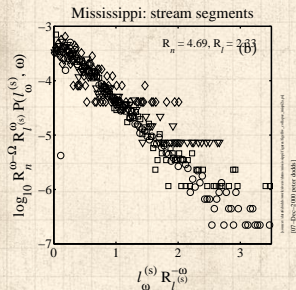
$P(l_{\omega})$ is a
convolution of
distributions for
the s_{ω}



Generalizing Horton's laws

Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \dots * N(s|\omega)$$




$$N(s|\omega) = \frac{1}{R_n^\omega R_l^\omega} F(s/R_l^\omega)$$

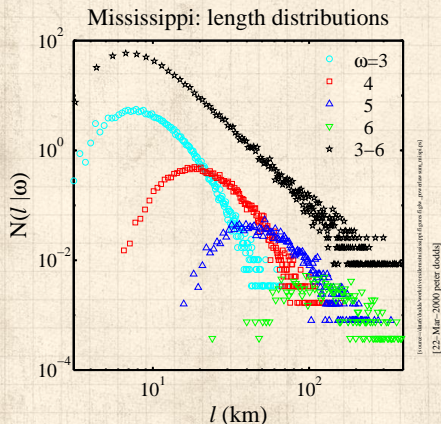
$$F(x) = e^{-x/\xi}$$


Mississippi: $\xi \simeq 900$ m.





Generalizing Horton's laws

 Next level up: Main stream length distributions must combine to give overall distribution for stream length




 $P(l) \sim l^{-\gamma}$


 Another round of convolutions ^[3]

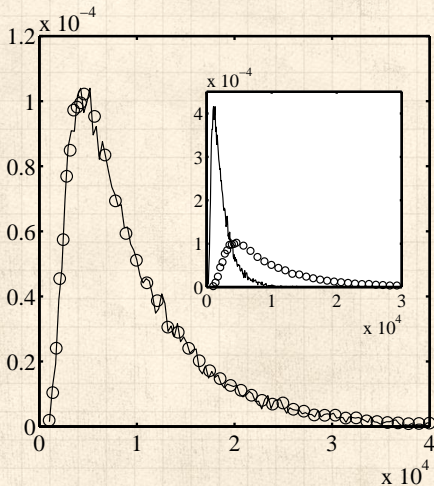
 Interesting ...



Generalizing Horton's laws

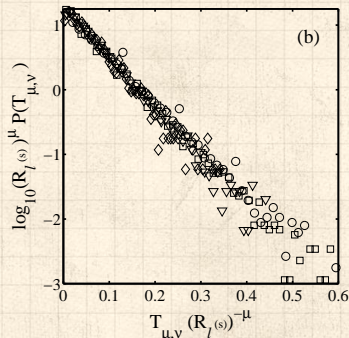
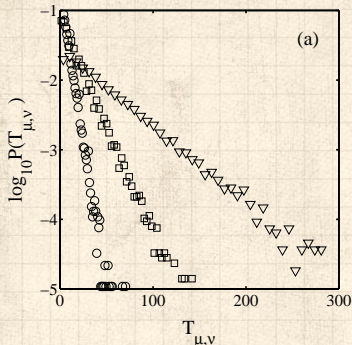
 Number and area distributions for the Scheidegger model [3]


 $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.




Generalizing Tokunaga's law

Scheidegger:



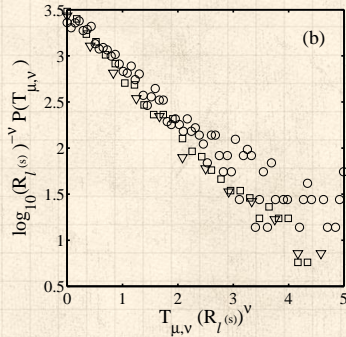
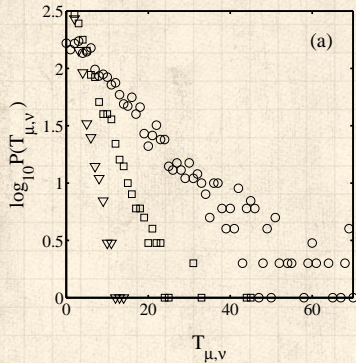
 Observe exponential distributions for $T_{\mu,\nu}$

 Scaling collapse works using R_s



Generalizing Tokunaga's law

Mississippi:



Same data collapse for Mississippi ...



Generalizing Tokunaga's law


So


$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t [T_{\mu,\nu}/(R_s)^{\mu-\nu-1}]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})$$

 Exponentials arise from randomness.

 Look at joint probability $P(s_\mu, T_{\mu,\nu})$.



Generalizing Tokunaga's law

Fluctuations

Models

Nutshell

References

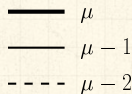
Network architecture:



Inter-tributary
lengths
exponentially
distributed



Leads to random
spatial
distribution of
stream segments



Generalizing Tokunaga's law


- Follow stream segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is **constant**:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1} \xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- \Rightarrow random spatial distribution of stream segments





Generalizing Tokunaga's law


 Joint distribution for generalized version of Tokunaga's law:


$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

where

 p_ν = probability of absorbing an order ν side stream


 \tilde{p}_μ = probability of an order μ stream terminating

 Approximation: depends on distance units of s_μ


 In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.




Generalizing Tokunaga's law

 Now deal with this thing:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

 Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.

 Obtain


$$P(x, y) = Nx^{-1/2} [F(y/x)]^x$$

where

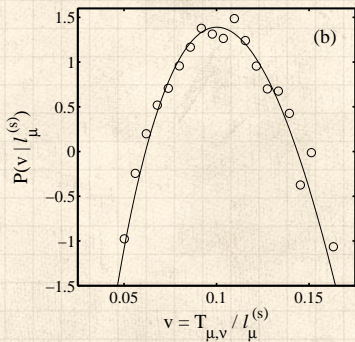
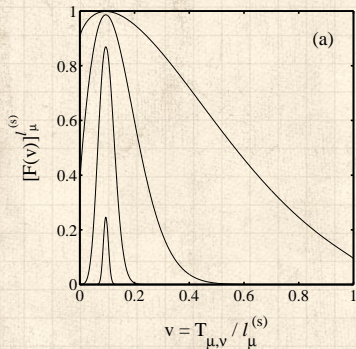
$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$




Generalizing Tokunaga's law

 Checking form of $P(s_{\mu}, T_{\mu, \nu})$ works:

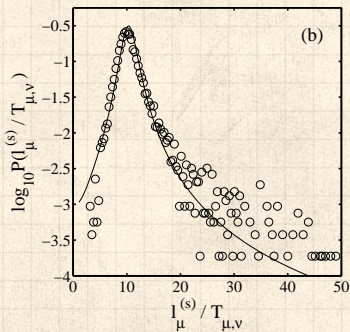
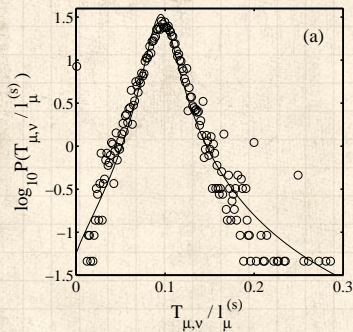
Scheidegger:



Generalizing Tokunaga's law

 Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

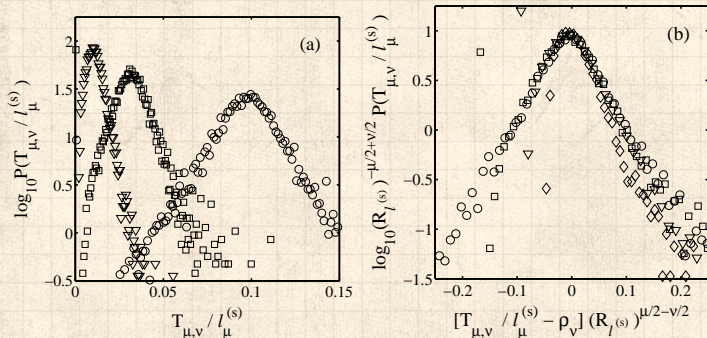
Scheidegger:




Generalizing Tokunaga's law

☰ Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

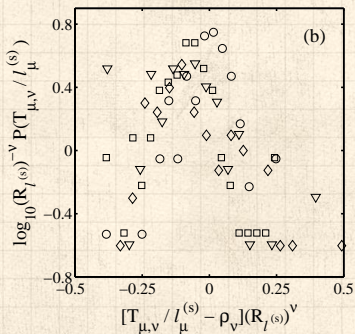
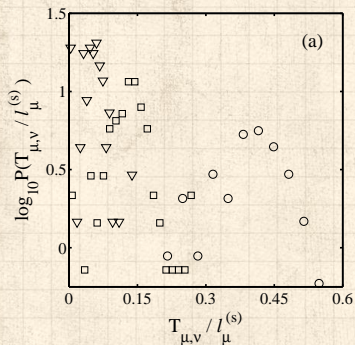
Scheidegger:



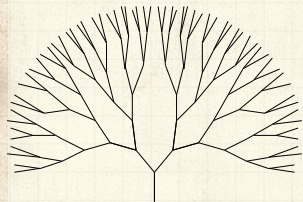
Generalizing Tokunaga's law







 Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Mississippi:



Random subnetworks on a Bethe lattice ^[13]

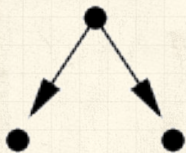


-  Dominant theoretical concept for several decades.
-  Bethe lattices are fun and tractable.
-  Led to idea of “Statistical inevitability” of river network statistics ^[7]
-  But Bethe lattices unconnected with surfaces.
-  In fact, Bethe lattices \simeq infinite dimensional spaces (oops).
-  So let's move on ...



Scheidegger's model

Directed random networks [11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$




Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]





Optimal channel networks


Rodríguez-Iturbe, Rinaldo, et al. ^[10]

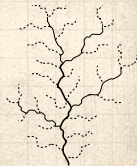
 Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\epsilon}$ is minimized, where

$$\dot{\epsilon} \propto \int d\vec{r} \text{ (flux)} \times \text{ (force)} \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^\gamma$$

 Landscapes obtained numerically give exponents near that of real networks.

 **But:** numerical method used matters.

 **And:** Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network ^[8]



Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$$h \Rightarrow \ell \propto a^h \text{ (Hack's law).}$$

$$d \Rightarrow \ell \propto L_{\parallel}^d \text{ (stream self-affinity).}$$






Branching networks II Key Points:

- 🧱 Horton's laws and Tokunaga's law all fit together.
- 🧱 For 2-d networks, these laws are 'planform' laws and ignore slope.
- 🧱 Abundant scaling relations can be derived.
- 🧱 Can take R_n , R_ℓ , and d as three independent parameters necessary to describe all 2-d branching networks.
- 🧱 For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- 🧱 Laws can be extended nicely to laws of distributions.
- 🧱 Numerous models of branching network evolution exist: nothing rock solid yet ...?





References I

- [1] H. de Vries, T. Becker, and B. Eckhardt.
Power law distribution of discharge in ideal
networks.
[Water Resources Research, 30\(12\):3541–3543,
1994. pdf](#) 
- [2] P. S. Dodds and D. H. Rothman.
Unified view of scaling laws for river networks.
[Physical Review E, 59\(5\):4865–4877, 1999. pdf](#) 
- [3] P. S. Dodds and D. H. Rothman.
Geometry of river networks. II. Distributions of
component size and number.
[Physical Review E, 63\(1\):016116, 2001. pdf](#) 



References II

- [4] P. S. Dodds and D. H. Rothman.
Geometry of river networks. III. Characterization
of component connectivity.
[Physical Review E, 63\(1\):016117, 2001. pdf](#) 
- [5] N. Goldenfeld.
Lectures on Phase Transitions and the
Renormalization Group, volume 85 of Frontiers in
Physics.
Addison-Wesley, Reading, Massachusetts, 1992.
- [6] J. T. Hack.
Studies of longitudinal stream profiles in Virginia
and Maryland.
[United States Geological Survey Professional
Paper, 294-B:45-97, 1957. pdf](#) 




References III

- [7] J. W. Kirchner.
Statistical inevitability of Horton's laws and the
apparent randomness of stream channel
networks.
[Geology](#), 21:591–594, 1993. [pdf](#) ↗
- [8] A. Maritan, F. Colaiori, A. Flammini, M. Cieplak,
and J. R. Banavar.
Universality classes of optimal channel networks.
[Science](#), 272:984–986, 1996. [pdf](#) ↗
- [9] S. D. Peckham.
New results for self-similar trees with applications
to river networks.
[Water Resources Research](#), 31(4):1023–1029,
1995.



References IV

- [10] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambridge, UK,
1997.
- [11] A. E. Scheidegger.
A stochastic model for drainage patterns into an
intramontane trench.
Bull. Int. Assoc. Sci. Hydrol., 12(1):15–20, 1967.
[pdf](#) 
- [12] A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.



References V

- [13] R. L. Shreve.
Infinite topologically random channel networks.
[Journal of Geology, 75:178–186, 1967.](#) pdf ↗
- [14] H. Takayasu.
Steady-state distribution of generalized
aggregation system with injection.
[Physical Review Letters, 63\(23\):2563–2565, 1989.](#)
pdf ↗
- [15] H. Takayasu, I. Nishikawa, and H. Tasaki.
Power-law mass distribution of aggregation
systems with injection.
[Physical Review A, 37\(8\):3110–3117, 1988.](#)





References VI

- [16] M. Takayasu and H. Takayasu.
Apparent independency of an aggregation system
with injection.
[Physical Review A, 39\(8\):4345–4347, 1989. pdf](#) ↗
- [17] D. G. Tarboton, R. L. Bras, and I. Rodríguez-Iturbe.
Comment on “On the fractal dimension of stream
networks” by Paolo La Barbera and Renzo Rosso.
[Water Resources Research, 26\(9\):2243–4, 1990.](#)
[pdf](#) ↗
- [18] E. Tokunaga.
The composition of drainage network in Toyohira
River Basin and the valuation of Horton’s first law.
[Geophysical Bulletin of Hokkaido University,](#)
[15:1–19, 1966. pdf](#) ↗



References VII

- [19] E. Tokunaga.
Consideration on the composition of drainage networks and their evolution.
[Geographical Reports of Tokyo Metropolitan University, 13:G1-27, 1978. pdf](#) 
- [20] E. Tokunaga.
Ordering of divide segments and law of divide segment numbers.
[Transactions of the Japanese Geomorphological Union, 5\(2\):71-77, 1984.](#)
- [21] S. D. Willett, S. W. McCoy, J. T. Perron, L. Goren, and C.-Y. Chen.
Dynamic reorganization of river basins.
[Science, 343\(6175\):1248765, 2014. pdf](#) 



References VIII

- [22] G. K. Zipf.
Human Behaviour and the Principle of
Least-Effort.
Addison-Wesley, Cambridge, MA, 1949.

The PoCSverse
Branching
Networks II
86 of 86

Horton \Leftrightarrow
Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References

