The Amusing Law of Benford

Last updated: 2023/08/22, 11:48:25 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number,
2023–2024| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont

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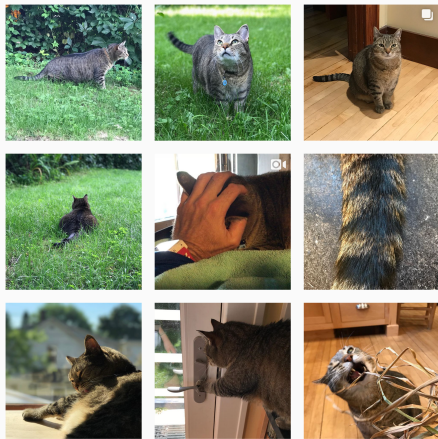
On Instagram at pratchett_the_cat

Outline

Benford's Law

References

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THE LAW OF DIGITS FIRST

Benford's Law ⧉—The Law of First Digits

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for certain sets of 'naturally' occurring numbers in base b

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- Newcomb almost always noted but Benford gets the stamp, according to Stigler's Law of Eponymy. ⤢

Benford's Law—The Law of First Digits

Observed for

- Fundamental constants (electron mass, charge, etc.)
- Utility bills
- Numbers on tax returns (ha!)
- Death rates
- Street addresses
- Numbers in newspapers

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- Cited as evidence of fraud ⬚ in the 2009 Iranian elections.

Benford's Law—The Law of First Digits

Real data:

predicted frequencies

| 30.1 | 17.6 | 12.5 | 9.7 | 7.9 | 6.7 | 5.8 | 5.1 | 4.6 |

frequencies (percent) vs first digit

■ Benford's law ■ newspapers ■ 1990 census ■ Dow Jones
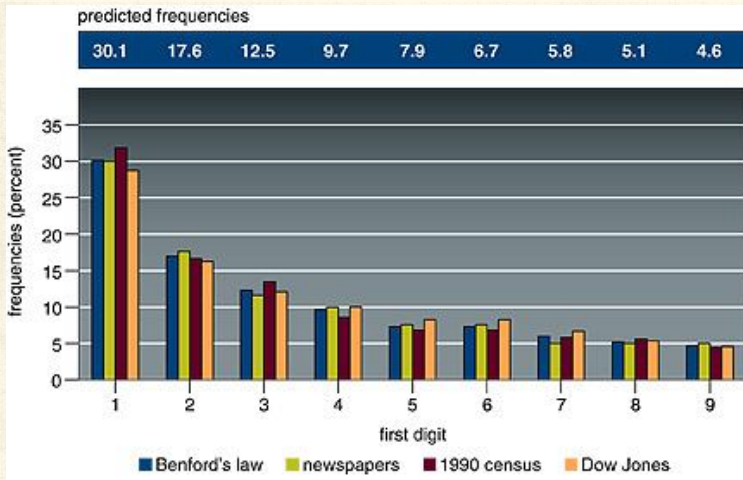
From 'The First-Digit Phenomenon' by T. P. Hill (1998) [1]

Benford's Law—The Law of First Digits

Physical constants of the universe:
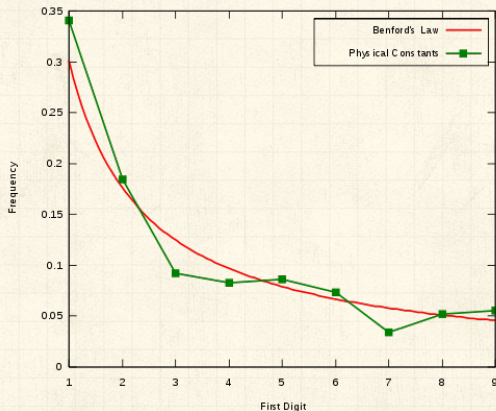
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Benford's Law—The Law of First Digits

Population of countries:
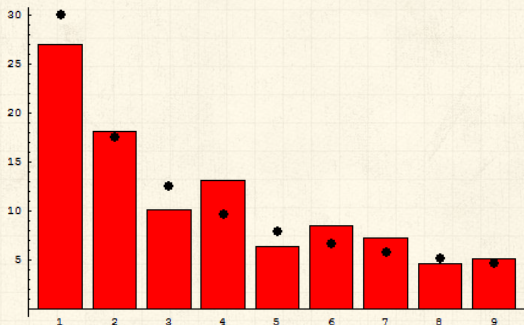
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Essential story

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Essential story

The PoCSverse
Benford's law
11 of 15

Benford's Law

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Observe this distribution if numbers are distributed uniformly in log-space:

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* Power law distributions at work again...
* Extreme case of $\gamma \simeq 1$.

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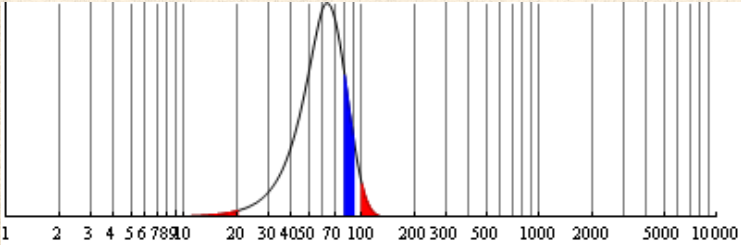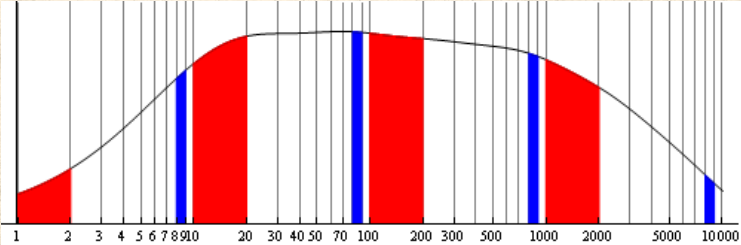
Benford's law

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"Citations to articles citing Benford's law: A Benford analysis" 🔗
Tariq Ahmad Mir,
Preprint available at
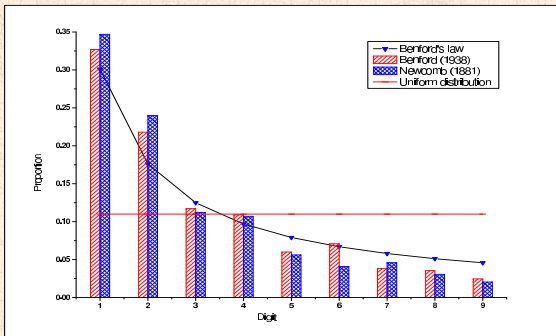https://arxiv.org/abs/1602.01205, 2016. [2]

Fig. 1: The observed proportions of first digits of citations received by the articles citing FB and SN on September 30, 2012. For comparison the proportions expected from BL and uniform distributions are also shown.

On counting and logarithms:

- Earlier: Listen to Radiolab's "Numbers." ⬀.
- Now: Benford's Law ⬀.

References I

[1] T. P. Hill.
The first-digit phenomenon.
American Scientist, 86:358–, 1998.

[2] T. A. Mir.
Citations to articles citing Benford's law: A Benford
analysis, 2016.
Preprint available at
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[3] S. Newcomb.
Note on the frequency of use of the different digits
in natural numbers.
American Journal of Mathematics, 4:39–40, 1881.
pdf ☑