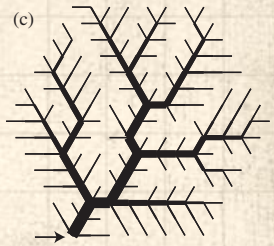
Optimal Supply Networks III: Redistribution

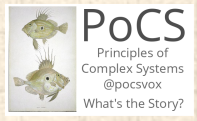
Last updated: 2022/08/28, 08:34:20 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont





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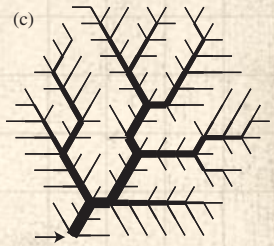

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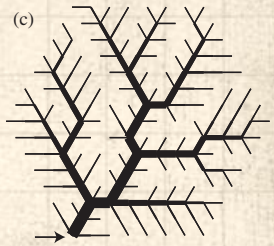
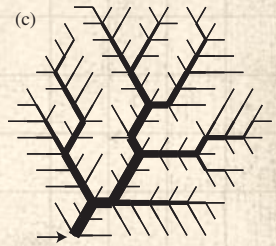


Outline

Distributed Sources

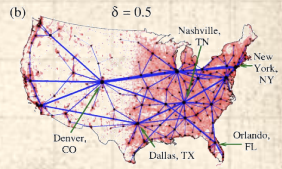
- Size-density law
- Cartograms
- A reasonable derivation
- Global redistribution
- Public versus Private

References



Many sources, many sinks

How do we distribute sources?



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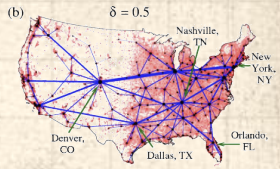

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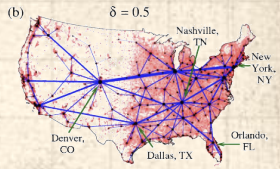

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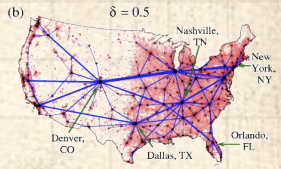

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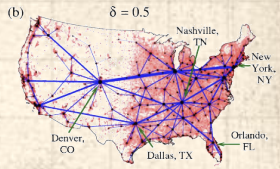

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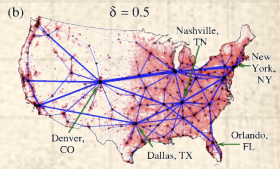

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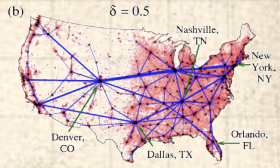
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?



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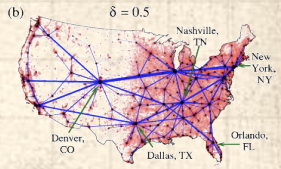
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.



Distributed Sources

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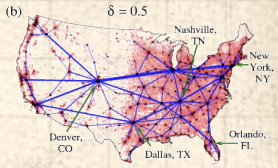
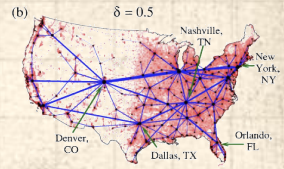


Optimal source allocation

Solidifying the basic problem



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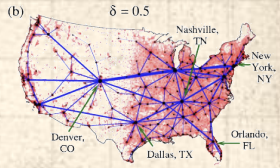
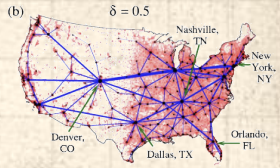
- Given a region with some population distribution ρ, most likely uneven.



Optimal source allocation

Solidifying the basic problem

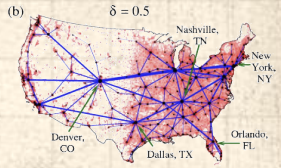
- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.



Optimal source allocation

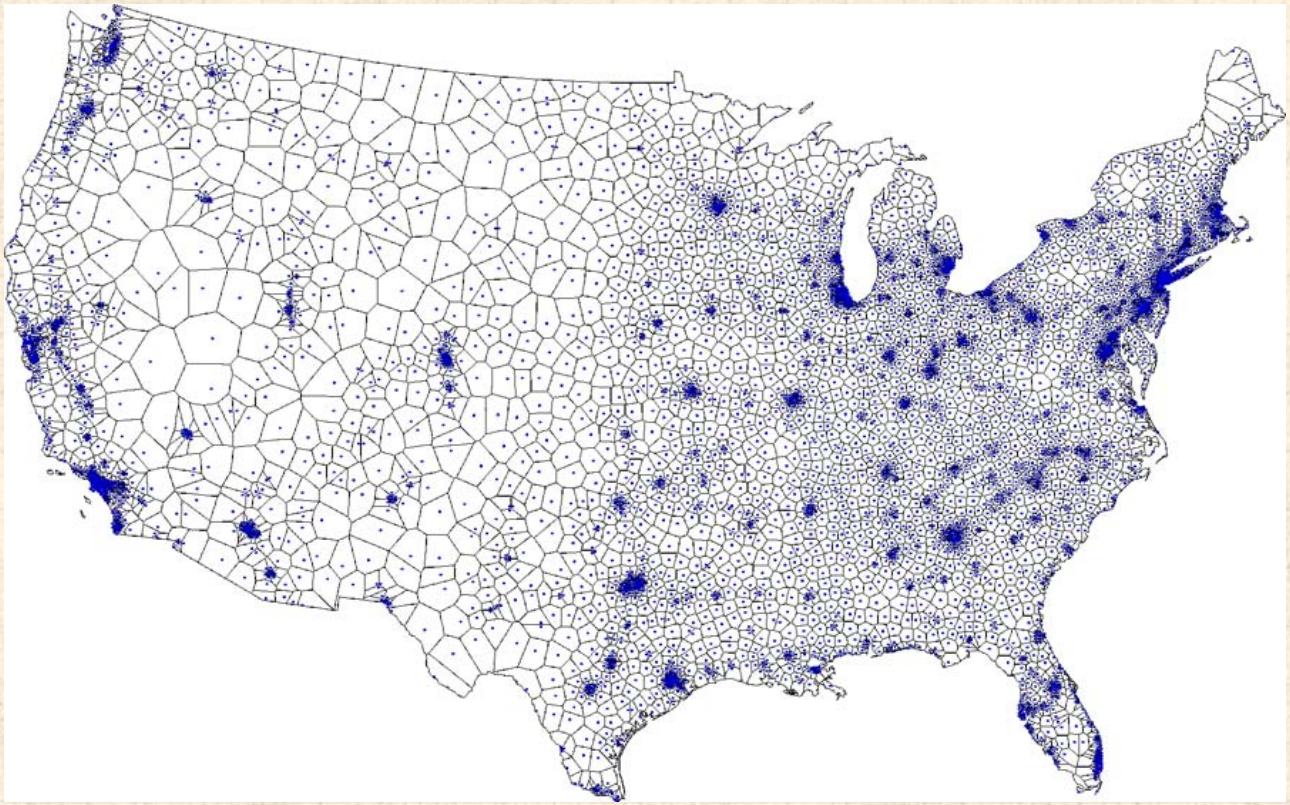
Solidifying the basic problem

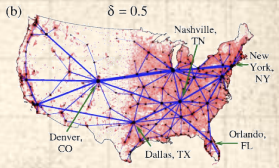
- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?



“Optimal design of spatial distribution networks”

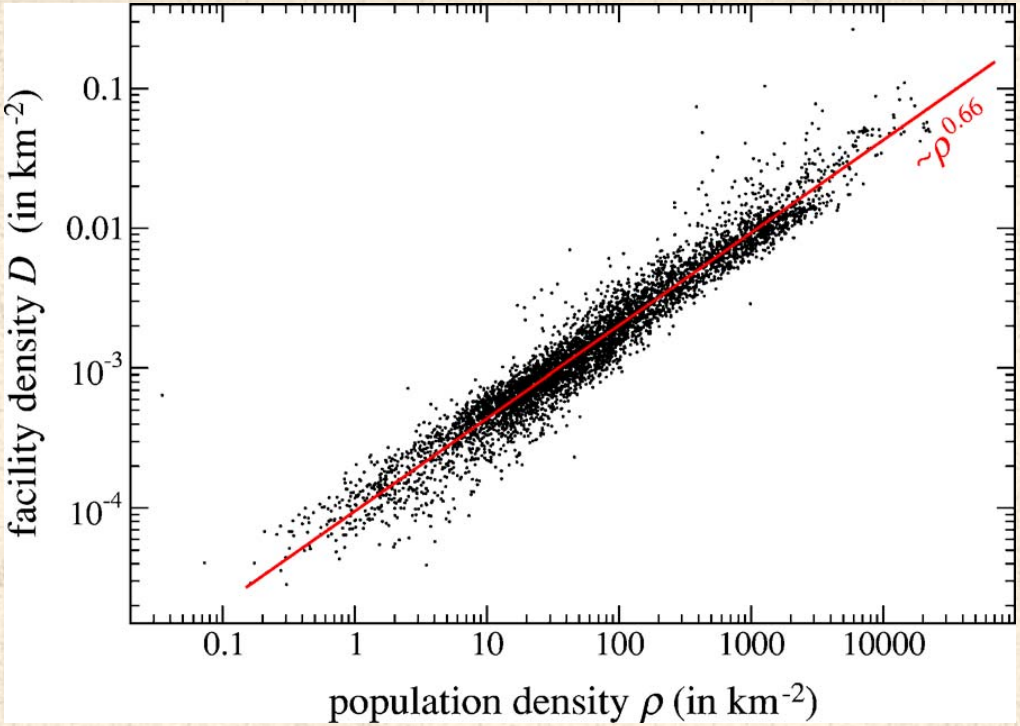
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



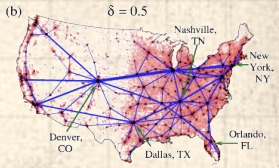


- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.

Optimal source allocation



- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.



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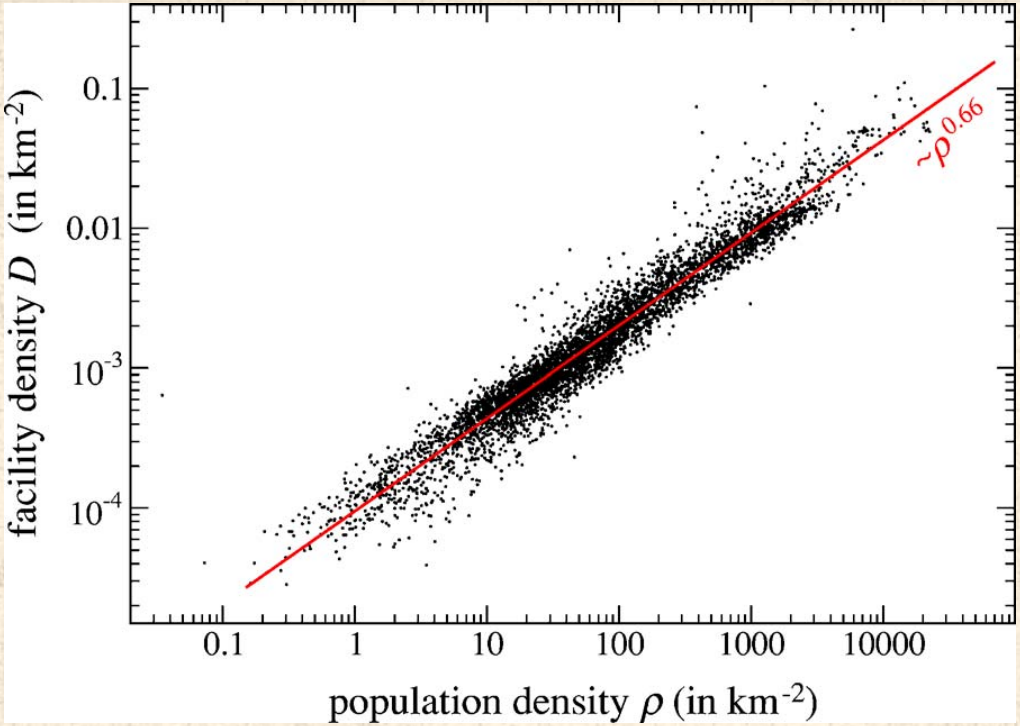
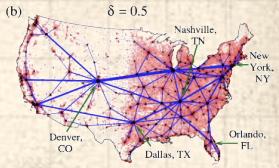


- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.
- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.



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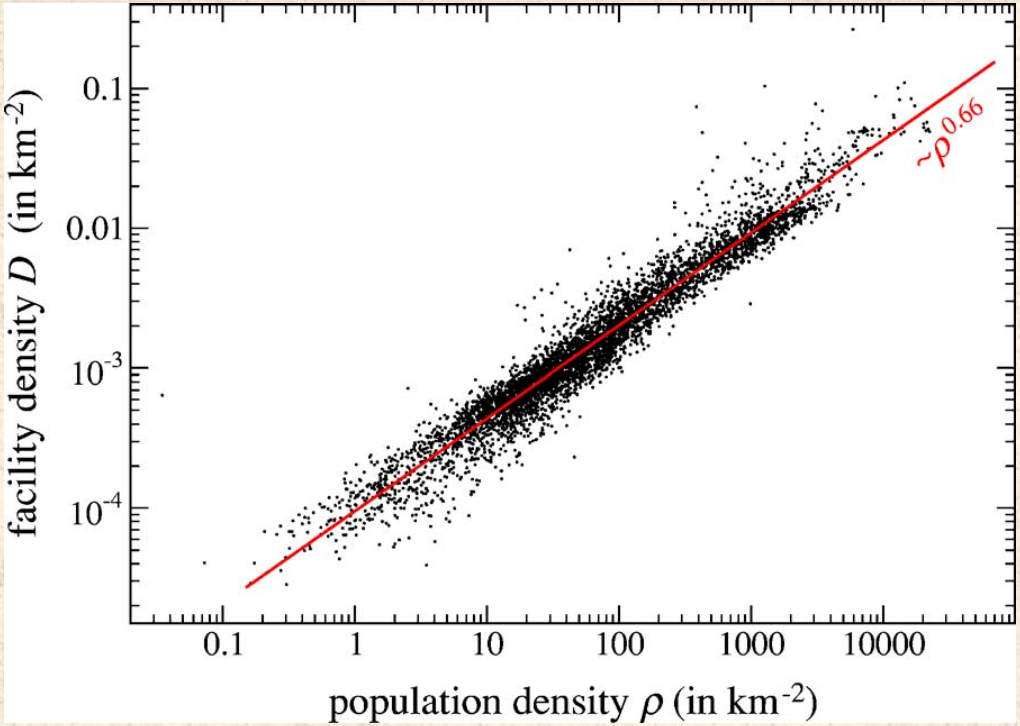

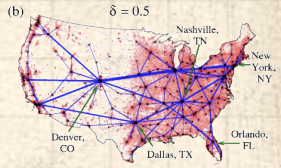

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- Looking good for a 2/3 power ...

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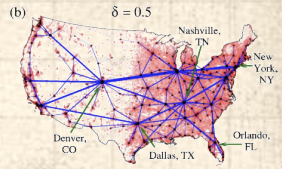
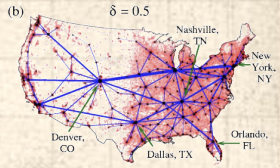


Optimal source allocation

Size-density law:

$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}$$



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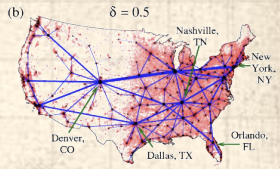

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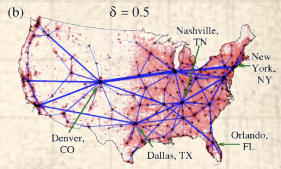

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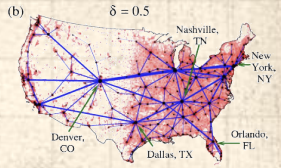
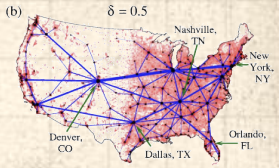
- Now sources & sinks are distributed throughout region.



Optimal source allocation

"Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries"
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]



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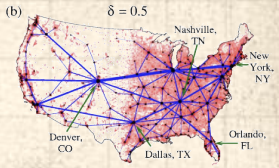

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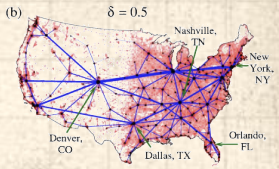
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer Simpson principle.



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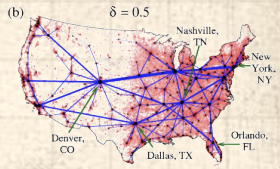

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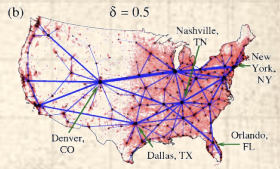

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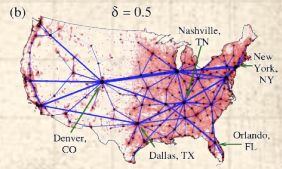

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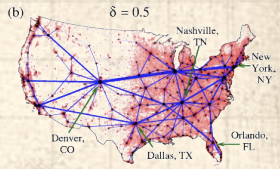

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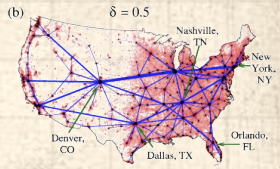
- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.



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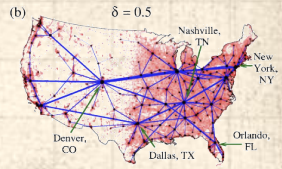

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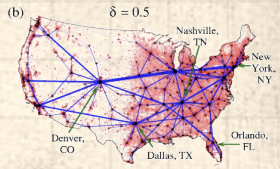

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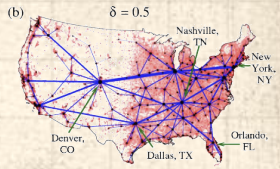

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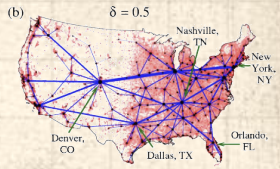

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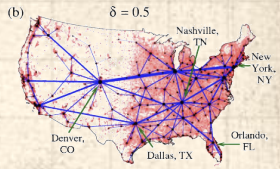

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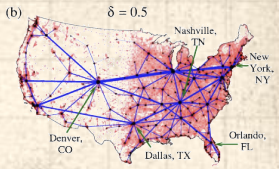
- Important assumption: uniform density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho_{\text{pop}} A)$$



Optimal source allocation

- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity τ.
- If burden of mainenance is shared then average cost per person is τ/P where P = population.
- Replace P by $\rho_{\text{pop}} A$ where ρ_{pop} is density.
- Important assumption: uniform density.
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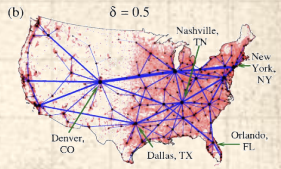
$$T = \bar{d}/\bar{v} + \tau/(\rho_{\text{pop}} A) = c A^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}} A).$$



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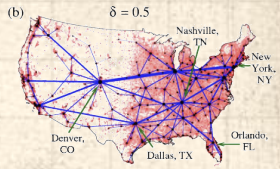
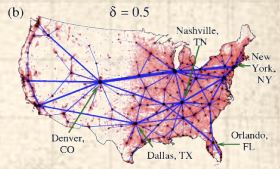


- Now Minimize with respect to A ...

Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A)\right)$$



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$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A)\right)$$

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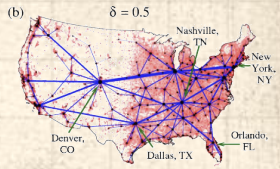

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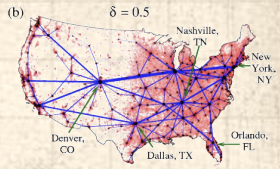

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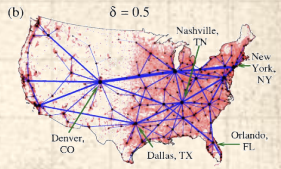

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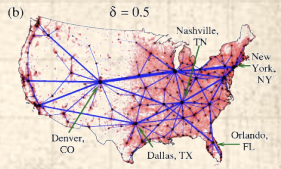

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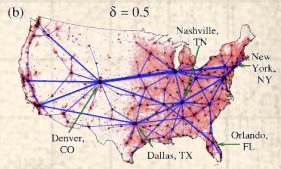

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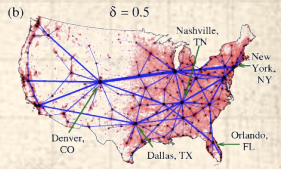
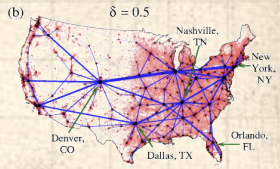


Optimal source allocation

An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)



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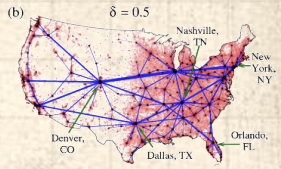
- Stephan's online book "The Division of Territory in Society" is here.
- (It used to be here.)
- The Readme is well worth reading (1995).

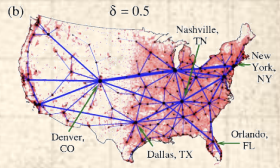


Outline

Distributed Sources

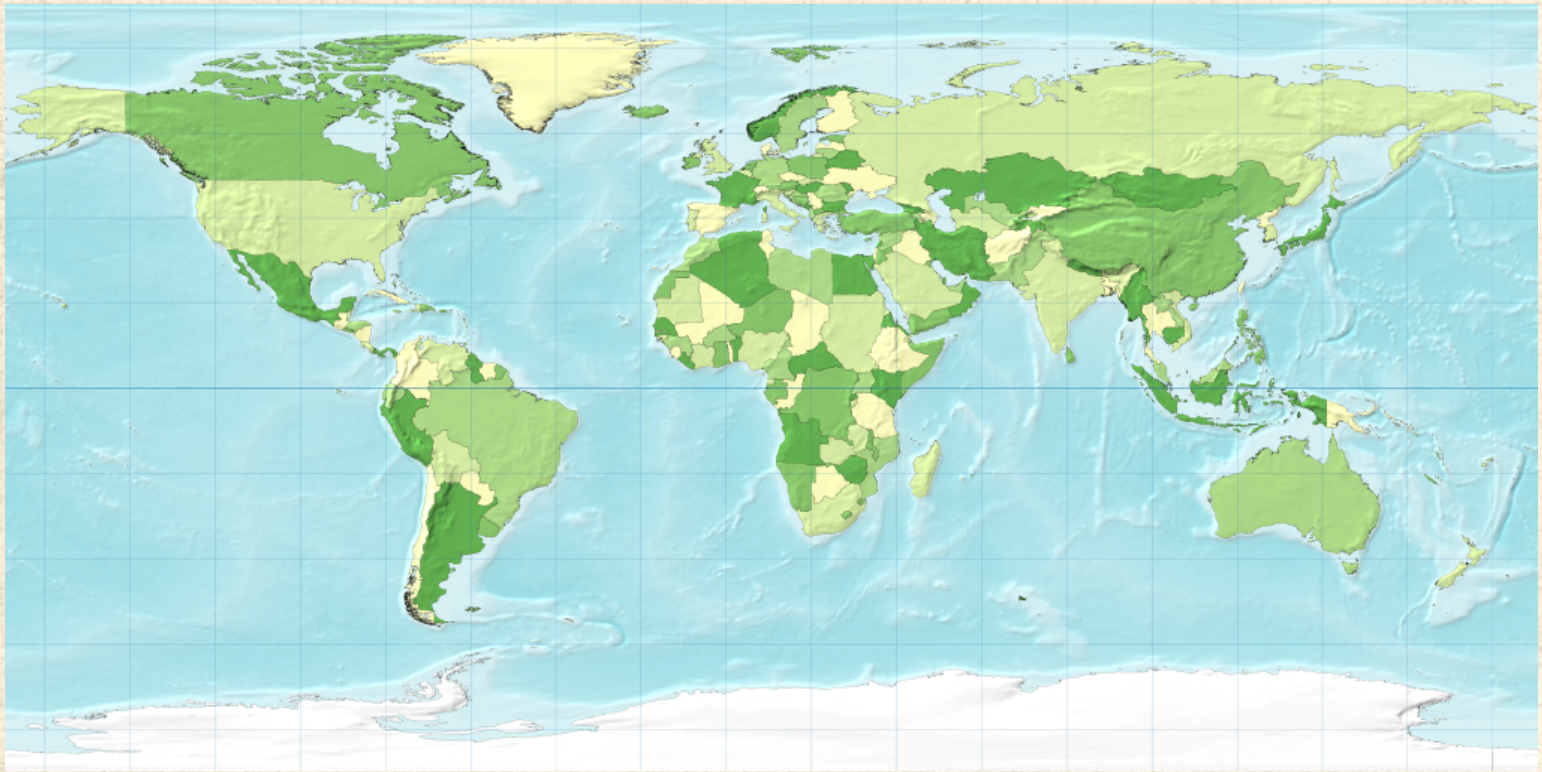
- Size-density law
- Cartograms
- A reasonable derivation
- Global redistribution
- Public versus Private

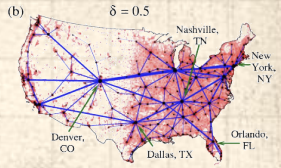
References



Cartograms

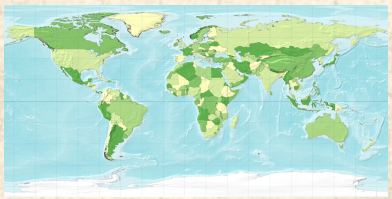
Standard world map:

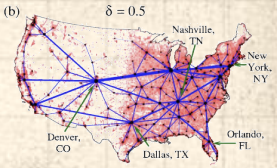




Cartograms

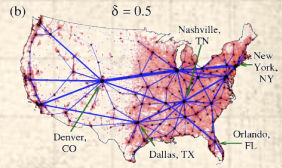
Cartogram of countries ‘rescaled’ by population:





Cartograms

Diffusion-based cartograms:



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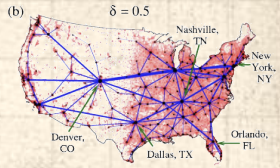

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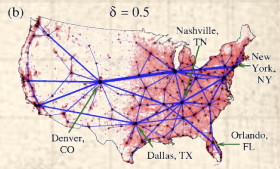

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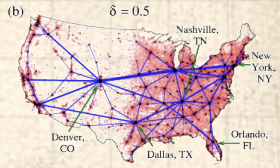

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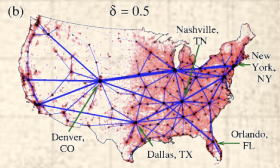

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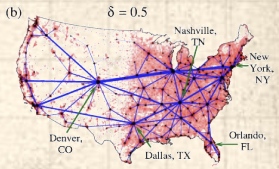
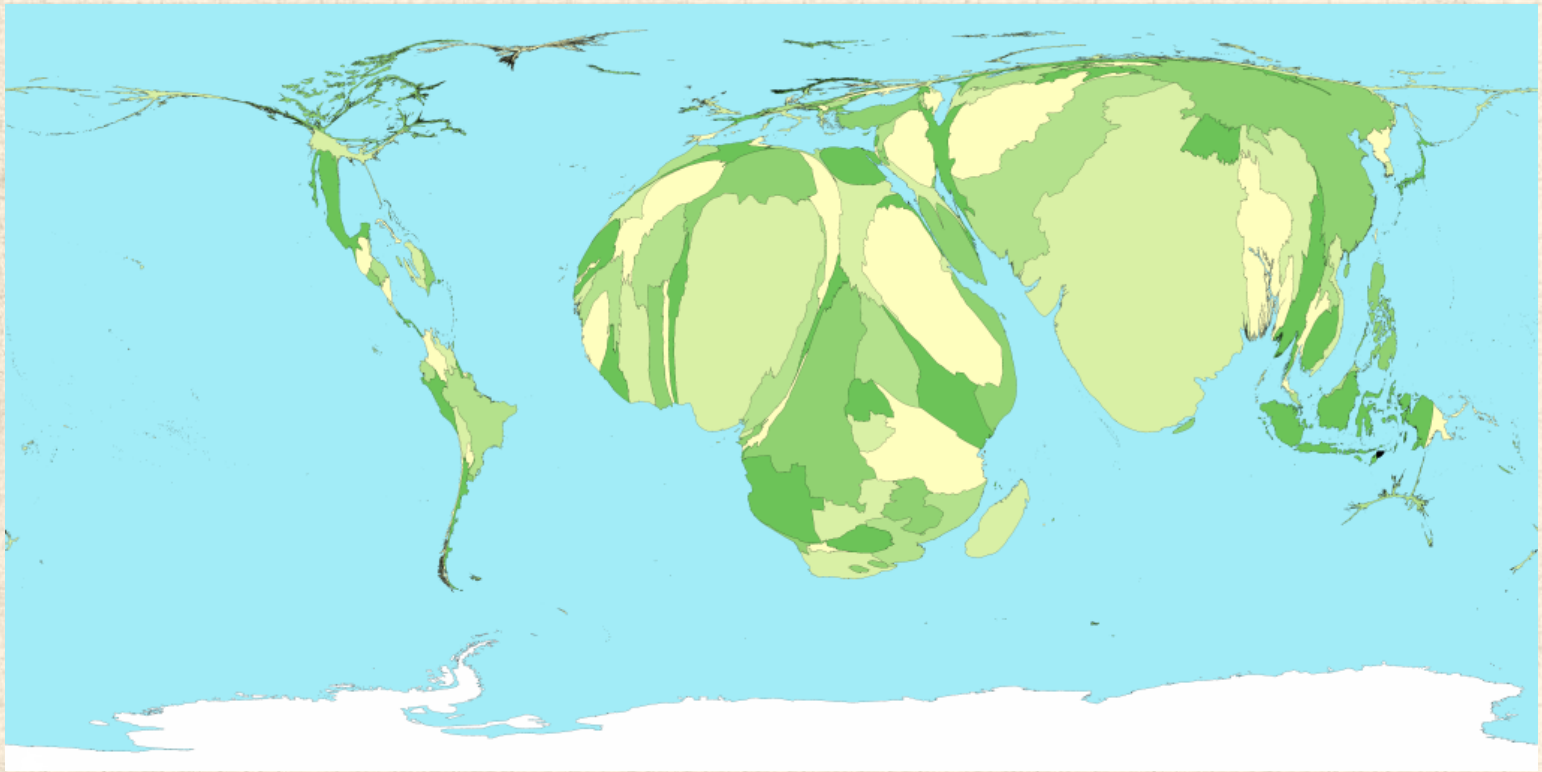
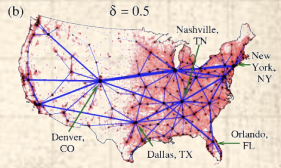


Cartograms

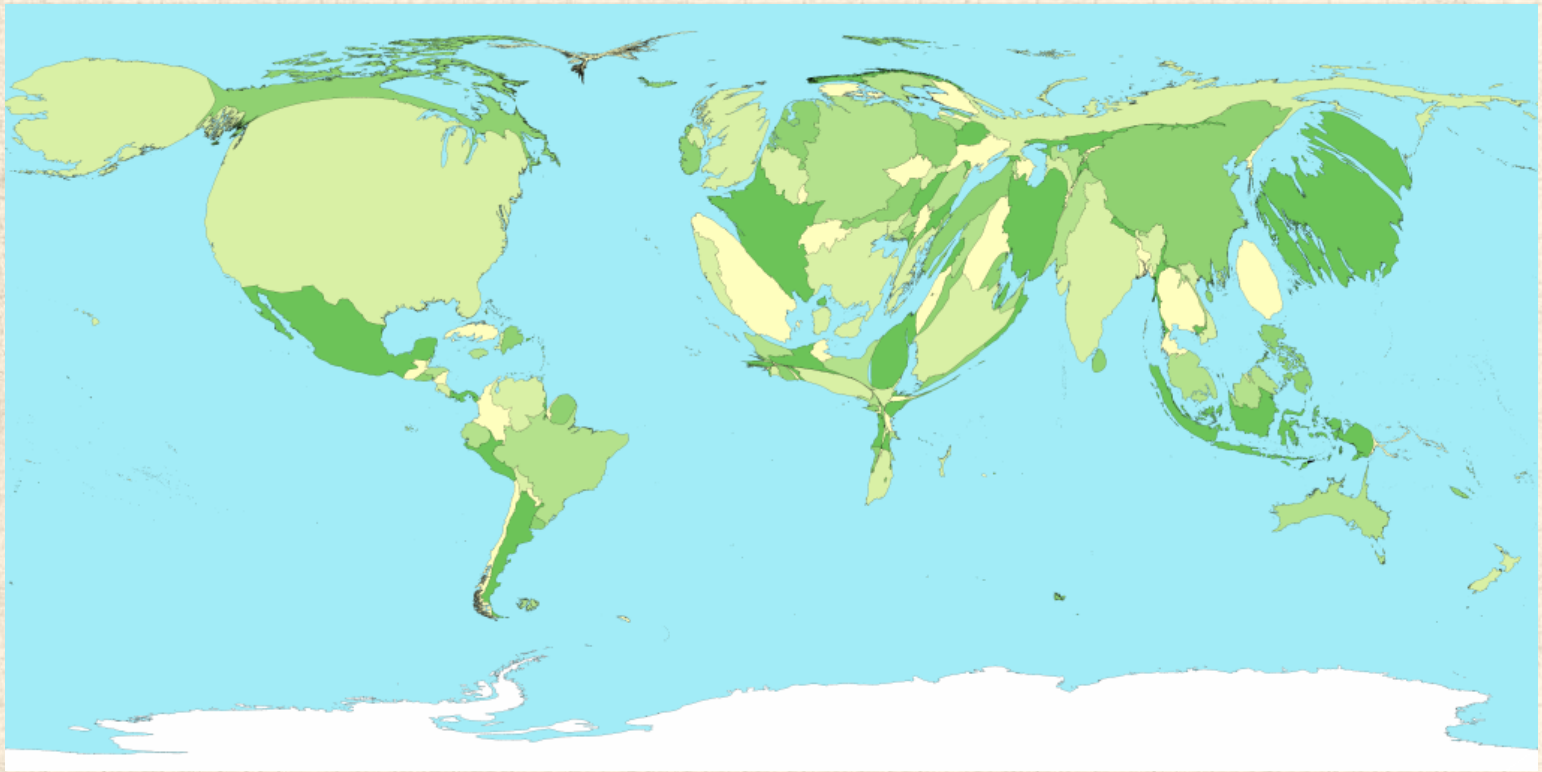
Child mortality:

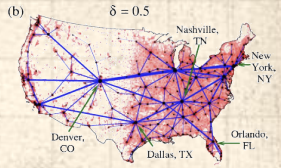




Cartograms

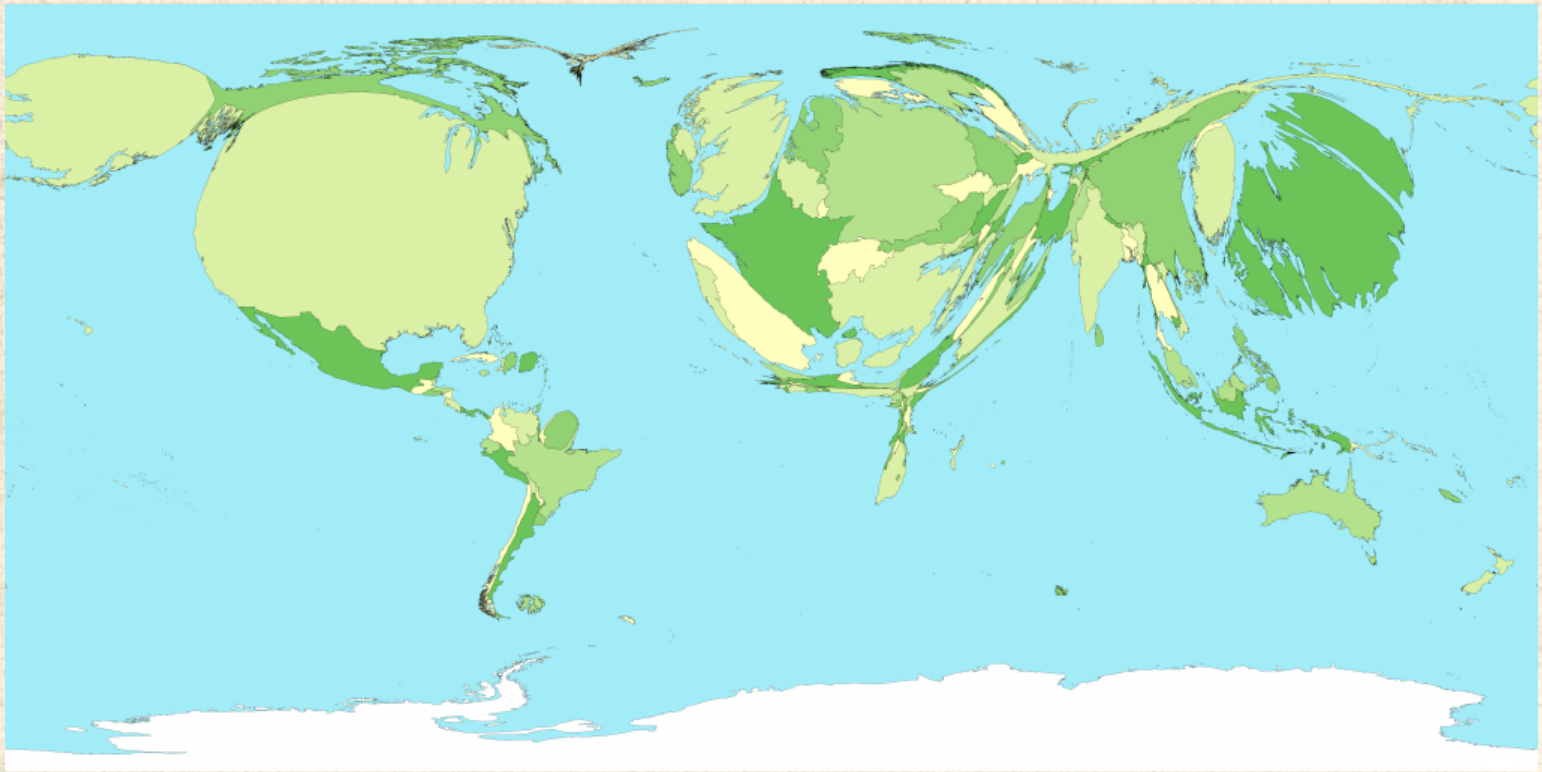
Energy consumption:

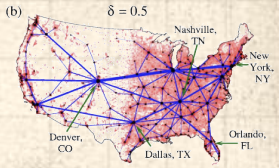




Cartograms

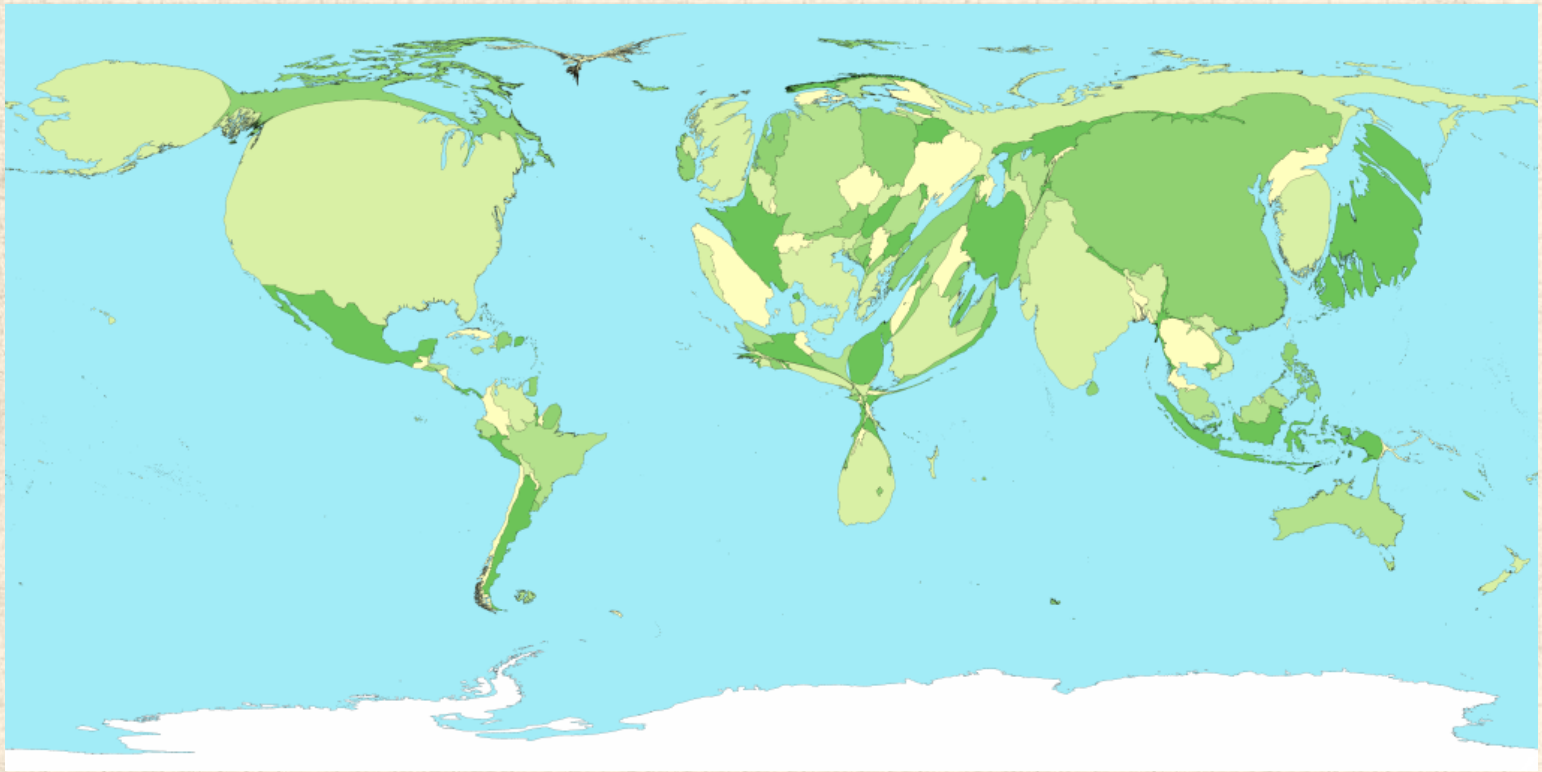
Gross domestic product:

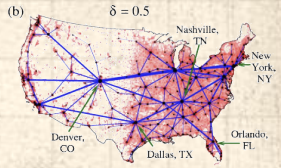




Cartograms

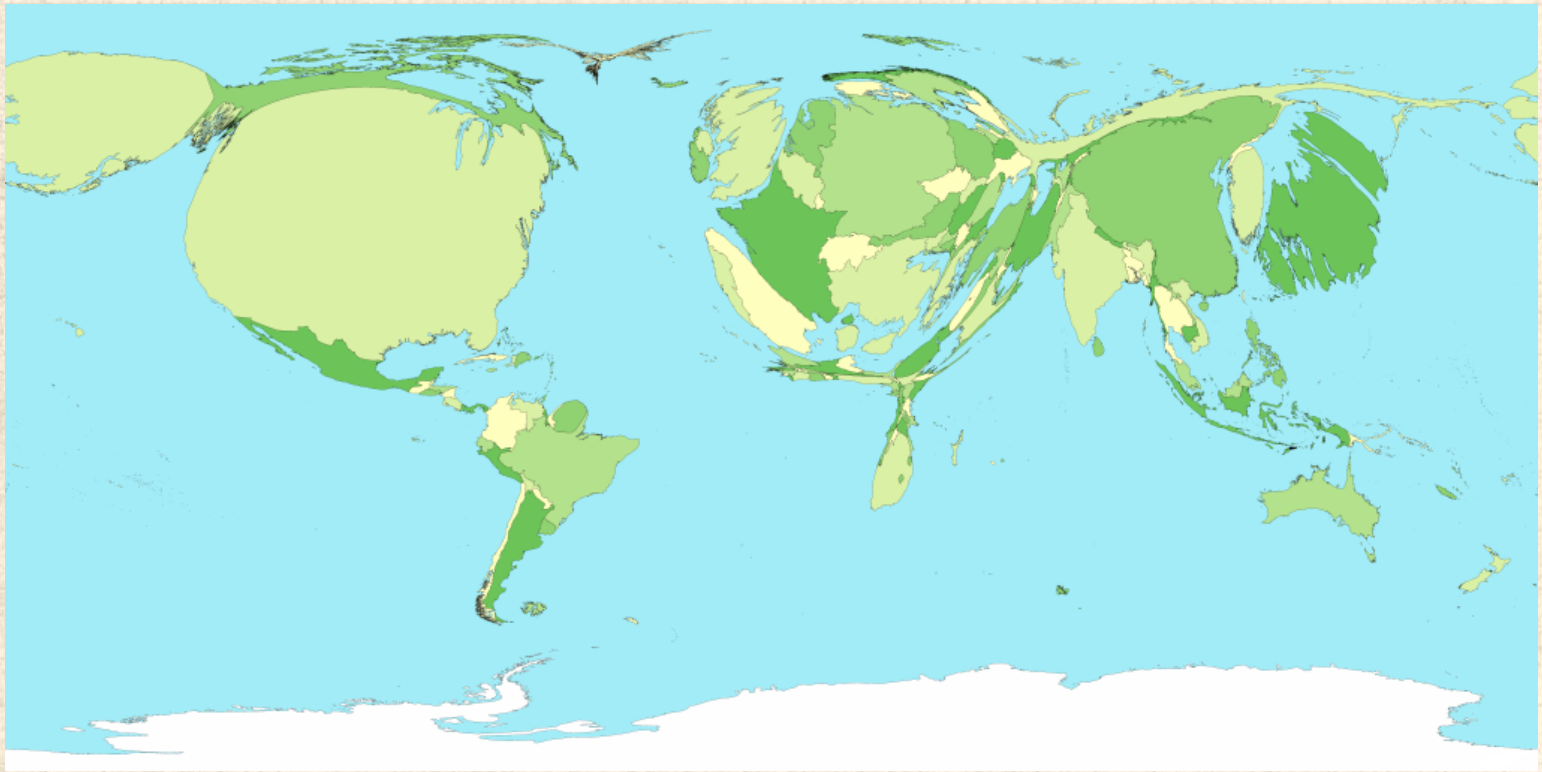
Greenhouse gas emissions:

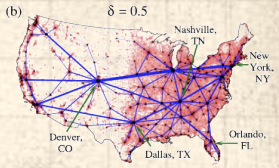




Cartograms

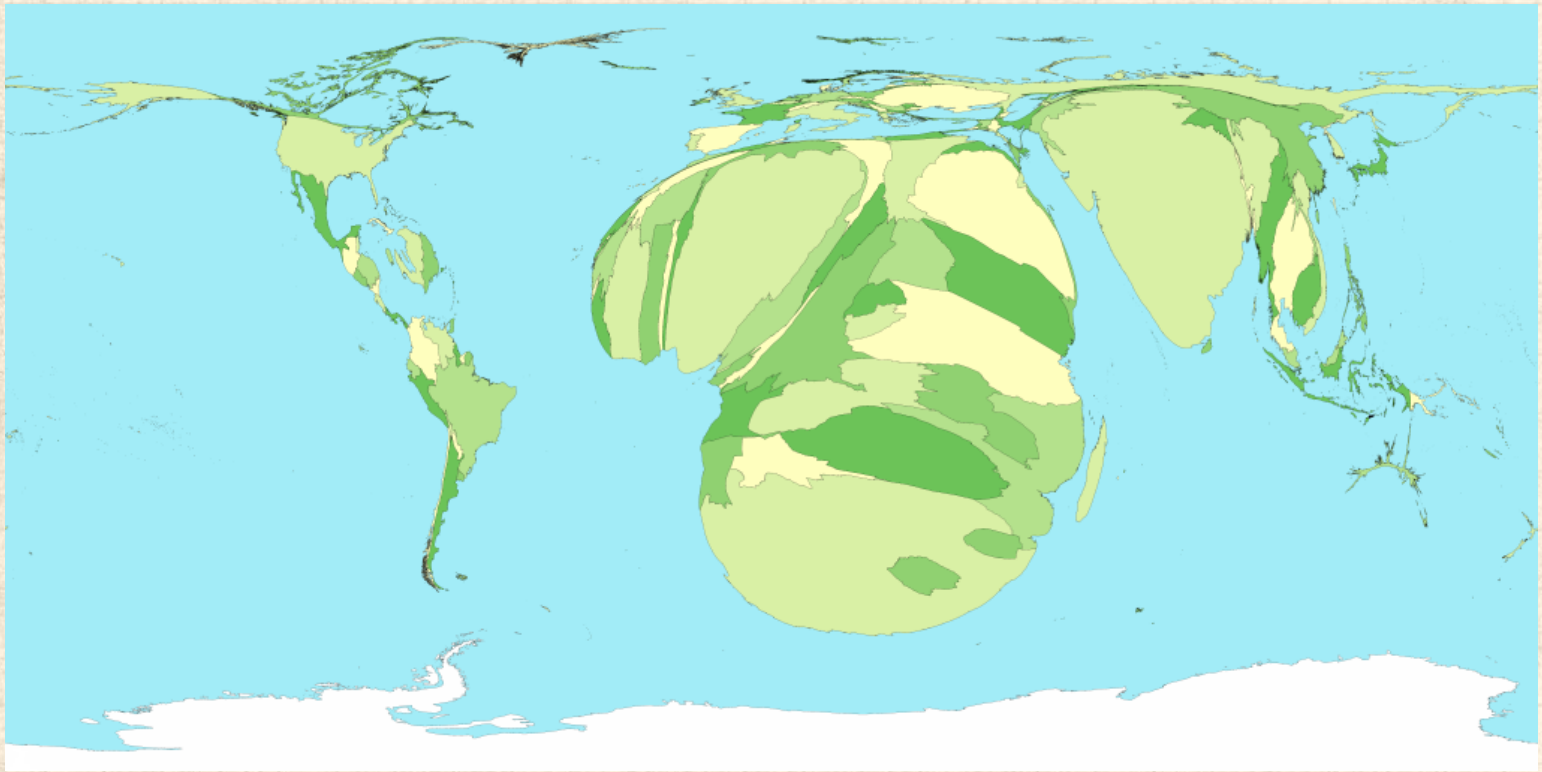
Spending on healthcare:

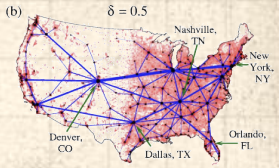




Cartograms

People living with HIV:

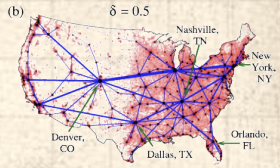




Cartograms

- The preceding sampling of Gastner & Newman's cartograms lives here.
- A larger collection can be found at worldmapper.org.

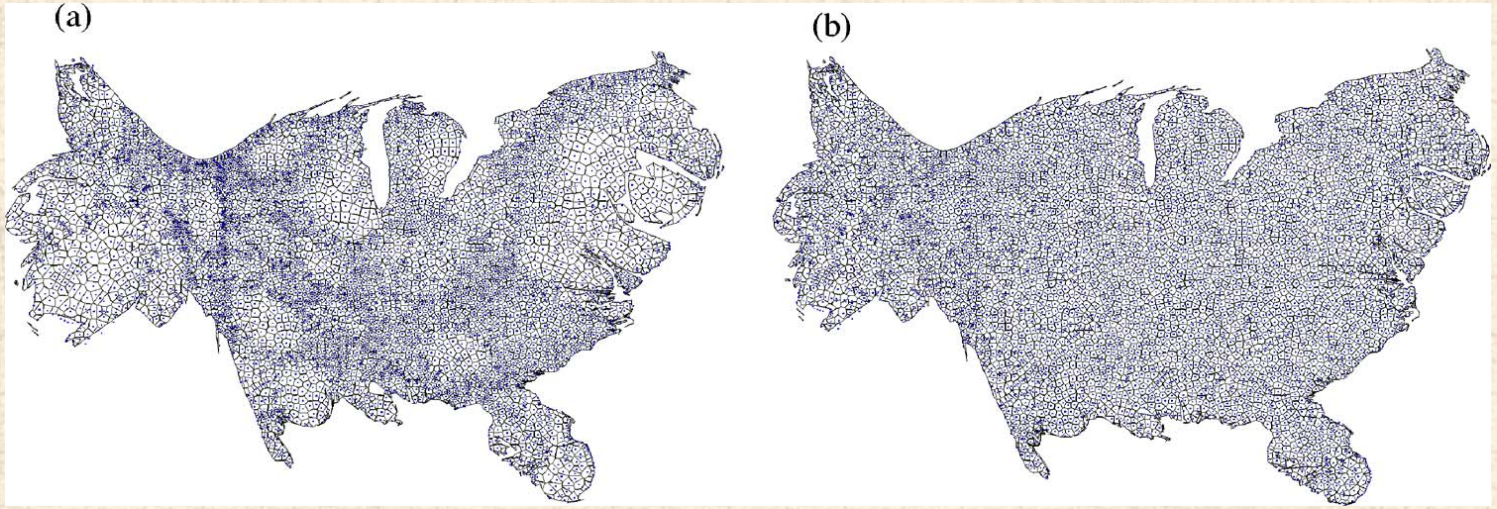
WORLDMAPPER The world as you've never seen it before



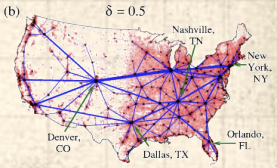
Size-density law

"Optimal design of spatial distribution networks"

Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



Left: population density-equalized cartogram.



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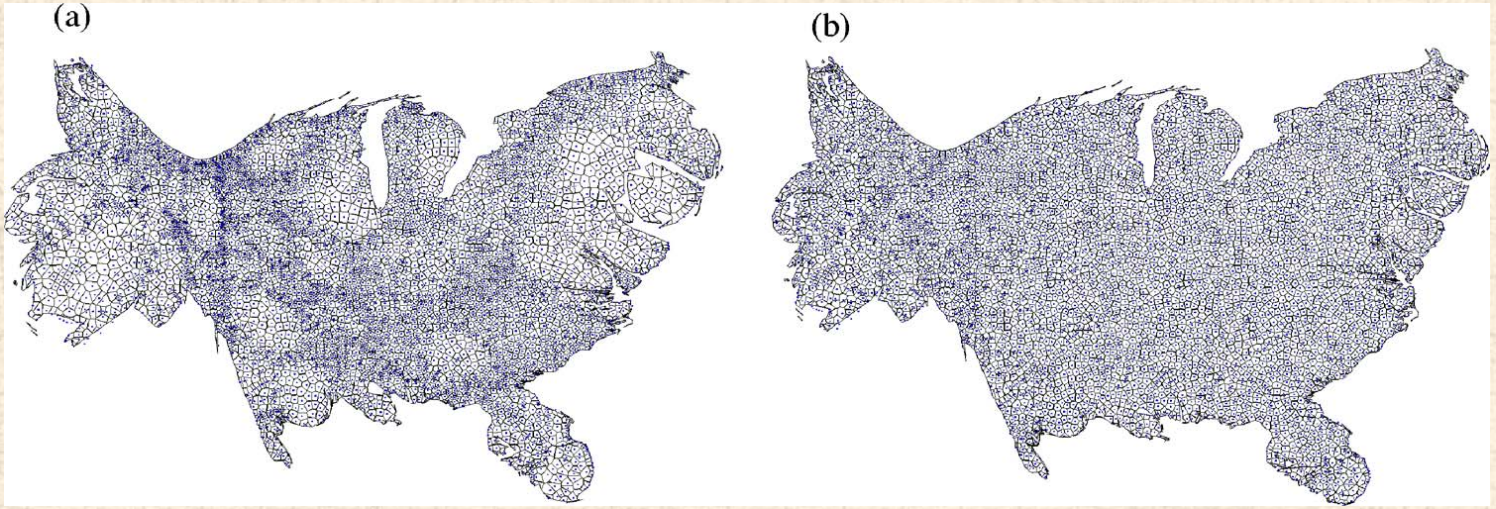
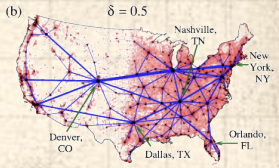


- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.



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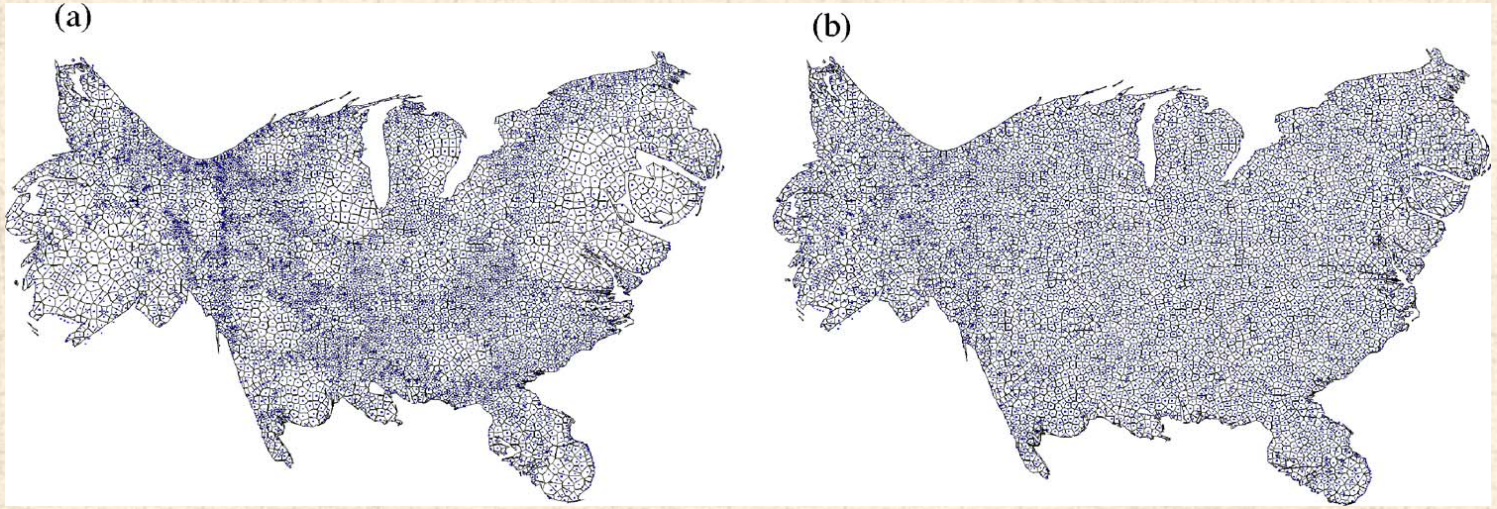
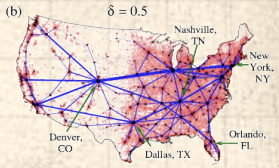


- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.



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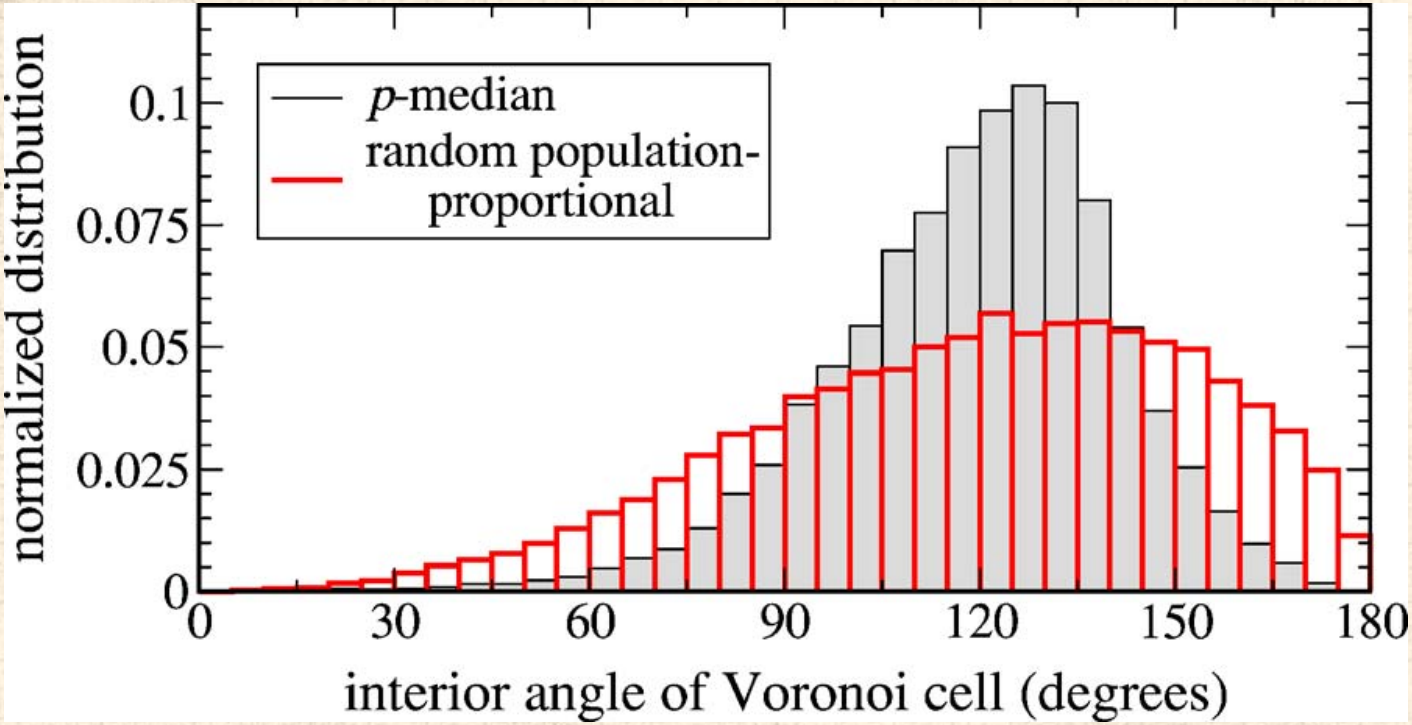
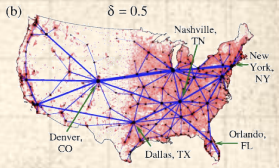


From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.



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- Global redistribution
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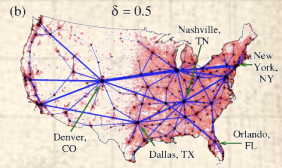

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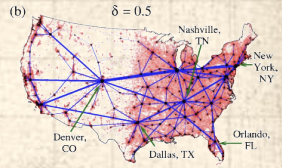
Deriving the optimal source distribution:



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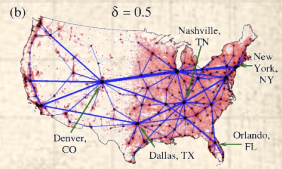

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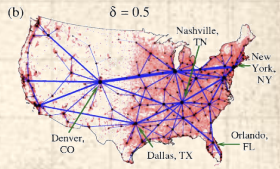

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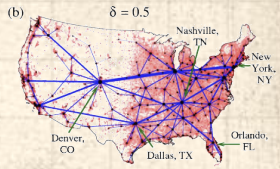

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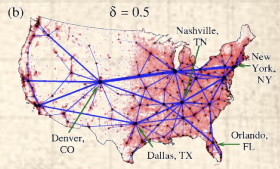

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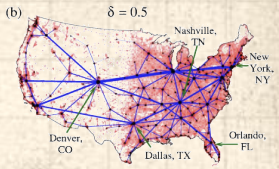

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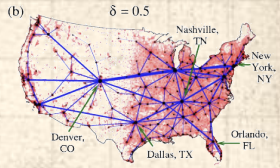

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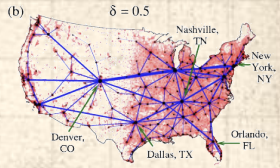

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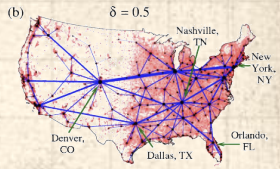

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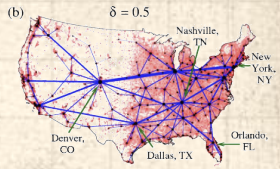

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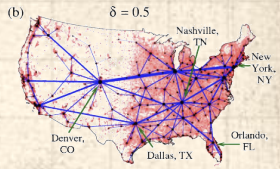

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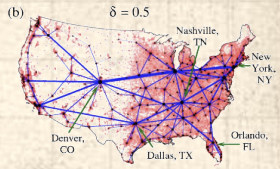

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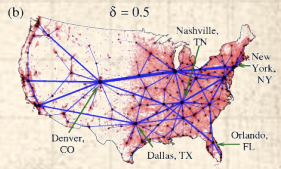

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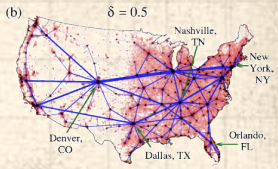

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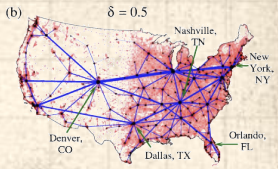

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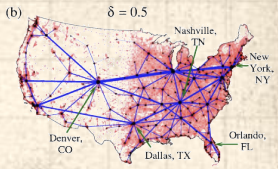

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- By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

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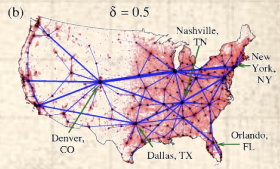

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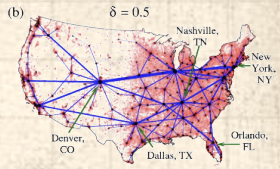

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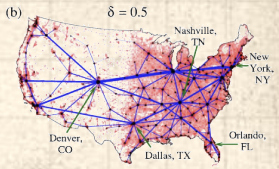

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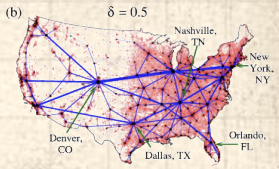

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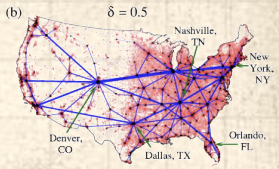

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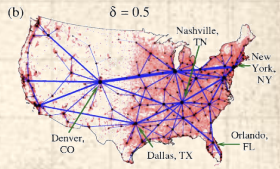

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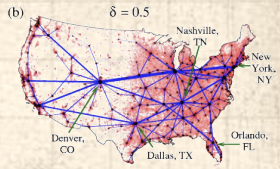

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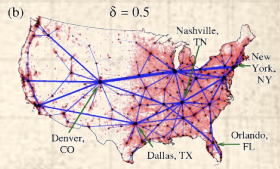

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- Normalizing (or solving for λ):

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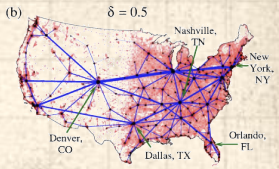
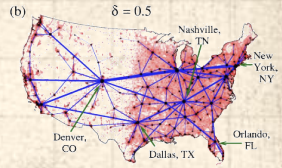


Outline

Distributed Sources

- Size-density law
- Cartograms
- A reasonable derivation
- Global redistribution
- Public versus Private

References



Global redistribution networks

One more thing:

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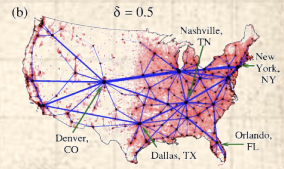

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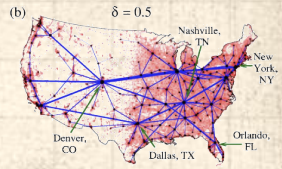

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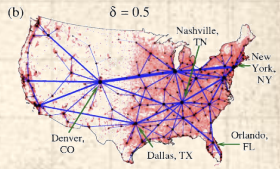

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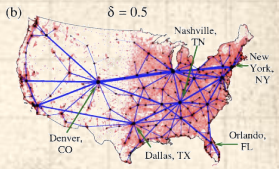

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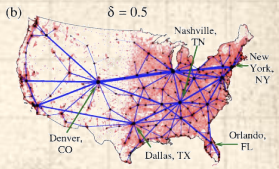

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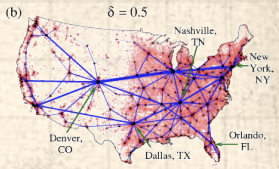

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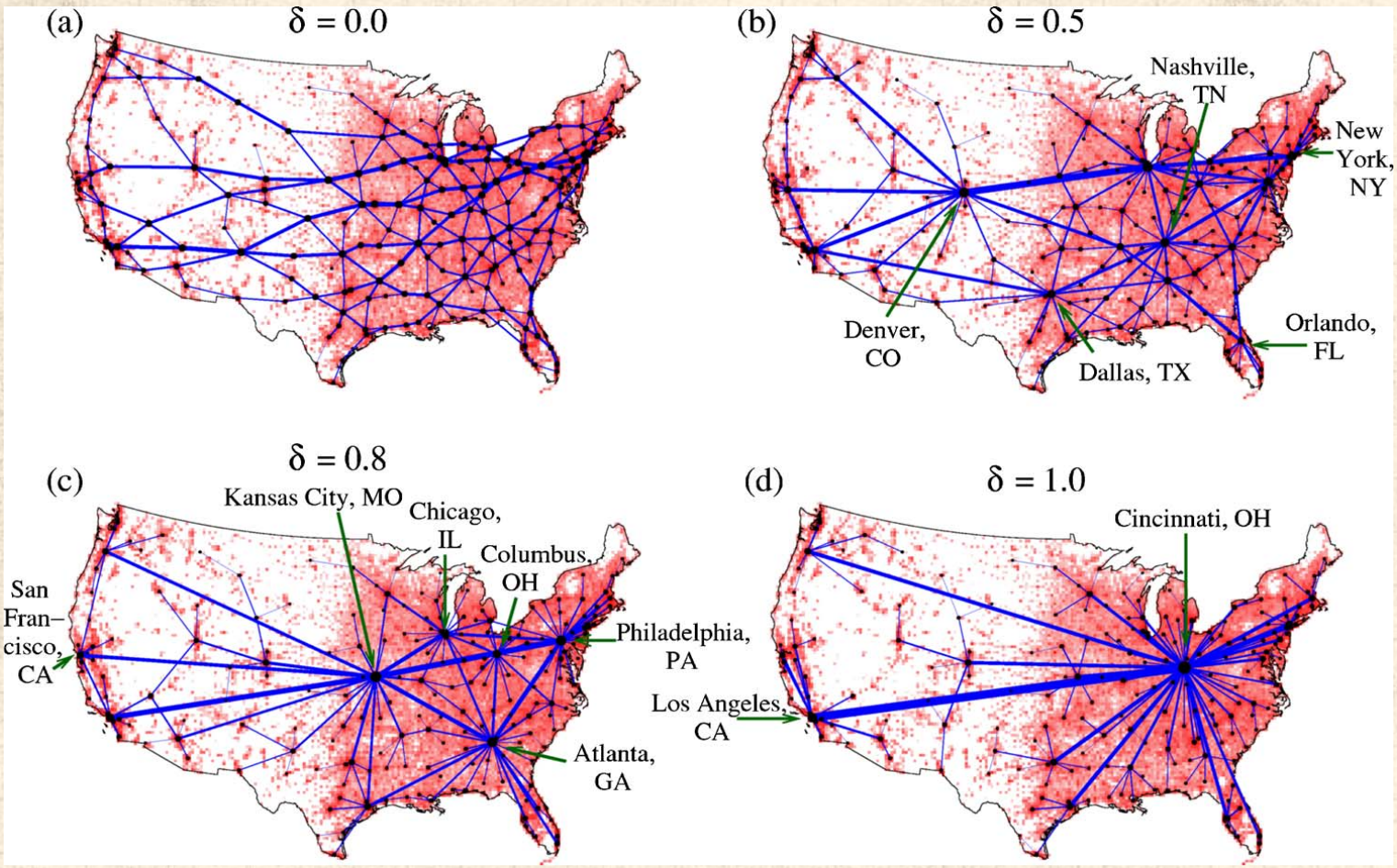
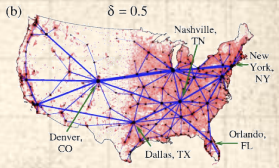
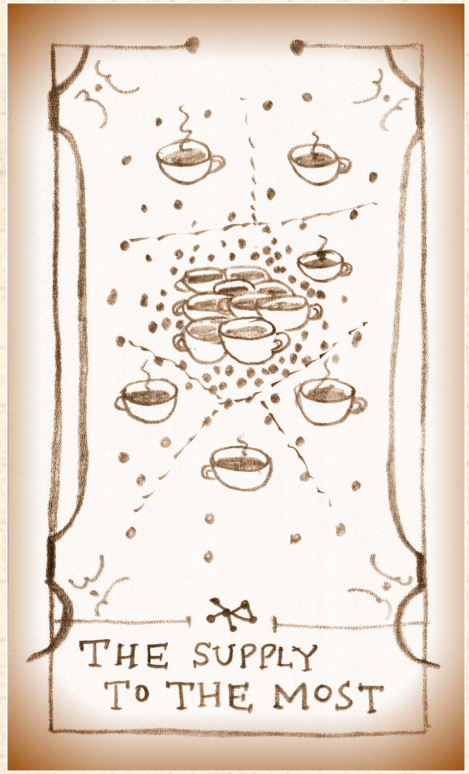
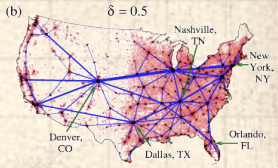


From Gastner and Newman (2006) [2]





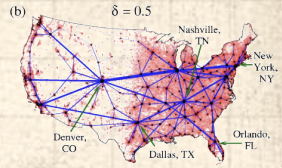


Outline

Distributed Sources

- Size-density law
- Cartograms
- A reasonable derivation
- Global redistribution
- Public versus Private

References



Public versus private facilities

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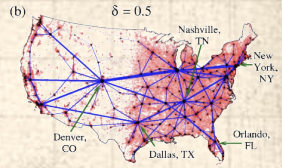

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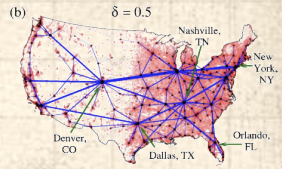

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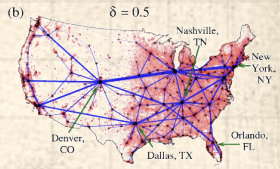
$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

does not universally hold with $\alpha = 2/3$.



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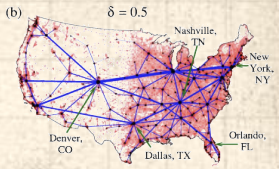

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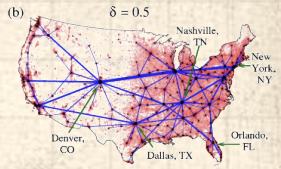
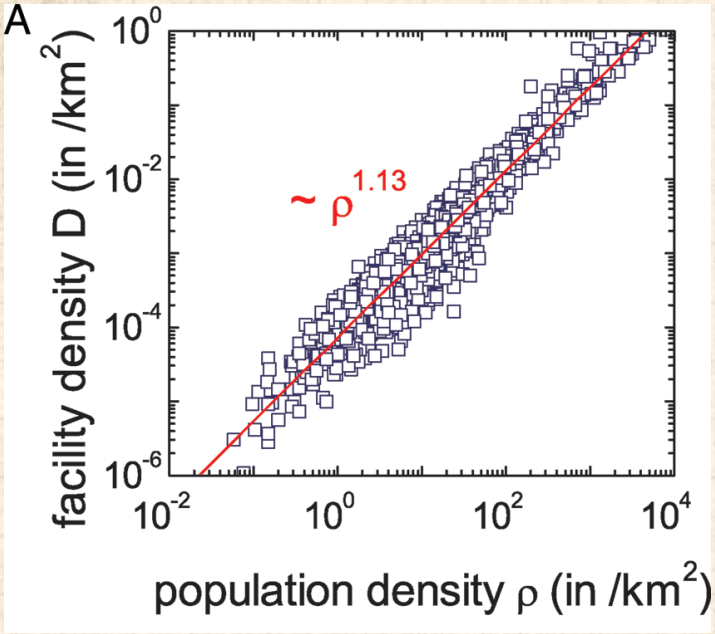
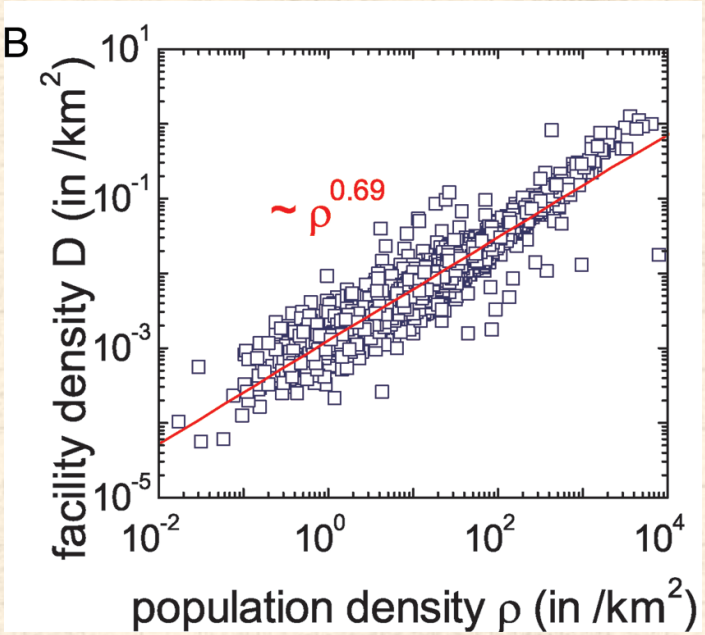
- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.

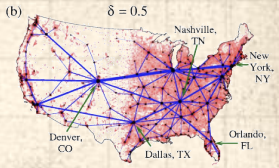


Public versus private facilities: evidence





- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.



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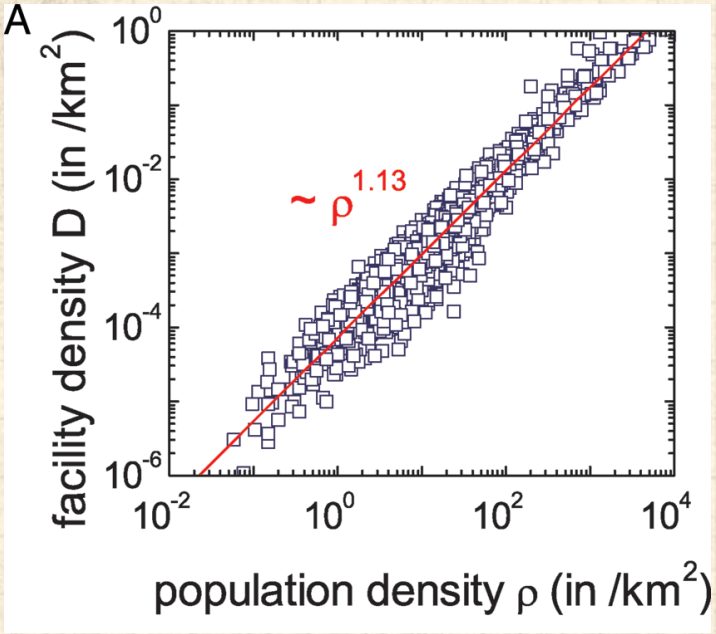
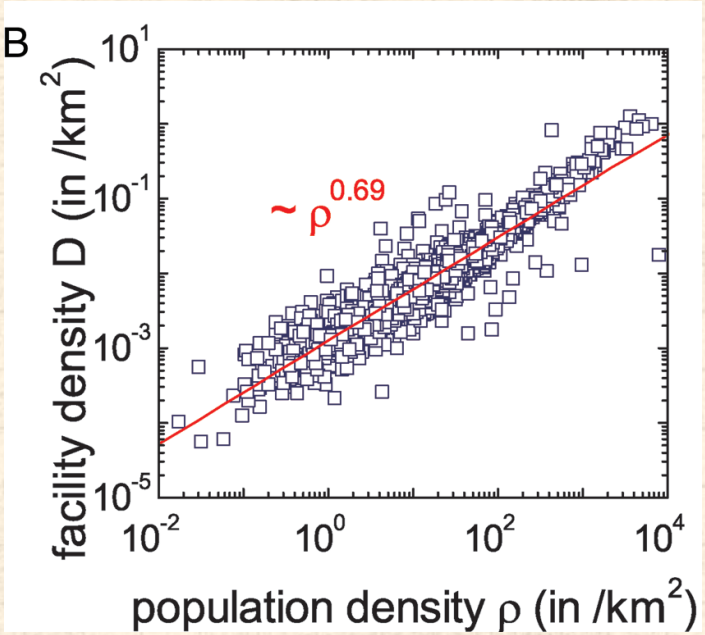
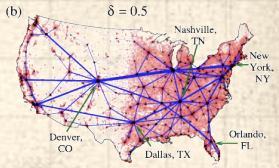




- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\text{pop}} \simeq 100$.



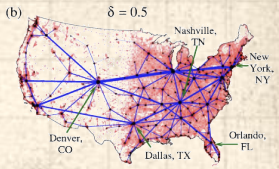
Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.



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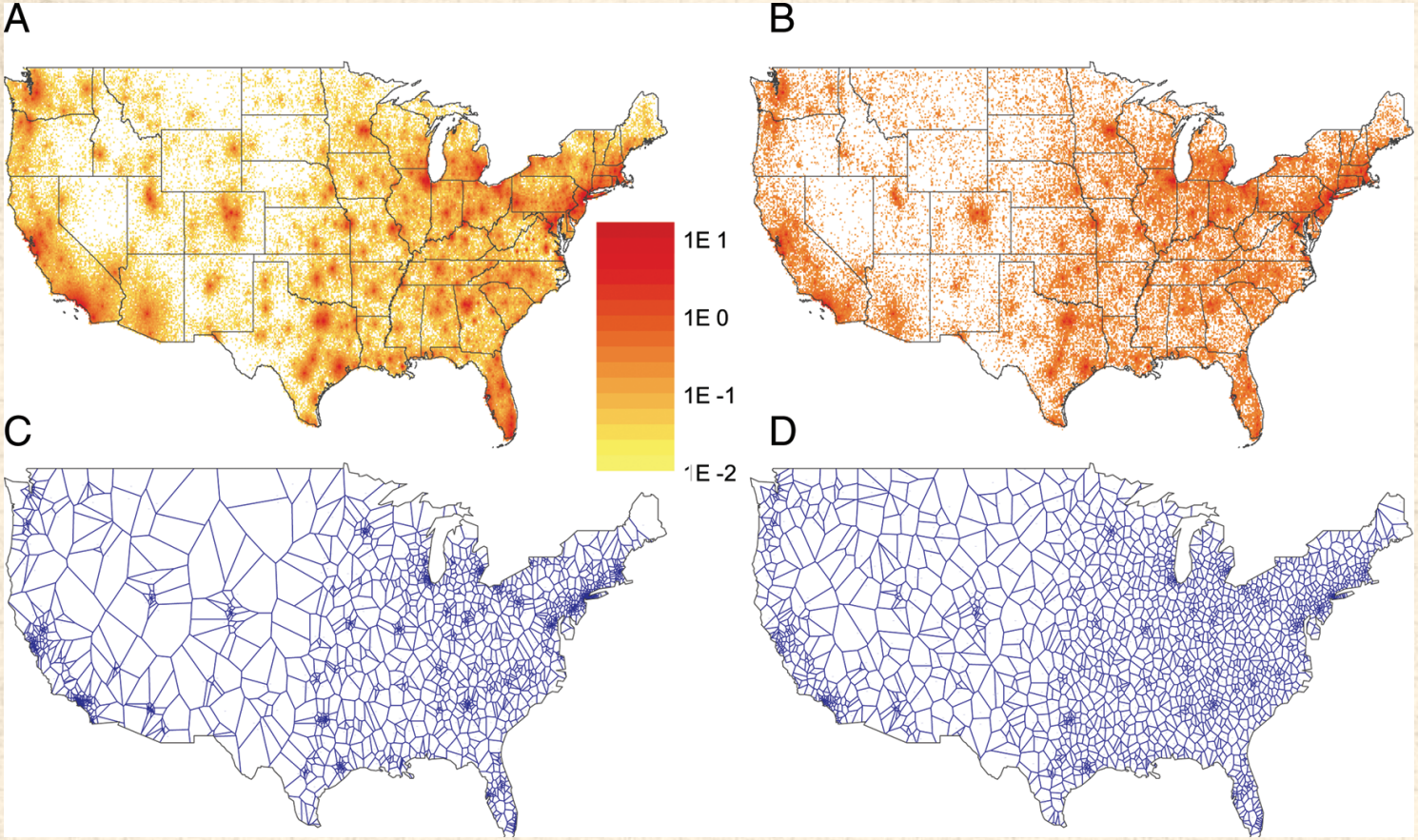
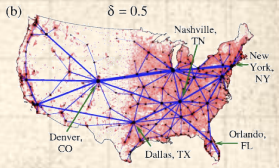


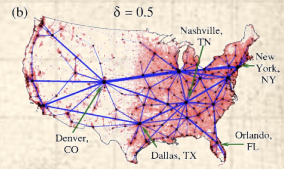
A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.



Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.



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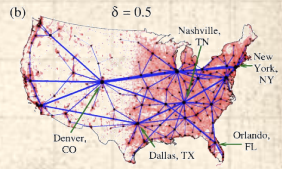

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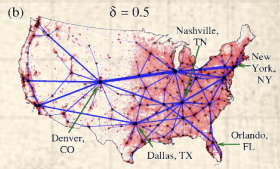

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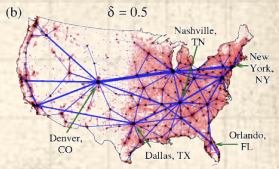
$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$



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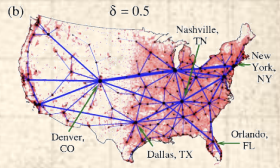
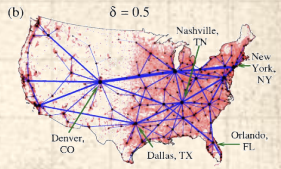
- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.



Public versus private facilities: the story

- Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\text{fac}}(\vec{x}) = n\frac{[\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_\Omega [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}\mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}.$$



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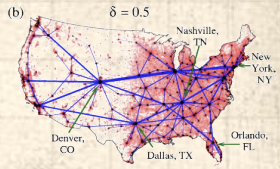
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.



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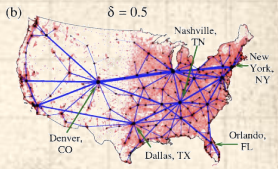
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.



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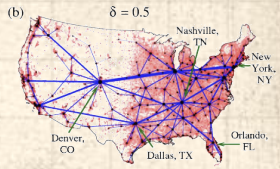

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